Navier Stokes Equations

1 Stress Tensor

\[ T_{ij} = -p\delta_{ij} + \tau_{ij} \]

Inviscid fluid:

\[ \tau_{ij} = 0 \]

Newtonian fluid:

\[ \tau_{ij} = \lambda \frac{\partial v_k}{\partial x_k} \delta_{ij} + \mu \left( \frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} \right) \]

Stokes Hypothesis:

\[ \lambda = -\frac{2}{3} \mu \]

You should now be able to do 6.1, 2, 3, 4, 5, 7.

2 Heat Flux

Fourier Law:

\[ q_i = -k \frac{\partial T}{\partial x_i} \quad \vec{q} = -k \nabla T \]

3 Navier-Stokes Equations

The governing equations for a Newtonian fluid satisfying Fourier’s law are called the Navier-Stokes equations.

Compressible flow:

The students can write them out themselves.

Incompressible flow:

Assuming constant viscosity, the net stress force per unit volume simplifies a bit:

\[ \frac{\partial T_{ji}}{\partial x_j} = \frac{\partial}{\partial x_j} \left[ -p\delta_{ij} + \mu \left( \frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} \right) \right] = -\frac{\partial p}{\partial x_i} + \mu \frac{\partial^2 v_i}{\partial x_j \partial x_j} \]

Conservation form:

\[ \frac{\partial v_i}{\partial x_i} = 0 \quad \frac{\partial \rho v_i}{\partial t} + \frac{\partial \rho v_j v_i}{\partial x_j} = -\frac{\partial p}{\partial x_i} + \mu \frac{\partial^2 v_i}{\partial x_j \partial x_j} \]

Nonconservation form:

\[ \frac{\partial v_i}{\partial x_i} = 0 \quad \rho \frac{\partial v_i}{\partial t} + \rho v_j \frac{\partial v_i}{\partial x_j} = -\frac{\partial p}{\partial x_i} + \mu \frac{\partial^2 v_i}{\partial x_j \partial x_j} \]

Vector form:

\[ \text{div} (\vec{v}) = 0 \quad \rho \frac{D\vec{v}}{Dt} = -\nabla p + \mu \nabla^2 \vec{v} \]

Bottom line: for incompressible flow, the net pressure force is the gradient of the pressure (as always) and the net viscous force the Laplacian of the velocity.

Exercise:

What happened to the energy equation?