1 Simple example

Student request: change notations. Mine seem better than the book’s, though. I think the book’s exposition (p207-210) is very confusing, partly by not using vector symbols to indicate vectors versus coordinates. I suggest you stick with my exposition.

To solve problems, it is often desirable or essential to change basis.

As an example, consider the vector of gravity $\vec{g}$. If I use a Cartesian coordinate system $\hat{i}, \hat{j}$ with the $x$-axis horizontal, the vector $\vec{g}$ will be along the negative $y$-axis. I will call this coordinate system, $(\hat{i}, \hat{j})$, the $E$-system.

Using the $E$-system, I can write the vector $\vec{g}$ as:

$$\vec{g} = 0\hat{i} - g\hat{j} \quad \text{or} \quad \vec{g}\big|_E = \begin{pmatrix} 0 \\ -g \end{pmatrix}$$

In other words, the coordinates of vector $\vec{g}$ in the $E$-coordinate system are $g_1\big|_E = 0$ and $g_2\big|_E = -g$.

But if, say, the ground is under an angle $\theta$ with the horizontal, it might be much more convenient to use a coordinate system $E^*, (\hat{i}^*, \hat{j}^*)$, with the $x$-axis aligned with the ground:
In this new coordinate system, the coordinates of \( \vec{g} \) will be different. With a bit of trig, you see:
\[
\vec{g} = -g \sin(\theta) \hat{i} - g \cos(\theta) \hat{j} \quad \text{or} \quad \vec{g}^\star_{E^*} = \begin{pmatrix} -g \sin(\theta) \\ -g \cos(\theta) \end{pmatrix}
\]
The coordinates of vector \( \vec{g} \) are now \( g_1 \big|_{E^*} = -g \sin(\theta) \) and \( g_2 \big|_{E^*} = -g \cos(\theta) \)

What if I need to change the coordinates of a lot of vectors from one coordinate system to the other? Is there a systematic way of doing this? The answer is yes; the following formula applies:
\[
\vec{v}_{E} = P \vec{v}^\star_{E^*} \quad \text{with} \quad P = \begin{pmatrix} \hat{i}^\star_{E} & \hat{j}^\star_{E} \end{pmatrix}
\]
So the transformation of coordinates can be done by multiplying by a matrix \( P \). This matrix consists of the basis vectors of the new coordinate system \( E^* \) expressed in terms of the old coordinate system \( E \).

In particular,
\[
\hat{i}^\star = \cos(\theta) \hat{i} + \sin(\theta) \hat{j} \quad \text{so} \quad \hat{i}^\star \big|_{E} = \begin{pmatrix} \cos(\theta) \\ \sin(\theta) \end{pmatrix}
\]
\[
\hat{j}^\star = -\sin(\theta) \hat{i} + \cos(\theta) \hat{j} \quad \text{so} \quad \hat{j}^\star \big|_{E} = \begin{pmatrix} -\sin(\theta) \\ \cos(\theta) \end{pmatrix}
\]
and matrix \( P \) becomes:
\[
P = \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix}
\]
Let’s test it: \( P \) times the coordinates of vector \( \vec{g} \) in the \( E^* \)-system should give the coordinates in the \( E \)-system:
\[
\begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix} \begin{pmatrix} -g \sin(\theta) \\ -g \cos(\theta) \end{pmatrix}
\]
Multiplying out gives 0 and \(-g\), which is exactly right.

Matrix \( P \) is called the \textit{transformation matrix from \( E \) to \( E^* \)}. Note however that it really transforms coordinates in the \( E^* \)-system to coordinates in the \( E \)-system. You just have to get used to that language: a transformation matrix from A to B transforms B coordinates into A coordinates. No, I do not know who thought of that first.
What if you really want to transform $E$ coordinates into $E^*$ coordinates? No big deal: just multiply by the inverse matrix $P^{-1}$.

2 General

The basis vectors do not have to be orthogonal, as in the example. In general, suppose I have a basis $S$, $\{\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_n\}$. Then any arbitrary vector $\vec{w}$ can be written as

$$\vec{w} = w_1 \big|_S \vec{u}_1 + w_2 \big|_S \vec{u}_2 + \ldots + w_n \big|_S \vec{u}_n$$

where $w_1 \big|_S, w_2 \big|_S, \ldots, w_n \big|_S$ are the coordinates of $\vec{w}$ in basis $S$. More briefly,

$$\vec{w} \big|_S = \begin{pmatrix} w_1 \big|_S \\ w_2 \big|_S \\ \vdots \\ w_n \big|_S \end{pmatrix}$$

Suppose I have another basis $S'$, $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$. Then the same vector $\vec{w}$ can also be written as

$$\vec{w} = w_1 \big|_{S'} \vec{v}_1 + w_2 \big|_{S'} \vec{v}_2 + \ldots + w_n \big|_{S'} \vec{v}_n$$

or

$$\vec{w} \big|_{S'} = \begin{pmatrix} w_1 \big|_{S'} \\ w_2 \big|_{S'} \\ \vdots \\ w_n \big|_{S'} \end{pmatrix}$$

The relationship between the two sets of coordinates is always

$$\vec{w} \big|_S = P \vec{w} \big|_{S'}$$

where $P$ is a matrix that is called the transformation matrix from $S$ to $S'$. (Although it really works the opposite way.)

Matrix $P$ takes the form:

$$P = \begin{pmatrix} \vec{v}_1 \big|_S \\ \vec{v}_2 \big|_S \\ \vdots \\ \vec{v}_n \big|_S \end{pmatrix}$$

It contains the basis vectors of the $S'$ system written in the $S$ system. (That is why if I multiply with $P$, I get a vector in the $S$ system.)

To get the transformation the other way, use the matrix $P^{-1}$. 