Part B  Tensors

\[ x = x'\cos\theta - y'\sin\theta \]
\[ y = -x'\sin\theta + y'\cos\theta \]

or

\[ \begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \]

Or, inverting

\[ x' = x\cos\theta - y\sin\theta \]
\[ y' = x\cos\theta - y\sin\theta \]

or

\[ \begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \]

The graph above represents a transformation of coordinates when the system is rotated at an angle \( \theta \) CCW.
Let $\vec{a}, \vec{b}$ be two vectors in a Cartesian coordinate system. If $\tilde{T}$ is a transformation, which transforms any vector into some other vector, we can write

\[
\tilde{T}\vec{a} = \vec{c} \\
\tilde{T}\vec{b} = \vec{d}
\]

where $\vec{c}$ and $\vec{d}$ are two different vectors.

If

\[
\tilde{T}(\vec{a} + \vec{b}) = \tilde{T}\vec{a} + \tilde{T}\vec{b} \quad (1.1a)
\]

\[
\tilde{T}(\alpha\vec{a}) = \alpha\tilde{T}\vec{a} \quad (1.2a)
\]

For any arbitrary vector $\vec{a}$ and $\vec{b}$ and scalar $\alpha$

$\tilde{T}$ is called a LINEAR TRANSFORMATION and a Second Order Tensor. (1.1a) and (1.2a) can be written as,

\[
\tilde{T}(\alpha\vec{a} + \beta\vec{b}) = \alpha\tilde{T}\vec{a} + \beta\tilde{T}\vec{b} \quad (1.3)
\]
2B2 Components of a Tensor

\[ \vec{T}_{e_1} = T_{11} \hat{e}_1 + T_{21} \hat{e}_2 + T_{31} \hat{e}_3 \]
\[ \vec{T}_{e_2} = T_{12} \hat{e}_1 + T_{22} \hat{e}_2 + T_{32} \hat{e}_3 \]
\[ \vec{T}_{e_3} = T_{13} \hat{e}_1 + T_{23} \hat{e}_2 + T_{33} \hat{e}_3 \]

\[ \sim \wedge \vec{T} e_i = T_{ji} e_j \]

In general \( T_{ij} = \hat{e}_i \cdot \vec{T} \hat{e}_j \) (2-2)

The components of \( T_{ij} \) can be written as

\[
[T] = \begin{bmatrix}
T_{11} & T_{12} & T_{13} \\
T_{21} & T_{22} & T_{23} \\
T_{31} & T_{32} & T_{33}
\end{bmatrix}
\]
Example 2B2.3

Q: Let \( R \) correspond to a right-hand rotation of a rigid body about the \( x_3 \) axis by an angle \( \theta \). Find a matrix of \( R \)

\[
\begin{align*}
\tilde{R}\hat{e}_1 &= \cos \theta \hat{e}_1 + \sin \theta \hat{e}_2 \\
\tilde{R}\hat{e}_2 &= -\sin \theta \hat{e}_1 + \cos \theta \hat{e}_2 \\
\tilde{R}\hat{e}_3 &= \hat{e}_3
\end{align*}
\]

Thus

\[
[R] = \begin{bmatrix}
\cos \theta & -\sin \theta & 0 \\
\sin \theta & \cos \theta & 0 \\
0 & 0 & 1
\end{bmatrix}
\]

Components of Tensor

\[
\begin{align*}
x &= x' \cos \theta - y' \sin \theta \\
y &= x' \sin \theta + y' \cos \theta
\end{align*}
\]

or

\[
\begin{bmatrix}
x \\
y
\end{bmatrix} = \begin{bmatrix}
\cos \theta & -\sin \theta \\
\sin \theta & \cos \theta
\end{bmatrix} \begin{bmatrix}
x' \\
y'
\end{bmatrix}
\]
On inverting,

\[ x' = x \cos \theta + y \sin \theta \]
\[ y' = -x \sin \theta + y \cos \theta \]

or

\[
\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}
\]

Above represents a transformation of coordinates where the equations can be written as

\[ x' = \beta_{ij} x_j \quad \text{and} \quad x_i = \beta_{ji} x_j \]

Recall

\[ \beta_{ij} = \beta_{ij}^T \]
A matrix is orthogonal if its transpose is equal to its inverse

\[ \beta_{ij}^T = (\beta_{ij})^{-1} \]

An example of an orthogonal matrix is the direction cosine matrix. The transformation associated with the matrix is an orthogonal transformation

If \( \tilde{T} \) is the transformation, with \( T_{ij} = \beta_{ij} \), then \( \tilde{T} \) transforms a vector \( \vec{a} \) in one coordinate system into another vector \( \vec{b} \)

\[ \text{e.g. } \vec{b}_1 = \tilde{T}\vec{a}_1 \text{ and } \vec{b}_2 = \tilde{T}\vec{a}_2 \]
2B2 Components of a Tensor (cont.)

The transformation $\tilde{T}$ is linear if

$$
\begin{cases}
\tilde{T}(\tilde{a}_1 + \tilde{a}_2) = \tilde{T}\tilde{a}_1 + \tilde{T}\tilde{a}_2 \\
\tilde{T}(\alpha\tilde{a}_1) = \alpha\tilde{T}\tilde{a}_1
\end{cases}
$$

For any arbitrary vectors $\tilde{a}_1, \tilde{a}_2$ and scalar $\alpha$, $\tilde{T}$ is then called a LINEAR TRANSFORMATION and is a second-order tensor.

If $\tilde{b} = \tilde{T}\tilde{a}$, then

$$
\begin{bmatrix}
  b_1 \\
  b_2 \\
  b_3
\end{bmatrix} =
\begin{bmatrix}
  T_{11} & T_{12} & T_{13} \\
  T_{21} & T_{22} & T_{23} \\
  T_{31} & T_{32} & T_{33}
\end{bmatrix}
\begin{bmatrix}
  a_1 \\
  a_2 \\
  a_3
\end{bmatrix}
$$
2B2 Components of a Tensor (cont.)

The square matrix is the matrix of the tensor $\mathbf{T}$ . In this instance, $\mathbf{T}$ can be considered as the operator for transforming vector $\mathbf{a}$ into $\mathbf{b}$ . However, $T_{11}\hat{e}_1 + T_{12}\hat{e}_2 + T_{13}\hat{e}_3 = T_{1j}\hat{e}_j$ can be considered as the components of a vector $\mathbf{T}\hat{e}_1$ . Similarly, by

$$\tilde{T}\hat{e}_2 = T_{21}\hat{e}_1 + T_{22}\hat{e}_2 + T_{23}\hat{e}_3$$

or, in general

$$\tilde{T}\hat{e}_i = T_{ji}\hat{e}_j$$

Viewing in this manner, the components of $\mathbf{T}$ are expanded through $\hat{e}_1, \hat{e}_2$ and $\hat{e}_3$ base vectors in one system.
2B2 Components of a Tensor (cont.)

Vector will have different components in (x, y, z) system and (x’, y’, z’) system. $\overline{AB}$ is still the same.

Similarly, by a tensor $\tilde{T}$ at P is the same in the two systems, but, will have different components like

$$[T] = [T_{ij}] \quad \text{and} \quad \langle \hat{e}_1, \hat{e}_2, \hat{e}_3 \rangle$$

$$[T]’ = [T’_{ij}] \quad \text{and} \quad \langle \hat{e}_1’, \hat{e}_2’, \hat{e}_3’ \rangle$$
2B4  Sum of Tensors

If $\tilde{T}$ and $\tilde{S}$ are two tensors, then

$$(\tilde{T} + \tilde{S})\tilde{a} = \tilde{T}\tilde{a} + \tilde{S}\tilde{a}$$

$$(\tilde{T} + \tilde{S})_{ij} = T_{ij} + S_{ij}$$

$$[\tilde{T} + \tilde{S}] = [\tilde{T}] + [\tilde{S}]$$
2B5 Product of Two Tensors

\[(\tilde{T}\tilde{S})\tilde{a} = \tilde{T} (\tilde{S}\tilde{a})\]

Components:

\[
\begin{align*}
(\tilde{T}\tilde{S})_{ij} &= T_{im} S_{mj} \\
(\tilde{S}\tilde{T})_{ij} &= S_{im} T_{mj}
\end{align*}
\]

or

\[
\begin{align*}
[\tilde{T}\tilde{S}] &= [\tilde{T}][\tilde{S}] \\
[\tilde{S}\tilde{T}] &= [\tilde{S}][\tilde{T}]
\end{align*}
\]

In general, the product of two tensors is not commutative:

\[
\tilde{T}\tilde{S} \neq \tilde{S}\tilde{T}
\]
2B6 Transpose of a Tensor

Transpose of $\tilde{\mathbf{T}}$ is denoted by $\tilde{\mathbf{T}}^T$.

If $\vec{a}$ and $\vec{b}$ are two vectors, then $\vec{a} \cdot \tilde{\vec{b}} = \vec{b} \cdot \tilde{\mathbf{T}}^T \vec{a}$

In component form, $T_{ij} = T_{ji}^T$

Note

$(\tilde{\mathbf{T}}\tilde{\mathbf{S}})^T = \tilde{\mathbf{S}}^T \tilde{\mathbf{T}}^T$

$(\tilde{\mathbf{A}}\tilde{\mathbf{B}}\tilde{\mathbf{C}})^T = \tilde{\mathbf{C}}^T \tilde{\mathbf{B}}^T \tilde{\mathbf{A}}^T$
2B7 Dyadic Product

Definition

\[ \vec{a} \otimes \vec{b} = \vec{W} \]

\[ (\vec{a} \otimes \vec{b})\hat{c} = \vec{a}(\vec{b} \cdot \hat{c}) \]

\[ W_{ij} = \hat{e}_i \cdot \vec{W} \hat{e}_j = \hat{e}_i (\vec{a} \otimes \vec{b}) \hat{e}_j \]

\[ = \hat{e}_i \cdot \vec{a}(\vec{b} \cdot \hat{e}_j) = a_i \cdot a_j = a_j a_j \]

\[ [\vec{W}] = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} [b_1 \quad b_2 \quad b_3] = \begin{bmatrix} a_1 b_1 & a_1 b_2 & a_1 b_3 \\ a_2 b_1 & a_2 b_2 & a_2 b_3 \\ a_3 b_1 & a_3 b_2 & a_3 b_3 \end{bmatrix} \]
2B8 Trace of a Tensor

\[ tr(\vec{a} \otimes \vec{b}) = \vec{a} \cdot \vec{b} \]

\[ tr(\vec{T}) = tr\left(T_{ij} \hat{e}_i \hat{e}_j\right) = T_{ij} tr(\hat{e}_i \cdot \hat{e}_j) \]

\[ = T_{ij} (\hat{e}_i \cdot \hat{e}_j) = T_{ij} \delta_{ij} = T_{ii} \]

\[ = T_{11} + T_{22} + T_{33} \]

\[ = tr(T^T) \]
A linear transformation which transforms every vector $\vec{a}$ into itself is an identity tensor $\vec{I}$

$$\vec{I}\vec{a} = \vec{a}$$

$$[\vec{I}] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$
**2B9 Inverse Tensor**

Given a tensor $\tilde{T}$, if $\tilde{S}$ exists such that

$$\tilde{S}\tilde{T} = I$$

then $\tilde{S}$ is the inverse of $\tilde{T}$, or

$$\tilde{S} = \tilde{T}^{-1}$$

Inverse, if $\tilde{T}$ exists if the matrix $[T]$ is non-singular,

$$\tilde{T}^{-1}\tilde{T} = I$$

Note that

$$\left(\tilde{S}\tilde{T}\right)^{-1} = \tilde{T}^{-1}\tilde{S}^{-1}$$
2B10 Orthogonal Tensor

Transformed vectors preserve their lengths and angles, thus if $\tilde{Q}$ is an orthogonal tensor, then

$$|\tilde{Q}\tilde{a}| = |\tilde{a}| \quad \text{and} \quad \cos(\tilde{a}, \tilde{b}) = \cos(\tilde{Q}\tilde{a}, \tilde{Q}\tilde{b})$$

Thus

$$\tilde{Q}\tilde{a} \cdot \tilde{Q}\tilde{b} = \tilde{a} \cdot \tilde{b}$$

for any $\tilde{a}$ and $\tilde{b}$,

$$(\tilde{Q}\tilde{a}) \cdot (\tilde{Q}\tilde{b}) = \tilde{b} \cdot \tilde{Q}^T (\tilde{Q}\tilde{a})$$

$$= \tilde{b} \cdot \tilde{Q}^T \tilde{Q}(\tilde{a}) = \tilde{b} \cdot \tilde{a} = \tilde{b} \cdot I\tilde{a}$$
Thus $\tilde{Q}^T \tilde{Q} = \tilde{I}$

Also, $\tilde{Q}^T \tilde{Q} = \tilde{Q}^{-1} \tilde{Q} = I$, Thus

$\tilde{Q}^T = \tilde{Q}^{-1}$

Thus for an orthogonal matrix, the transpose is also its inverse.

Example:

A rigid body rotation is an ORTHOGONAL tensor,

$[R][R]^T = I$

$\det[R] = 1$

$[R] = \begin{bmatrix}
\cos \theta & -\sin \theta & 0 \\
\sin \theta & \cos \theta & 0 \\
0 & 0 & 1
\end{bmatrix}$
2B11 Transformation Matrix Between Two Coordinate Systems

Consider two systems $\overline{X} \equiv \{e_1, e_2, e_3\}$ and $\overline{X}' \equiv \{e'_1, e'_2, e'_3\}$, 

$\{e'_i\}$ is obtained from $\{e_i\}$ using a rigid body rotation of $\{e_i\}$.

We are interested to relate the unit vectors $\{e'_i\}$ in $\overline{X}'$ from that of $\overline{X}$.

$\hat{e}'_i = \tilde{Q}\hat{e}_i = Q_{mi}\hat{e}_m$

We see that,

$\hat{e}'_1 = Q_{11}\hat{e}_1 + Q_{21}\hat{e}_2 + Q_{31}\hat{e}_3$

$\hat{e}'_2 = Q_{12}\hat{e}_1 + Q_{22}\hat{e}_2 + Q_{32}\hat{e}_3$

$\hat{e}'_3 = Q_{13}\hat{e}_1 + Q_{23}\hat{e}_2 + Q_{33}\hat{e}_3$
2B11 Transformation Matrix Between Two Coordinate Systems (cont.)

\[ [Q] \text{ is the transformation given by} \]

We note that for \( \tilde{Q} \tilde{Q}^T = \tilde{I} \), \([\tilde{Q}]\) is an orthogonal matrix

\[
Q_{11} = \hat{e}_1 \tilde{Q} \hat{e}_1' \\
Q_{13} = \hat{e}_1 \tilde{Q} \hat{e}_3'
\]

(e.g.)

For example,
2B12 Transformation laws for Vectors.

Consider one vector \( \vec{a} \)

Let \( \vec{a} \) belong to \( \{ \hat{e}_i \} \). Thus \( a_i = \vec{a} \cdot \hat{e}_i \) in the \( \overline{X} \) frame.

If \( \overline{X}' \) is another system, and we wish to express \( \vec{a} \) within \( \overline{X}' \), let

\[
\hat{e}_i' = Q_{mi} \hat{e}_m
\]

\[
a_i' = \vec{a} \cdot Q_{mi} \hat{e}_m = Q_{mi} (\vec{a} \cdot \hat{e}_m)
\]

\[
a_i' = Q_{mi} a_m
\]

\[
[a]' = [Q]^T [a]
\]

\[
\begin{bmatrix}
  a'_1 \\
  a'_2 \\
  a'_3 \\
\end{bmatrix}
= 
\begin{bmatrix}
  Q_{11} & Q_{12} & Q_{13} \\
  Q_{21} & Q_{22} & Q_{23} \\
  Q_{31} & Q_{32} & Q_{33} \\
\end{bmatrix}
\begin{bmatrix}
  a_1 \\
  a_2 \\
  a_3 \\
\end{bmatrix}
\]

Thus \( \{ a'_1, a'_2, a'_3 \} \) are the components of \( \vec{a} \) with respect to the \( \overline{X}' \) system.
Example:

If \( \{e_i\} \) is obtained by rotating \( \{e_i\} \) ccw with respect to \( e_3 \)-axis find the components of \( \bar{a} = 2\hat{e}_1 \) in terms of \( \{e_i\} \).

Answer

\[
[a] = [Q]^T [a]
\]

\[
[Q] = \begin{bmatrix}
0 & -1 & 0 \\
1 & 0 & 0 \\
0 & 0 & 1
\end{bmatrix}
\]

\[
[a] = \begin{bmatrix}
0 & 1 & 0 \\
-1 & 0 & 0 \\
0 & 0 & 1
\end{bmatrix} \begin{bmatrix}
2 \\
0 \\
0
\end{bmatrix} = \begin{bmatrix}
0 \\
-2 \\
0
\end{bmatrix}
\]
2B13  Transformation law for tensor

Let \( \tilde{T} \in \{ e_i \} \) and \( \tilde{T}' \in \{ e'_i \} \)

Recall that the components of a tensor \( \tilde{T} \) are:

\[
\begin{align*}
T_{ij} &= e_i \tilde{T} e_j \\
T'_{ij} &= e'_i \tilde{T}' e'_j
\end{align*}
\]

Since \( e'_i = Q_{mi} e_m \), we have

\[
\begin{align*}
T'_{ij} &= Q_{mi} \hat{e}_m \tilde{T} Q_{nj} \hat{e}_n \\
&= Q_{mi} Q_{nj} (\hat{e}_m \tilde{T} \hat{e}_n) \\
T'_{ij} &= Q_{mi} Q_{nj} T_{mn}
\end{align*}
\]
2B13  Transformation law for tensor (cont.)

Thus

\[
\begin{bmatrix}
T'_1 & T'_2 & T'_3 \\
T''_1 & T''_2 & T''_3 \\
T'''_1 & T'''_2 & T'''_3
\end{bmatrix} =
\begin{bmatrix}
Q_{11} & Q_{12} & Q_{13} \\
Q_{12} & Q_{22} & Q_{32} \\
Q_{13} & Q_{23} & Q_{33}
\end{bmatrix}
\begin{bmatrix}
T_1 & T_2 & T_3 \\
T''_1 & T''_2 & T''_3 \\
T'''_1 & T'''_2 & T'''_3
\end{bmatrix}
\begin{bmatrix}
Q_{11} & Q_{12} & Q_{13} \\
Q_{12} & Q_{22} & Q_{32} \\
Q_{13} & Q_{23} & Q_{33}
\end{bmatrix}
\]

\[
[T]' = [Q]^T [T] [Q]
\]

Note that the tensor \( \tilde{T} \) is the same, but has different components \([T]'\) in the new frame \(\{e'_i\}\) compared to \([T]\) in \(\{e_i\}\)
Example
2B13 Transformation law for tensor (cont.)

Let \[
[T] = \begin{bmatrix}
0 & 1 & 0 \\
1 & 2 & 0 \\
0 & 0 & 1
\end{bmatrix}
\]

If \([Q]\) represents rotation as in the earlier example such that
\[
[Q] = \begin{bmatrix}
0 & -1 & 0 \\
1 & 0 & 0 \\
0 & 0 & 1
\end{bmatrix},
\]
then
\[
[T]' = [Q]^T [T] [Q]
\]

\[
= \begin{bmatrix}
2 & -1 & 0 \\
-1 & 0 & 0 \\
0 & 0 & 1
\end{bmatrix}
\]
\textbf{2B14  Tensors by Transformation laws}

\{\hat{e}_i\} \text{ are the components of the system } \bar{X}, \text{ and } \{\hat{e'}_i\} \text{ are that of new system } \bar{X}' . \quad \tilde{Q} \text{ or } Q_{ij} \text{ define direction cosines tensor transforming from } \bar{X} \text{ to } \bar{X}' . \quad Q_{ij} = \cos(\hat{e}_i, \hat{e'}_j) ; \quad Q \text{ is an orthogonal transformation; } \tilde{Q} \tilde{Q}^T = \tilde{I}
The components transform as follows:

\[ \alpha' = \alpha \]
\[ a_i' = Q_{mi} a_m \]
\[ T_{ij}' = Q_{mi} Q_{nj} T_{mn} \]
\[ T_{ijk}' = Q_{mi} Q_{nj} Q_{rk} T_{mnr} \]
\[ T_{ijk...} = Q_{mi} Q_{nj} Q_{ok} \ldots Q_{sp} T_{mno...} \]

- Scalar
- Vector
- Second Order Tensor
- Third Order Tensor
- \( n \)th Order Tensor
2B14 Tensors by Transformation laws (cont.)

a.) If \( a_i \) are components of vector and \( b_i \) are components of another vector, then \( T_{ij} = a_i b_j \) is a second order tensor.

b.) If \( a_i \) is a vector and \( T_{ij} \) are tensor components, then \( w_{ijk} = a_i T_{jk} \) is a the third order tensor.

c.) The quotient rule

If \( a_i \) are components of a vector, \( T_{ij} \) is the second-order tensor, then
Similarly $T_{ij}$ and $E_{ij}$ are tensors, then

$$ a_i = T_{ij} b_j $$

$b_j$ is a vector.

Similarly $T_{ij}$ and $E_{ij}$ are tensors, then

$$ T_{ij} = C_{ijkl} E_{kl} $$

means $C_{ijkl}$ is a fourth-order tensor,