Gram-Schmidt

Description:

Gram-Schmidt orthogonalization is a way of converting a given arbitrary basis \( \{ \vec{u}_1, \vec{u}_2, \ldots, \vec{u}_n \} \) into an equivalent orthonormal basis:

This often leads to better accuracy (e.g. in least square problems) and/or simplifications.

Modified Gram-Schmidt Procedure

Given a set of linearly independent vectors, \( \vec{u}_1, \vec{u}_2, \ldots \), turn them into an equivalent orthonormal set \( \hat{\vec{v}}_1, \hat{\vec{v}}_2, \ldots \) as follows:

Step 1:

1. Normalize the first vector \( \vec{u}_1 \). That will be your \( \hat{\vec{v}}_1 \)

\[
\hat{\vec{v}}_1 = \frac{\vec{u}_1}{||\vec{u}_1||}
\]

2. For the remaining vectors \( \vec{u}_2, \vec{u}_3, \ldots \), eliminate their component in the direction of \( \hat{\vec{v}}_1 \) using the following formula:

\[
\vec{u}_j' = \vec{u}_j - \hat{\vec{v}}_1 \left( \hat{\vec{v}}_1^H \vec{u}_j \right)
\]
Note that $i'_1^H \vec{u}_j = ||i'_1|| ||\vec{u}_j|| \cos \theta = ||\vec{u}_j|| \cos \theta$ is the component of $\vec{u}_j$ in the direction of $i'_1$:

![Diagram](image.png)

Also $i'_1 i'_1^H \vec{u}_j = \text{proj}(i'_1, \vec{u}_j)$. The matrix $i'_1 i'_1^H$ is called the projection operator onto $i'_1$.

Ignore $i'_1$ in the remaining process.

**Step 2:**

1. Normalize the second vector $\vec{u}_2^*$. That will be your $i'_2$

   $$i'_2 = \frac{\vec{u}_2^*}{||\vec{u}_2^*||}$$

2. For the remaining vectors $\vec{u}_3, \vec{u}_4, \ldots$, eliminate their component in the direction of $i'_2$ using the following formula:

   $$\vec{u}_j^{**} = \vec{u}_j^* - i'_2 (i'_2^H \vec{u}_j^*)$$

Ignore $i'_2$ in the remaining process.

Repeat the process along the same lines until you run out of vectors.

*Graphical example:*
Normalize $\vec{u}_1$:

Eliminate the components in the $\vec{u}_1$ direction from the rest:

Normalize $\vec{u}_2$:

Eliminate the components in the $\vec{u}_2$ direction from the rest:
Normalize $\vec{u}_3$: 