Transpose matrices

1 General

Transposing a matrix turns the columns into rows and vice-versa

\[
A = \begin{pmatrix}
a_{11} & a_{12} & \cdots & a_{1n} \\
a_{21} & a_{22} & \cdots & a_{2n} \\
a_{31} & a_{32} & \cdots & a_{3n} \\
\vdots & \vdots & \ddots & \vdots \\
a_{m1} & a_{m2} & \cdots & a_{mn}
\end{pmatrix} \quad A^T = \begin{pmatrix}
a_{11} & a_{21} & a_{31} & \cdots & a_{m1} \\
a_{12} & a_{22} & a_{32} & \cdots & a_{m2} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
a_{1n} & a_{2n} & a_{3n} & \cdots & a_{mn}
\end{pmatrix}
\]

Similarly, transposing turns a column vector into a row vector and vice-versa.

Another way of thinking about it is that the elements are flipped over around the “main diagonal”, which runs from top left to bottom right:

\[
\begin{pmatrix}
a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\
a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\
a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn}
\end{pmatrix}
\]

(The sum of the elements on the main diagonal is called the trace of the matrix.)

Note that \((A^T)^T = A\).

Transpose in index notation:

\[a^T_{ij} = a_{ji}\] for all \(i\) and \(j\)

Note that in index notation, the main diagonal consists of the elements where \(i = j\). These stay put during transposing.

Transposing matrix products:

\[(AB)^T = B^T A^T\]

For complex matrices, the normal generalization of transpose is “Hermitian conjugate”, where you take the complex conjugate of each complex number, in addition to interchanging rows and columns: \(A^H \equiv \bar{A}^T\), or \(a^H_{ij} = \bar{a}_{ji}\).

Example:

\[
\begin{pmatrix}
1 + 2i & 3 + 4i \\
5 + 6i & 7 + 8i
\end{pmatrix}^H = \begin{pmatrix}
1 - 2i & 5 - 6i \\
3 - 4i & 7 - 8i
\end{pmatrix}
\]
2 Special matrices

Symmetric matrices satisfy
\[ S^T = S \]
Symmetric matrices are very common in engineering. For example, most statics deals with symmetric matrices, as does solid body dynamics, and a lot of the simpler fluid flows.

Complex matrices for which \( A^H = A \) are called “Hermitian matrices.” They are all over quantum mechanics.

Skew-symmetric matrices satisfy
\[ K^T = -K \]
Skew-symmetric matrices determine the velocity field in solid body motion, and other fields involving cross products.

Example: the following is a skew symmetric matrix:
\[
\begin{pmatrix}
0 & 3 \\
-3 & 0
\end{pmatrix}
\]

Diagonal matrices have only nonzero elements on the main diagonal:
\[
D = 
\begin{pmatrix}
d_{11} & 0 & 0 & \ldots & 0 \\
0 & d_{22} & 0 & \ldots & 0 \\
0 & 0 & d_{33} & \ldots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \ldots & d_{nn}
\end{pmatrix}
\]

An example is the unit matrix. In index notation, a matrix is diagonal iff \( d_{ij} = 0 \) if \( i \neq j \).

Upper triangular matrices have only nonzero elements on and above the main diagonal:
\[
U = 
\begin{pmatrix}
u_{11} & u_{12} & u_{13} & \ldots & u_{1n} \\
0 & u_{22} & u_{23} & \ldots & u_{2n} \\
0 & 0 & u_{33} & \ldots & u_{3n} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \ldots & u_{nn}
\end{pmatrix}
\]

In index notation, \( u_{ij} = 0 \) if \( j < i \).

Lower triangular matrices:
\[
L = 
\begin{pmatrix}
l_{11} & 0 & 0 & \ldots & 0 \\
l_{21} & l_{22} & 0 & \ldots & 0 \\
l_{31} & l_{32} & l_{33} & \ldots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
l_{n1} & l_{n2} & l_{n3} & \ldots & l_{nn}
\end{pmatrix}
\]
In index notation, $l_{ij} = 0$ if $j > i$.

The transpose of an upper triangular matrix is a lower triangular one and vice-versa.