Chapter 1: General Notes

4. Stress (5 Lectures)
   - Definition
   - Stress vector and Tensor
   - Cauchys Formula
   - Equations of Equilibrium
   - Plane stress
   - Principal stress
   - Shearing stress
   - Boundary Conditions

Exam 2

5. Constitutive Equations of (4 Lectures)
   - Definition
   - Thermodynamic Constraints
   - Hooke's Law
   - Elasticity Tensor
   - Isotropy, Orthotropy, Anisotropy
   - Uniaxial and Multiaxial behavior
   - Experimental Determination of elastic constants
   - Newtonian Viscous Fluid

6. General Field Equations (6 Lectures)
   - Basic Equations
   - Green's and Divergence Theorems
   - General Principles
• Formulation and Solution of Boundary Value Problems

Final
Concept of Continuous Media

Continuum mechanics deals with forces (stresses) and motion (or deformation, strain) of solids, liquids and gases disregarding their molecular structure. It is assumed that continuous mathematical functions can describe the medium valid at all interior points of the body. This concept allows us to define stress, i.e., force/unit area at all points. This definition implies that mass density \( \rho \) is continuous at all points

\[
\rho = \rho(x_1, x_2, x_3, t) = \lim_{\Delta V \to 0} \left( \frac{\Delta m}{\Delta V} \right)
\]

where \( x_1, x_2, x_3 \) are the coordinate position at time \( t \), and \( \Delta m \) is the mass identified with a volume element of \( \Delta V \).

Applications of the theory lead to the study of the theory of elasticity, plasticity and fluid mechanics.
Force in a continuous body

Refer to the figure describing the body with surface $\Delta S$ and volume element $\Delta V$. This body is truly representing

- An airplane
- An automobile
- Thin foil in an electronic circuit
- Fluid flow around a jet

This body is acted upon by

1. **Surface forces**
   - Concentrated (at a point)
   - Distributed (over a surface)

2. **Body forces**
   - Gravitational
   - Inertial
   - Thermoelastic

3. **Momental forces (rotational effect)**
A set of three figures showing the deformation at time $t=0$, $t$ and $t$.

The same body deforms with time under the action of external forces. The point $P$ embedded in the volume element $\Delta V$ traverses a path called the trajectory. This path is described by the displacement function $\bar{u} = \bar{u}(x_1, x_2, x_3, t)$ is continuous within the space and time. Thus

$$ \text{Strain} = \frac{\partial \bar{u}}{\partial x} \quad \text{and Velocity} = \frac{\partial \bar{u}}{\partial t} $$

can be defined. Note that if the function $\bar{u} = \bar{u}(x_1, x_2, x_3, t)$ is not continuous, then the derivatives cannot even be defined.
Validity of continuum theory

In the continuum theory, one can take a piece of steel and assign some property. For example we can say that the steel has an Young's modulus of $E=30 \text{ E}_6 \text{ psi}$. That property is valid for a volume element of the size of the test piece. The question is that if we keep subdividing the volume element till it becomes very small will that property still retain its meaning. It may still hold good if the volume element is $1 \text{ mm}^3$. How about if the element is of the order of a few nanometers, i.e., in the scale of atomic distance. Obviously the idea is the material is continuous breaks down at that scale.

In a general sense, the concept of continuum depends on the problem. For example a discontinuity on the same order of the problem being modeled will not yield the right result. For example a material discontinuity (rarefied atmosphere) of a few centimeters in the outer space can be ignored when modeling the flight of a rocket of characteristic dimension of a few meters; whereas a cavity the size of a few micrometers cannot be ignored when attempting to solve wave propagation problem where the characteristic dimensions are also in the same order. As a general rule, if the discontinuity is not more than two orders of magnitude that of the characteristic dimension in the problem then the concept of continuum mechanics can be safely applied.
Additional notes on continuum theory

The concept of a continuum is very critical in the study of materials under motion. Materials in this context refers to solids, fluids or gases. Motion refers to the changes that take place in the materials when subjected to static or dynamic (e.g. cyclid) loading conditions. The effect of the loading process may be realized in a few microseconds as in a ballistic impact conditions, or in a few milleniums as in the movement of geo plates on the earth surfaces. These two effects are strain-rate effects. The temperatures of the body may be very very hot as in 3000 C in a flame, 1000 C in a high temperature gamma titanium aluminde to near absolute temperature in a microkelvin tanks.

A view of the material at the atomic scale:

We know that every physical object is made up of molecules, atoms and even smaller particles. These particles are not continuously distributed over the object. Microscopic observations reveal that there are gaps (empty spaces) between particles. Consider an atomic structure of a metal in which the atoms are separated by interatomic distance of the order of 4 to 5 nm (4×10^{-9} m). The nucleus of the atom where most of the mass (neutron and protons) are concentrated are at least three order lower, thus leaving a vast empty space where the electrons revolve. In essence the physical space occupied by materials is very very small. However, this effect is never felt in the everyday experience of dealing with materials. For all practical purposes, we ignore that the material is a continuously occupied by matter.

Micro, Meso and Mesoscopic scales of the materials.

Though there are many possible scales description of materials in terms of characteristic lengths is very useful. For that purpose if we analyze the problem at the scale of micrometers (10^{-6} m) or less then the descriptions refers to microscopic scale of the materials. Though in the realms of nuclear physics a scale of (10^{-9} m or 10^{-3} \mu m = 1 nm), sometimes referred to as nanoscopic scale, is used in the study of mechanics of continuous media we will still refer to them as microscopic description. Understanding the effect of point (vacancy, interstitials), and line (edge or screw dislocations) defects on the field falls under this category. In the mesoscopic analysis, we are interested in scales between 1 \mu m and 1 mm (10^{-6} to 10^{-3} m). In this scale, we can analyze the effect of individual grains, void, cavities, cracks and grain boundaries. In the macroscopic scale, (> 1 mm) we include the study of structures anywhere between electronic devices, to automobiles to large space shuttles.

Physical scale of the problem
Every physical problem in nature, based on mechanics or otherwise has a length scale associated with it. All of those problems are described by a set of governing field equations be it be based on mechanics, thermodynamics, magnetic or electrical fields. With each of the specific problem, there is a characteristic length scale. For example if one were to study the effect of cracks on the failure strength of the material, the size of the crack is the characteristic length. In this case it ranges from a few tenths of a mm (100 µm) to a few mm. If we are to study the effect of the deflection of a large bridge under dynamic loading, then a few mm is the characteristic length. Even for the same physical problem, the length scale varies depending on what is the specific issue we are analyzing. If we like to study the viscous drag of air on an airplane, we will focus on the boundary layer which is a less than a mm. On the other hand, if we like to evaluate the lift of the same plane, the projected area is the critical parameter leading to a characteristic length a few decimeters.

**Validity of Continuum assumptions.**

In order to validate the assumptions of continuity, we need to compare the characteristic length of the problem with that of the discontinuity in the material. For the sake of simplicity, we can assume that the assumptions of continuity is valid if the material discontinuity is at least two orders of magnitude lower than that of the characteristic length of the problem. For example the atomic level discontinuity can be ignored in fracture mechanics problem since the latter has a characteristic length scale of 1000 µm compared to the atomic discontinuity of 0.1 µm.

**Ramifications of continuum theory**

In general, mechanics of continuum medium attempts to relate the deformation of a body from an undeformed to deformed state under the action of all external and internal forces. The assumption of continuity of material particles that make up of the body implies that there is a one to one correspondence between the original and current configurations. That is for every particle \( a_i \) in the original configurations a corresponding \( x_i \) in the deformed configuration and there is one and only particle that has a correspondence in both the states. Thus the deformation \( \Psi = \Psi(t) \) has one to one correspondence such that the inverse \( \Gamma = \Psi^{-1}(t) \) exists and is unique. Also both the functions can have derivatives of any given order because of the continuity assumptions. This assumption is important in the definition of deformation gradient and strain quantities.
2. Cartesian Tensor Theory

Part A: Indicial Notation

2A1 Summation Convention, Dummy Indices

Consider the sum

\[ S = a_1 x_1 + a_2 x_2 + \ldots + a_n x_n \]

\[ = \sum_{i=1}^{n} a_i x_i = \sum_{j=1}^{n} a_j x_j = \sum_{k=1}^{n} a_k x_k \]

Thus, the repeated indices (i,j,k…) are called dummy indices and the dummy indices are summed. [Einstein's convention]

\[ \int_{a}^{b} f(x) \, dx = \int_{a}^{b} f(\xi) \, d\xi \]

Very similar to x and \( \xi \) are dummies.

In this convention, an index should not be repeated more than once,

\[ \sum_{i=1}^{N} a_i b_i c_i \]

i.e. should not be written in indicial form.

Assume n=3. Then

\[ a_i x_i = a_k x_k = a_1 x_1 + a_2 x_2 + a_3 x_3 \]

\[ a_i \hat{e}_i = a_1 \hat{e}_1 + a_2 \hat{e}_2 + a_3 \hat{e}_3 \]

concisely represents a vector.
Double and Triple Sum

\[ \sum_{i=1}^{3} \sum_{j=1}^{3} a_{ij} x_i x_j = a_{11} x_1 x_1 + a_{12} x_1 x_2 + a_{13} x_1 x_3 + a_{21} x_2 x_1 + a_{22} x_2 x_2 + a_{23} x_2 x_3 + a_{31} x_3 x_1 + a_{32} x_3 x_2 + a_{33} x_3 x_3 + \equiv a_{ij} x_i x_j \]

and similarly, by

\[ a_{ijk} x_i x_j x_k \equiv \sum_{i=1}^{3} \sum_{j=1}^{3} \sum_{k=1}^{3} a_{ijk} x_i x_j x_k \]
2A2  Free Indices

Consider the system of equations,
\[\begin{align*}
x_1' &= a_{11}x_1 + a_{12}x_2 + a_{13}x_3 \\
x_2' &= a_{21}x_1 + a_{22}x_2 + a_{23}x_3 \\
x_3' &= a_{31}x_1 + a_{32}x_2 + a_{33}x_3
\end{align*}\]

We can write
\[\begin{align*}
x_1' &= a_{1i}x_i \\
x_2' &= a_{2i}x_i \\
x_3' &= a_{3i}x_i
\end{align*}\]

And further as
\[x_j' = a_{ji}x_i \quad i=1,2,3 \quad j=1,2,3\]

The index, \(j\), appears only once in \(x_j'\) and \(a_{ji}\) and is called a free index. The free index takes \(1,2,3\ldots\) one at a time. The free index should appear on the left and right side of the equation.

\[a_i = b_j \quad \text{is meaningless}\]
Consider a right-handed, orthogonal Cartesian coordinate system. Let $\hat{e}_1, \hat{e}_2, \hat{e}_3$ be the base vectors.

\[
\overline{x} = \overline{x}_i \hat{e}_i = \hat{x}_i \hat{e}_i
\]

From the orthogonality of the base vectors:
\[
\hat{e}_1 \cdot \hat{e}_1 = 1 \\
\hat{e}_2 \cdot \hat{e}_2 = 1 \\
\hat{e}_3 \cdot \hat{e}_3 = 1 \\
\]

But \[
\hat{e}_1 \cdot \hat{e}_2 = \hat{e}_2 \cdot \hat{e}_3 = \hat{e}_3 \cdot \hat{e}_1 = 0
\]

Introduce the symbol called the Kronecker Delta
\[
\delta_{ij} = \begin{cases} 
1 & i = j \\
0 & i \neq j 
\end{cases}
\]

and thus \[
\hat{e}_i \cdot \hat{e}_j = \delta_{ij}
\]

We can see the matrix
\[
I = \begin{bmatrix} 
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1 
\end{bmatrix}
\]
Also:

i. \( \delta_{ii} = 3 \)

ii. \( \delta_{im} a_m = \delta_{11} a_1 + \delta_{11} a_1 + \delta_{11} a_1 = a_i \)

\[ \text{In general} \quad \delta_{im} a_m = a_i \]

iii. \( \delta_{im} T_{mj} = T_{ij} \)

iv. \( \delta_{im} \delta_{mj} = \delta_{ij} \)

v. \( \delta_{im} \delta_{mj} \delta_{jn} = \delta_{in} \)

vi. \( \delta_{ij} = \delta_{ji} \iff \text{symmetric} \)
2A4  Permutation Symbol

Denote
\[
\varepsilon_{ijk} = \begin{cases} 
1 & \text{for } i, j, k \text{ having even permutation} \\
-1 & \text{" " odd permutation} \\
0 & \text{" " no permutation} 
\end{cases}
\]

Even/Odd changes from 1,2,3

\[
\begin{align*}
\varepsilon_{321} & \rightarrow \varepsilon_{213} \rightarrow \text{odd} \rightarrow -1 \\
\varepsilon_{123} & \rightarrow \varepsilon_{312} \rightarrow \varepsilon_{231} \rightarrow \text{even} \rightarrow 1 \\
\varepsilon_{223} & \rightarrow \text{no} \rightarrow 0
\end{align*}
\]

Consider

Fast way to determine

\[
\begin{align*}
\left\uparrow \quad 3 \quad \left\downarrow \quad \uparrow \quad 1 \quad \downarrow \right. \\
1 & \rightarrow 2 \\
3 & \leftarrow 2 \\
CCW & = \text{odd} \\
CW & = \text{even}
\end{align*}
\]

i.  \( \varepsilon_{ijk} = -\varepsilon_{jik} = \varepsilon_{jki} \)
ii. Consider the cross product
\[
\hat{e}_1 \times \hat{e}_2 = \hat{e}_3 \\
\hat{e}_2 \times \hat{e}_3 = \hat{e}_1 \\
\hat{e}_3 \times \hat{e}_1 = \hat{e}_2 \\
\hat{e}_2 \times \hat{e}_1 = -\hat{e}_3
\]
in general
\[\hat{e}_i \times \hat{e}_j = \varepsilon_{ijk} \hat{e}_k\]

iii. Thus
\[
\bar{a} \times \bar{b} = a_i \hat{e}_i \times b_j \hat{e}_j \\
= a_i b_j (\hat{e}_i \times \hat{e}_j) \\
= a_i b_j \varepsilon_{ijk} \hat{e}_k
\]
\[
\varepsilon_{ijm} \varepsilon_{klm} = \delta_{ik} \delta_{jl} - \delta_{il} \delta_{jk}
\]

iv. \((\varepsilon - \delta \text{ identity})\)
2A5 Operations with indicial quantities

a. Substitution
Consider \( a_i = u_{im}b_m \) and \( b_i = v_{im}c_m \)

To express \( a_i \) in terms of \( b_i \), change the free index,
\[
b_i = v_{im}c_m \quad i \rightarrow m \quad m \rightarrow n
\]
\[
b_m = v_{mn}c_n
\]
\[
a_i = u_{im}v_{mn}c_n \leftarrow 2 \text{ repeated indeces} = 9 \text{ terms}
\]
\[
\text{1 free index} = 3 \text{ equations}
\]

b. Multiplication
\[
p = a_m b_m
\]
\[
q = c_m d_m \leftarrow \text{change dummy index}
\]
\[
pq = a_m b_m c_n d_n
\]
Again
\[
\bar{a} \cdot \bar{b} = (a_i \hat{e}_i) \cdot (b_j \hat{e}_j)
\]
\[
= a_i b_j \hat{e}_i \cdot \hat{e}_j
\]
\[
= \delta_{ij} a_i b_j = a_i b_i
\]

c. Factoring
\[
T_{ij} n_j - \lambda n_i = 0
\]
Use \( n_i = \delta_{ij} n_j \)
\[
n_j \left( T_{ij} - \lambda \delta_{ij} \right) = 0
\]
d. Contraction

\[ T_{ij} = \lambda \theta \delta_{ij} + 2 \mu E_{ij} \]
\[ T_{ii} = \lambda \theta \delta_{ii} + 2 \mu E_{ii} \]
\[ T_{ii} = 3\lambda \theta + 2 \mu E_{ii} \]
Part B  Tensors

\[ x = x' \cos \theta - y' \sin \theta \quad \text{or} \quad \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix} \begin{bmatrix} x' \\ y' \end{bmatrix} \]

Or, inverting

\[ x' = x \cos \theta - y \sin \theta \quad \text{or} \quad \begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \]

Above represents a transformation of coordinates when the system is rotated at an angle \( \theta \) CCW.
Let $\overrightarrow{a}, \overrightarrow{b}$ be two vectors in a Cartesian coordinate system. If $\tilde{T}$ is a transformation, which transforms any vector into some other vector, we can write

\[
\tilde{T}\overrightarrow{a} = \overrightarrow{c} \\
\tilde{T}\overrightarrow{b} = \overrightarrow{d}
\]

where $\overrightarrow{c}$ and $\overrightarrow{d}$ are two different vectors.

If

\[
\tilde{T} (\alpha \overrightarrow{a}) = \alpha \tilde{T}\overrightarrow{a} \quad (1.2a)
\]

For any arbitrary vector $\overrightarrow{a}$ and $\overrightarrow{b}$ and scalar $\alpha$, then $\tilde{T}$ is called a LINEAR TRANSFORMATION and a Second Order Tensor. (1.1a) and (1.2a) can be written as,

\[
\tilde{T} \left( \alpha \overrightarrow{a} + \beta \overrightarrow{b} \right) = \alpha \tilde{T}\overrightarrow{a} + \beta \tilde{T}\overrightarrow{b} \quad (1.3)
\]
2B2 Components of a Tensor

\[ \tilde{T}e_i = T_{11} \hat{e}_1 + T_{21} \hat{e}_2 + T_{31} \hat{e}_3 \]
\[ \tilde{T}e_2 = T_{12} \hat{e}_1 + T_{22} \hat{e}_2 + T_{32} \hat{e}_3 \]
\[ \tilde{T}e_3 = T_{13} \hat{e}_1 + T_{23} \hat{e}_2 + T_{33} \hat{e}_3 \]

\[ \tilde{T} e_i = T_{ji} e_j \]

\[ T_{11} = \hat{e}_1 \cdot \tilde{T} e_1 \]
\[ T_{12} = \hat{e}_1 \cdot \tilde{T} e_2 \]

In general \( T_{ij} = \hat{e}_i \cdot \tilde{T} e_j \) (2-2)

The components of \( T_{ij} \) can be written as,

\[ [T] = \begin{bmatrix} T_{11} & T_{12} & T_{13} \\ T_{21} & T_{22} & T_{23} \\ T_{31} & T_{32} & T_{33} \end{bmatrix} \]

Example 2B2.3

\[ \tilde{R} \hat{e}_1 = \cos \theta \hat{e}_1 + \sin \theta \hat{e}_2 \]
\[ \tilde{R} \hat{e}_2 = -\sin \theta \hat{e}_1 + \cos \theta \hat{e}_2 \]
\[ \tilde{R} \hat{e}_3 = \hat{e}_3 \]
Thus

\[
[R] = \begin{bmatrix}
\cos \theta & -\sin \theta & 0 \\
\sin \theta & \cos \theta & 0 \\
0 & 0 & 1
\end{bmatrix}
\]

Components of Tensor

\[
x = x' \cos \theta - y' \sin \theta \\
y = x' \sin \theta + y' \cos \theta
\]

or

\[
\begin{cases}
x' = \cos \theta x - \sin \theta y \\
y' = \sin \theta x + \cos \theta y
\end{cases}
\]

On inverting,

\[
x' = x \cos \theta + y \sin \theta \\
y' = -x \sin \theta + y \cos \theta
\]

or

\[
\begin{cases}
x' = \sin \theta y - \sin \theta x \\
y' = -\sin \theta y + \sin \theta x
\end{cases}
\]

Above represents a transformation of coordinates where the equations can be written as

\[
x' = \beta_{ij} x_j \quad \text{and} \quad x_i = \beta_{ji} x'_j
\]

Recall \( \beta_{ij} = (\beta_{ji})^T \)

A matrix is orthogonal if its transpose is equal to its inverse

\[
(\beta_{ij})^T = (\beta_{ji})^{-1}
\]

An example of an orthogonal matrix is the direction cosine matrix. The transformation associated with the matrix is an orthogonal transformation.

If \( \tilde{T} \) is the transformation, with \( T_{ij} = \beta_{ij} \), then \( \tilde{T} \) transforms a vector \( \vec{a} \) in one coordinate system into another vector \( \vec{b} \).
e.g. \( \vec{b}_1 = \vec{T} \vec{a}_1 \) and \( \vec{b}_2 = \vec{T} \vec{a}_2 \)

The transformation \( \vec{T} \) is linear if
\[
\vec{T}(\vec{a}_1 + \vec{a}_2) = \vec{T} \vec{a}_1 + \vec{T} \vec{a}_2
\]
\[
\vec{T}(\alpha \vec{a}) = \alpha \vec{T} \vec{a}
\]

For any arbitrary vectors \( \vec{a}_1, \vec{a}_2 \) and scalar \( \alpha \), \( \vec{T} \) is then called a LINEAR TRANSFORMATION and is a second-order tensor.

If \( \vec{b} = \vec{T} \vec{a} \), then
\[
\begin{pmatrix}
  b_1 \\
  b_2 \\
  b_3
\end{pmatrix} =
\begin{bmatrix}
  T_{11} & T_{12} & T_{13} \\
  T_{21} & T_{22} & T_{23} \\
  T_{31} & T_{32} & T_{33}
\end{bmatrix}
\begin{pmatrix}
  a_1 \\
  a_2 \\
  a_3
\end{pmatrix}
\]

The square matrix is the matrix of the tensor \( \vec{T} \). In this instance, \( \vec{T} \) can be considered as the operator for transforming vector \( \vec{a} \) into \( \vec{b} \).

However, \( T_{11} \hat{e}_1 + T_{12} \hat{e}_2 + T_{13} \hat{e}_3 = T_{1j} \hat{e}_j \) can be considered as the components of a vector \( \vec{T} \hat{e}_1 \). Similarly, by
\[
\vec{T} \hat{e}_2 = T_{21} \hat{e}_1 + T_{22} \hat{e}_2 + T_{23} \hat{e}_3
\]
\[
\text{or, in general}
\vec{T} \hat{e}_i = T_{ji} \hat{e}_j
\]

Viewing in this manner, the components of \( \vec{T} \) are expanded through \( \hat{e}_1, \hat{e}_2 \) and \( \hat{e}_3 \) base vectors in one system.

Vector will have different components in (x,y,z) system and (x’,y’,z’) system. \( \overline{AB} \) is still the same.

Similarly, by a tensor \( \vec{T} \) at P is the same in the two systems, but, will have different components like
\[ [T] = [T_{ij}] \text{ and } \langle \hat{e}_1, \hat{e}_2, \hat{e}_3 \rangle \]
\[ [T]' = [T'_{ij}] \text{ and } \langle \hat{e}'_1, \hat{e}'_2, \hat{e}'_3 \rangle \]
If $\tilde{T}$ and $\tilde{S}$ are two tensors, then

$$(\tilde{T} + \tilde{S})\tilde{a} = \tilde{T}\tilde{a} + \tilde{S}\tilde{a}$$

$$(\tilde{T} + \tilde{S})_{ij} = T_{ij} + S_{ij}$$

$$[\tilde{T} + \tilde{S}] = [\tilde{T}] + [\tilde{S}]$$
2B5 Product of Two Tensors

\[(\tilde{T}\tilde{S})\tilde{a} = \tilde{T}(\tilde{S}\tilde{a})\]

Components:

\[(\tilde{T}\tilde{S})_{ij} = T_{im}S_{mj}\]

\[\begin{bmatrix} \tilde{T}\tilde{S} \end{bmatrix} = \begin{bmatrix} \tilde{T} \end{bmatrix}[\tilde{S}]\]

\[(\tilde{S}\tilde{T})_{ij} = S_{im}T_{mj}\]

or

\[\begin{bmatrix} \tilde{S}\tilde{T} \end{bmatrix} = \begin{bmatrix} \tilde{S} \end{bmatrix}[\tilde{T}]\]

In general, the product of two tensors is not commutative:

\[\tilde{T}\tilde{S} \neq \tilde{S}\tilde{T}\]
2B6 Transpose of a Tensor

Transpose of $\mathbf{T}$ is denoted by $\mathbf{T}^T$. If $\mathbf{a}$ and $\mathbf{b}$ are two vectors, then

$$\mathbf{a} \cdot \mathbf{Tb} = \mathbf{b} \cdot \mathbf{T}^T \mathbf{a}$$

In component form, $T_{ij} = T_{ji}^T$

Note $(\mathbf{T}^\mathbf{S})^T = \mathbf{S}^T \mathbf{T}^T$ or $(\mathbf{A}\mathbf{B}\mathbf{C})^T = \mathbf{C}^T \mathbf{B}^T \mathbf{A}^T$
2B7 Dyadic Product

\[ \vec{a} \otimes \vec{b} = \vec{W} \]

Definition \((\vec{a} \otimes \vec{b}) \vec{c} = \vec{a} (\vec{b} \cdot \vec{c})\)

\[ W_{ij} = \hat{e}_i \cdot \vec{W} \hat{e}_j = \hat{e}_i (\vec{a} \otimes \vec{b}) \hat{e}_j = \hat{e}_i \vec{a} (\vec{b} \cdot \hat{e}_j) = a_i \cdot b_j = a_i b_j \]

\[
\begin{bmatrix} \vec{W} \\ a_1 \\ a_2 \\ b_1 \\ b_2 \\ b_3 \\ a_3 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}
\]
2B8 Trace of a Tensor

\[ \text{tr}(\bar{a} \otimes \bar{b}) = \bar{a} \cdot \bar{b} \]

\[ \text{tr}(\tilde{T}) = \text{tr}(T_{ij} \hat{\mathbf{e}}_i \hat{\mathbf{e}}_j) = T_{ij} \text{tr}(\hat{\mathbf{e}}_i \cdot \hat{\mathbf{e}}_j) \]

\[ = T_{ij} (\hat{\mathbf{e}}_i \cdot \hat{\mathbf{e}}_j) = T_{ij} \delta_{ij} = T_{ii} \]

\[ = T_{11} + T_{22} + T_{33} \]

\[ = \text{tr}(T^T) \]
Identity Tensor

A linear transformation which transforms every vector $\vec{a}$ into itself is an identity tensor $\vec{I}$

$$\vec{I} \vec{a} = \vec{a}$$

$$\begin{bmatrix} \vec{I} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Inverse of a vector

Given a tensor $\vec{T}$, if $\vec{S}$ exists such that

$$\vec{S} \vec{T} = I$$

then $\vec{S}$ is the inverse of $\vec{T}$, or

$$\vec{S} = \vec{T}^{-1}$$

Inverse, if $\vec{T}$ exists if the matrix $[T]$ is non-singular,

$$\vec{T}^{-1} \vec{T} = I$$

Note that $(\vec{S} \vec{T})^{-1} = \vec{T}^{-1} \vec{S}^{-1}$
2B10 Orthogonal Tensor

Transformed vectors preserve their lengths and angles, thus if \( \tilde{Q} \) is an orthogonal tensor, then

\[
|\tilde{Q}\vec{a}| = |\vec{a}|
\]

\[
\cos(\vec{a},\vec{b}) = \cos(\tilde{Q}\vec{a},\tilde{Q}\vec{b}), \text{ Thus}
\]

\[
\tilde{Q}\vec{a} \cdot \tilde{Q}\vec{b} = \vec{a} \cdot \vec{b}
\]

for any \( \vec{a} \) and \( \vec{b} \),

\[
(\tilde{Q}\vec{a}) \cdot (\tilde{Q}\vec{b}) = \vec{b} \cdot \tilde{Q}^T(\tilde{Q}\vec{a})
\]

\[
= \vec{b} \cdot \tilde{Q}^T\tilde{Q}(\vec{a}) = \vec{b} \cdot \vec{a} = \vec{b} \cdot I\vec{a}
\]

Thus \( \tilde{Q}^T\tilde{Q} = I \)

Also, \( \tilde{Q}^T\tilde{Q} = \tilde{Q}^{-1}\tilde{Q} = I \), Thus

\[
\tilde{Q}^T = \tilde{Q}^{-1}
\]

Thus for an orthogonal matrix, the transpose is also its inverse.

Example:

A rigid body rotation is an ORTHOGONAL tensor,

\[
[R][R]^T = I \quad [R] = \begin{bmatrix}
\cos \theta & -\sin \theta & 0 \\
\sin \theta & \cos \theta & 0 \\
0 & 0 & 1
\end{bmatrix}
\]

\[
\det[R] = 1
\]
Consider two systems \( \mathbf{X} \equiv \{e_1, e_2, e_3\} \) and \( \mathbf{X}' \equiv \{e'_1, e'_2, e'_3\} \),
\( \{e'_i\} \) is obtained from \( \{e_i\} \) using a rigid body rotation of \( \{e_i\} \).

We are interested to relate the unit vectors \( \{e'_i\} \) in \( \mathbf{X}' \) from that of \( \mathbf{X} \).

\[ \hat{e}'_i = \tilde{Q} \hat{e}_i = Q_{mi} \hat{e}_m \]

We see that,

\[ \hat{e}'_1 = Q_{11} \hat{e}_1 + Q_{21} \hat{e}_2 + Q_{31} \hat{e}_3 \]
\[ \hat{e}'_2 = Q_{12} \hat{e}_1 + Q_{22} \hat{e}_2 + Q_{32} \hat{e}_3 \]
\[ \hat{e}'_3 = Q_{13} \hat{e}_1 + Q_{23} \hat{e}_2 + Q_{33} \hat{e}_3 \]

\([\tilde{Q}]\) is the transformation given by

We note that for \( \tilde{Q} \cdot \tilde{Q}^r = \tilde{I} \), \([\tilde{Q}]\) is an orthogonal matrix

\( Q_{11} = \hat{e}_1 \cdot \tilde{Q} \hat{e}'_1 \)
\( Q_{13} = \hat{e}_1 \cdot \tilde{Q} \hat{e}'_3 \)

For example,
Consider one vector $\vec{a}$
Let $\vec{a}$ belong to $\{\hat{e}_i\}$. Thus $a_i = \vec{a}.\hat{e}_i$ in the $X$ frame.
If $X'$ is another system, and we wish to express $\vec{a}$ within $X'$, let $\hat{e}_i = Q_{mi}\hat{e}_m$

$$a'_i = \vec{a}.Q_{mi}\hat{e}_m = Q_{mi}(\vec{a}.\hat{e}_m)$$
$$a'_i = Q_{mi}a_m$$

$$[a]' = [Q]^T[a]$$

$$\begin{bmatrix}
    a'_1 \\
    a'_2 \\
    a'_3
\end{bmatrix} =
\begin{bmatrix}
    Q_{11} & Q_{21} & Q_{31} \\
    Q_{12} & Q_{22} & Q_{32} \\
    Q_{13} & Q_{23} & Q_{33}
\end{bmatrix}
\begin{bmatrix}
    a_1 \\
    a_2 \\
    a_3
\end{bmatrix}$$

Thus $\{a'_1, a'_2, a'_3\}$ are the components of $\vec{a}$ with respect to the $X'$ system.
Example:

If \( \{ \hat{e}_i' \} \) is obtained by rotating \( \{ \hat{e}_i \} \) ccw with respect to \( e_3 \)-axis find the components of \( \bar{a} = 2\hat{e}_1 \) in terms of \( \{ \hat{e}_i' \} \)

Answer

\[
[Q] = \begin{bmatrix}
0 & -1 & 0 \\
1 & 0 & 0 \\
0 & 0 & 1
\end{bmatrix}
\]

\[
[a]' = [Q]^T [a]
\]

\[
= \begin{bmatrix}
0 & 1 & 0 \\
-1 & 0 & 0 \\
0 & 0 & 1
\end{bmatrix} \begin{bmatrix}
2 \\
0 \\
0
\end{bmatrix} = \begin{bmatrix}
0 \\
-2 \\
0
\end{bmatrix}
\]
Let \( \tilde{T} \in \{ \tilde{e}_i \} \) and \( \tilde{T}' \in \{ \tilde{e}_i' \} \)

Recall that the components of a tensor \( \tilde{T} \) are:

\[
\begin{cases}
T_{ij} = e_i \cdot \tilde{T} e_j \\
T'_{ij} = e_i' \cdot \tilde{T}' e_j
\end{cases}
\]

Since \( e'_i = Q_{mi} e_m \), we have

\[
T'_{ij} = Q_{mi} e_m \cdot \tilde{T} Q_{nj} e_n
= Q_{mi} Q_{nj} (\hat{e}_m \cdot \tilde{T} \hat{e}_n)
\]

Thus

\[
\begin{bmatrix}
T'_{11} & T'_{12} & T'_{13} \\
T'_{21} & T'_{22} & T'_{23} \\
T'_{31} & T'_{32} & T'_{33}
\end{bmatrix}
= \begin{bmatrix}
Q_{11} & Q_{12} & Q_{13} \\
Q_{12} & Q_{22} & Q_{32} \\
Q_{13} & Q_{23} & Q_{33}
\end{bmatrix}
\begin{bmatrix}
T_{11} & T_{12} & T_{13} \\
T_{21} & T_{22} & T_{23} \\
T_{31} & T_{32} & T_{33}
\end{bmatrix}
= \begin{bmatrix}
Q_{11} & Q_{12} & Q_{13} \\
Q_{12} & Q_{22} & Q_{32} \\
Q_{13} & Q_{23} & Q_{33}
\end{bmatrix}
\begin{bmatrix}
Q_{11} & Q_{12} & Q_{13} \\
Q_{12} & Q_{22} & Q_{32} \\
Q_{13} & Q_{23} & Q_{33}
\end{bmatrix}
\]

\[
[T]' = [Q]^T [T] [Q]
\]

Note that the tensor \( \tilde{T} \) is the same, but has different components

\([T]'\) in the new frame \( \{ e'_i \} \) compared to \([T]\) in \( \{ e_i \} \)
Example

Let

\[
[T] = \begin{bmatrix}
0 & 1 & 0 \\
1 & 2 & 0 \\
0 & 0 & 1 \\
\end{bmatrix}
\]

If \([Q]\) represents rotation as in the earlier example such that

\[
[Q] = \begin{bmatrix}
0 & -1 & 0 \\
1 & 0 & 0 \\
0 & 0 & 1 \\
\end{bmatrix}, \text{ then}
\]

\[
[T]' = [Q]^T [T][Q] = \begin{bmatrix}
2 & -1 & 0 \\
-1 & 0 & 0 \\
0 & 0 & 1 \\
\end{bmatrix}
\]
\{e_i\} are the components of the system \(\vec{X}\), and \{\hat{e}_i\} are that of new system \(\vec{X}'\). \(\vec{Q}\) or \(Q_{ij}\) define direction cosines tensor transforming from \(\vec{X}\) to \(\vec{X}'\). \(Q_{ij} = \cos(e_i, e'_j)\); \(Q\) is an orthogonal transformation; \(\vec{Q}\vec{Q}^T = \vec{I}\)

The components transform as follows:

\[
\begin{align*}
\alpha' &= \alpha & \text{Scalar} \\
\hat{a}_i' &= Q_{mi}a_m & \text{Vector} \\
T'_{ij} &= Q_{mi}Q_{nj}T_{mn} & \text{Second Order Tensor} \\
T'_{ijk} &= Q_{mi}Q_{nj}Q_{rk}T_{mnr} & \text{Third Order Tensor} \\
T_{ijk...} &= Q_{mi}Q_{nj}Q_{ok}...Q_{sp}T_{mno...} & n^{th} \text{ Order Tensor}
\end{align*}
\]
Multiplication Rule

a.) If \( a_i \) are components of vector and \( b_j \) are components of another vector, then \( T_{ij} = a_i b_j \) is a second order tensor.

b.) If \( a_i \) is a vector and \( T_{ij} \) are tensor components, then \( \omega_{ijk} = a_i T_{jk} \) is a third order tensor.

c.) The quotient rule

If \( a_i \) are components of a vector, \( T_{ij} \) is the second-order tensor, then

\[
a_i = T_{ij} b_j
\]

\( b_j \) is a vector.

Similarly \( T_{ij} \) and \( E_{ij} \) are tensors, then

\[
T_{ij} = C_{ijkl} E_{kl}
\]

means \( C_{ijkl} \) is a fourth-order tensor,
2B15 Symmetric and Anti-symmetric Tensors

If $\tilde{T}$ is tensor with $T_{ij}$ is its components, then $\tilde{T}^T$ is its transpose, with $T_{ji}$ components. If $\tilde{T}$ is symmetric, then

(a) $\tilde{T} = \tilde{T}^T$ or $T_{ij} = T_{ji}$

Thus a symmetric $[\tilde{T}] = \begin{bmatrix} T_{11} & T_{12} & T_{13} \\ T_{12} & T_{22} & T_{23} \\ T_{13} & T_{23} & T_{33} \end{bmatrix}$ only has 6 components.

(b) If $\tilde{T}$ is anti-symmetric, then $\tilde{T} = -\tilde{T}^T$

$$T_{ij} = -T_{ji} = -T_{ji}$$

Thus $[\tilde{T}] = \begin{bmatrix} 0 & T_{12} & T_{13} \\ -T_{12} & 0 & T_{23} \\ -T_{13} & -T_{23} & 0 \end{bmatrix}$ only has 3 independent components.

(c) Any $\tilde{T}$ can be decomposed into

$$\tilde{T}^S = \frac{\tilde{T} + \tilde{T}^T}{2}$$

$$\tilde{T} = \tilde{T}^S + \tilde{T}^A$$

$$\tilde{T}^A = \frac{\tilde{T} - \tilde{T}^T}{2}$$
2B16 Dual Vector of an Anti-symmetric Tensor

For an antisymmetric tensor,

\[ T_{12} = -T_{21} \]
\[ T_{13} = -T_{31} \]
\[ T_{23} = -T_{32} \]
\[ T_{11} = T_{22} = T_{33} = 0 \]

We can form a vector with the 3 independent components to

Let \( \tilde{T}^A \equiv (\text{Dual Vector}) \)

\[ \tilde{T}^A = \tilde{t}^A \times \tilde{a} \ ; \]

From basics,

\[ \tilde{a} \cdot (\tilde{b} \times \tilde{c}) = \tilde{b} \cdot (\tilde{c} \times \tilde{a}) \]

\[ T_{12} = \hat{e}_1 \cdot \tilde{T}\hat{e}_2 = \hat{e}_1 \cdot \tilde{t}^A \times \hat{e}_2 \]
\[ = \tilde{t}^A \cdot \hat{e}_2 \times \hat{e}_1 = -\tilde{t}^A \cdot \hat{e}_3 = -t^A_3 \]

Similarly

\[ T_{31} = -t^A_2 \text{ and } T_{23} = -t^A_1 \]

It can be seen that

\[ \tilde{t}^A = -\frac{1}{2} \varepsilon_{ijk} T_{jk} \hat{e}_i \text{ or } t_i = -\frac{1}{2} \varepsilon_{ijk} T_{jk} \]

\[ [\tilde{T}^A] = \begin{bmatrix} 0 & -T_{12} & -T_{13} \\ T_{12} & 0 & -T_{23} \\ T_{13} & T_{23} & 0 \end{bmatrix} \]
2B17 Eigenvalues and Eigenvectors of A Tensor

If \( \mathbf{T} \) is a tensor, and \( \mathbf{a} \) is a vector, then if

\[
\mathbf{T} \mathbf{a} = \lambda \mathbf{a}
\]

then \( \mathbf{a} \) is an eigenvector, \( \lambda \) is the corresponding eigenvalue.

If \( \mathbf{n} \) is a unit eigenvector, then

\[
\mathbf{T} \mathbf{n} = \lambda \mathbf{n} = \lambda \mathbf{I} \mathbf{n}
\]

\[
(\mathbf{T} - \lambda \mathbf{I}) \mathbf{n} = 0
\]

Let \( \mathbf{n} = \alpha_i \hat{e}_i \), in terms of components,

\[
(T_{ij} - \lambda \delta_{ij}) \alpha_j = 0
\]

or

\[
(T_{11} - \lambda) \alpha_1 + T_{12} \alpha_2 + T_{13} \alpha_3 = 0
\]
\[
T_{21} \alpha_1 + (T_{22} - \lambda) \alpha_2 + T_{23} \alpha_3 = 0
\]
\[
T_{31} \alpha_1 + T_{32} \alpha_2 + (T_{33} - \lambda) \alpha_3 = 0
\]

\( \alpha_1, \alpha_2, \alpha_3 \) are direction cosines of the vector, then

\[
\alpha_1^2 + \alpha_2^2 + \alpha_3^2 = 1
\]

We have a unique vector (eigenvector) for each eigenvalue.
For any given \( \tilde{T} \),

\[
|\tilde{T} - \lambda \hat{I}| = 0
\]

or

\[
\begin{vmatrix}
T_{11} - \lambda & T_{12} & T_{13} \\
T_{21} & T_{22} - \lambda & T_{23} \\
T_{31} & T_{32} & T_{33} - \lambda
\end{vmatrix} = 0
\]

Above yields a cubic equation in \( \lambda \), called the characteristic equation. Roots of the equation are the eigenvalues of \( \tilde{T} \).
2B18-19 Eigenvalues and Eigen vectors of tensor $\tilde{T}$

Recall the definition that $\tilde{T}$ is a linear transformation, transforming $\vec{a}$ into $\vec{b}$,

$$\vec{a}, \vec{b} \in R \text{ for every } \vec{a}$$
$$\tilde{T}\vec{a} = \vec{b} \text{ all in } X$$

However, if the vector $\vec{a}$ is transformed parallel to itself, then

$$\tilde{T}\vec{a} = \lambda \vec{a}$$

$\vec{a}$ is called an EIGEN VECTOR, and $\lambda$ is the corresponding eigen value.

We consider $\vec{a}$ as a unit vector, transformed into a vector parallel to itself.

Let $\hat{n}$ be a unit eigen vector, then $\tilde{T}\hat{n} = \lambda \hat{n} = \lambda I\hat{n}$

or $(\tilde{T} - \lambda I)\hat{n} = 0$ with $\hat{n} \cdot \hat{n} = 1$

Let $\hat{n} = n_i \hat{e}_i = n_1 \hat{e}_1 + n_2 \hat{e}_2 + n_3 \hat{e}_3$

$$\left(T_{ij} - \lambda \delta_{ij}\right)n_i = 0 \text{ with } n_i n_j = 1$$
The equation can be written as

\[
(T_{11} - \lambda)n_1 + T_{12}n_2 + T_{13}n_3 = 0
\]
\[
T_{21}n_1 + (T_{22} - \lambda)n_2 + T_{23}n_3 = 0
\]
\[
T_{31}n_1 + T_{32}n_2 + (T_{33} - \lambda)n_3 = 0
\]

and \( n_1^2 + n_2^2 + n_3^2 = 1 \)

or in matrix form

\[
|\mathbf{T} - \lambda \mathbf{I}| = 0
\]

\[
\begin{vmatrix}
T_{11} - \lambda & T_{12} & T_{13} \\
T_{21} & T_{22} - \lambda & T_{23} \\
T_{31} & T_{32} & T_{33} - \lambda
\end{vmatrix} = 0
\]

The above equation is a cubic equation in \( \lambda \) (\( T_{ij} \) known).

Giving three roots \( \lambda_1, \lambda_2 \) and \( \lambda_3 \) with three corresponding eigen vectors.
Let $[\sigma]$ denote the stress state at $P \equiv \tilde{\sigma}$

$$\tilde{\sigma} \cdot \hat{n} = \vec{s}$$

Physically $\vec{s}$ denotes stress vector in the direction of $\hat{n}$.

Note the direction of $\hat{n}$ and $\vec{s}$ are not same, since $\vec{s}$ has components parallel and vertical to $\hat{n}$.

However, if $\vec{s}$ and $\hat{n}$ direction are same, then they are the EIGEN DIRECTIONS.
Important: Eigen values will be the same irrespective of $X$ or $X'$

Let $[T] = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 4 \\ 0 & 4 & -3 \end{bmatrix}$

To find: Eigen values and eigen vectors

The characteristic equation is

$$|T - \lambda I| = \begin{vmatrix} 2 - \lambda & 0 & 0 \\ 0 & 3 - \lambda & 4 \\ 0 & 4 & -3 \end{vmatrix} = (2 - \lambda)(\lambda^2 - 25) = 0$$

or $\lambda = 2, 5, -5$

$\lambda_1 = 5$

Let $\lambda_2 = 2$

$\lambda_3 = -5$

Consider $\lambda_1 = 5$, we need to find the eigen vector

$$\left(2 - \lambda_1\right)n_1 + 0n_2 + 0n_3 = 0$$

$$0 + (3 - \lambda_1)n_2 + 4n_3 = 0$$

$$0 + 4n_2 + (-3 - \lambda_1)n_3 = 0$$

$$-3n_1 = 0 \Rightarrow n_1 = 0$$

$$-2n_2 + 4n_3 = 0$$

$$4n_2 - 8n_3 = 0$$

dependent equation

$$n_1^2 + n_2^2 + n_3^2 = 1 \Rightarrow n_3 = \frac{1}{\sqrt{5}} \text{ and } n_2 = \frac{2}{\sqrt{5}}$$

or $\hat{n}_{(\lambda_1)} = \pm \frac{1}{\sqrt{5}} \left(2\hat{e}_2 + \hat{e}_3\right)$

Similarly $\hat{n}_{(\lambda_2)} = \pm \hat{e}_1$
Similarly \( \hat{n}_{(\lambda_3)} = \pm \frac{1}{\sqrt{5}} (-\hat{e}_2 + 2\hat{e}_3) \)
Principal Values and Directions of Real Symmetric Tensors

In elasticity stress tensor, strain tensor, rate of deformation tensor are all real and symmetric. Eigen values of any real symmetric tensor are all real. So, there are at least three eigen vectors called PRINCIPAL DIRECTIONS and corresponding eigen values called PRINCIPAL VALUES. Also in this special case, the principal directions are MUTUALLY PERPENDICULAR.

Case(i) If the characteristic equation $|\tilde{T} - \lambda \hat{I}|$ has three distinct roots.

There will be three distinct eigen values and three eigen vectors mutually perpendicular to each other.

Case (ii) If the characteristic equation has one distinct root $\lambda_1$ and two repeated roots $\lambda_2 = \lambda_3$

There will be two distinct eigen values. There will be an eigen vector corresponding to $\lambda_1$. Any line vertical to $\lambda_1$ will also be an eigen vector. Or any vector lying in the plane vertical to $\hat{n}_{(\lambda_1)}$ will be an eigen vector.

Case(iii) All the three roots are equal.

There is only one eigen value. Any vector in the domain is an eigen vector.

(e.g.) $\delta_{ij}$ tensor

Corresponding to the three mutually vertical directions of eigen vectors, the tensor $\tilde{T}$ can be transformed by coordinate transformation with $\hat{n}_1, \hat{n}_2, \hat{n}_3$, as the unit base vectors

$$\begin{bmatrix} \tilde{T} \end{bmatrix}_{\hat{n}_1, \hat{n}_2, \hat{n}_3} = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$$

to give a diagonal matrix with eigen values.

With the convention $\lambda_1 \geq \lambda_2 \geq \lambda_3$
Recall that for a tensor $\tilde{T}$, the characteristic equation is

$$|T_{ij} - \lambda \delta_{ij}| = 0$$

The cubic equation can be written as

$$\lambda^3 - I_1 \lambda^2 + I_2 \lambda - I_3 = 0$$

with 3 roots $\lambda_1, \lambda_2, \lambda_3$ and three invariants $I_1, I_2, I_3$

$$I_1 = T_{11} + T_{12} + T_{13} = T_{kk}$$

$$I_2 = \frac{1}{2} \left( T_{ii} T_{jj} - T_{ij} T_{ji} \right)$$

$$= \begin{vmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{vmatrix} + \begin{vmatrix} T_{22} & T_{23} \\ T_{32} & T_{33} \end{vmatrix} + \begin{vmatrix} T_{11} & T_{13} \\ T_{31} & T_{33} \end{vmatrix}$$

$$I_3 = \begin{vmatrix} T_{11} & T_{12} & T_{13} \\ T_{21} & T_{22} & T_{23} \\ T_{31} & T_{32} & T_{33} \end{vmatrix} = \text{det}[T]$$

In terms of eigenvalues,

$$I_1 = \lambda_1 + \lambda_2 + \lambda_3$$

$$I_2 = \lambda_1 \lambda_2 + \lambda_2 \lambda_3 + \lambda_3 \lambda_1$$

$$I_3 = \lambda_1 \lambda_2 \lambda_3$$

Note that scalar invariants do not change under coordinate transformations.
Let $\tilde{T} = \tilde{T}(t)$ be a tensor-valued function of a scalar $t$ say, differentiating with respect to time.

$$\frac{d\tilde{T}}{dt} = \lim_{t \to 0} \frac{\tilde{T}(t + \Delta t) - \tilde{T}(t)}{\Delta t}$$

The usual rules of differential calculus hold good.

$$\frac{d}{dt} (\tilde{T} + \tilde{S}) = \frac{d\tilde{T}}{dt} + \frac{d\tilde{S}}{dt}$$
$$\frac{d}{dt} (\alpha(t) \cdot \tilde{T}) = \frac{d\alpha}{dt} \tilde{T} + \frac{d\tilde{T}}{dt} \cdot \alpha$$
$$\frac{d}{dt} (\tilde{T} \cdot a) = \frac{d\tilde{T}}{dt} \cdot a + \tilde{T} \frac{da}{dt}$$
$$\frac{d}{dt} (\tilde{T}^T) = \left(\frac{d\tilde{T}}{dt}\right)^T$$

Next: 2C1 Example Problem 1
Consider $\frac{d\tilde{T}}{dt}$

We can show that $\left(\frac{d\tilde{T}}{dt}\right)_{ij} = \frac{d\tilde{T}_{ij}}{dt}$

The new tensor can be obtained by differentiating the individual components

$$\dot{\tilde{T}} = \frac{d}{dt}(\tilde{T})$$

Thus $\dot{T}_{ij} = \frac{d}{dt}(T_{ij})$
If \( R(t) \) is a time dependant rotation tensor, \( \vec{r}_o \) is transformed into \( r(t) \) by

\[
\vec{r}(t) = \tilde{R} \cdot \vec{r}_o
\]

\[
\frac{d\vec{r}}{dt} = \dot{\vec{r}} = \vec{\omega} \times \vec{r}
\]

Where \( \vec{\omega} \) is the dual vector of \( T^R \)

Proof:

\[
\frac{d\vec{r}}{dt} = \frac{d\tilde{R}}{dt} \vec{r}_o = \tilde{R} R^T \vec{r}_o = \vec{\omega} \times \vec{r}
\]

Recall that \( \vec{\omega} \) is the dual vector of \( \tilde{R} R^T \)

Next: 2C2 Scalar Field, Gradient of a Scalar Function
Let \( \phi(\vec{r}) \) be a scalar valued function of \( \vec{r} \). \( \phi(\vec{r}) \) can be density, temperature at \( \vec{r} \);

Let us define \( \nabla = \left( \frac{\partial}{\partial x} i + \frac{\partial}{\partial y} j + \frac{\partial}{\partial z} \right) \).

We note that the gradient of \( \phi(\vec{r}) \) is a vector field

\[
\nabla \phi = \phi(\vec{r} + d\vec{r}) - \phi(\vec{r}) = \nabla \phi \cdot d\vec{r}
\]

Unit vector along gradient = \( \frac{d\vec{r}}{dr} = \hat{n} = \hat{e} \)

Next: 2C2(b) (Continued)
\[ \nabla \phi = \frac{\partial \phi}{\partial x_1} \hat{e}_1 + \frac{\partial \phi}{\partial x_2} \hat{e}_2 + \frac{\partial \phi}{\partial x_3} \hat{e}_3 \]

\[ \text{grad} \phi \cdot d\vec{r} = 0 \]
\[ \nabla \phi \cdot d\vec{r} = 0 \quad \text{For } d\vec{r} \text{ on “iso-} \phi \text{”} \]
\[ \nabla \phi \cdot d\vec{r} \text{ is maximum when } \nabla \phi \text{ is parallel to } d\vec{r} \]
2C2 Example Problem:

Previous: 2C2 (b) (Continued)

\[ \phi = x_1 x_2 + x_3 \] Find \( \hat{n} \) normal to const \( \phi \) at \( \bar{r} = (2,1,0) \)

\[ \nabla \phi = \frac{\partial \phi}{\partial x_1} \hat{e}_1 + \frac{\partial \phi}{\partial x_2} \hat{e}_2 + \frac{\partial \phi}{\partial x_3} \hat{e}_3 = x_1 \hat{e}_1 + x_2 \hat{e}_2 + x_3 \hat{e}_3 \]

At \((2,1,0)\)

\[
\begin{align*}
\nabla \phi &= \hat{e}_1 + 2\hat{e}_2 + \hat{e}_3 \\
\hat{n} &= \frac{1}{\sqrt{6}}(\hat{e}_1 + 2\hat{e}_2 + \hat{e}_3)
\end{align*}
\]

Next: 2C3 Vector field, Gradient of a vector field
Let \( \vec{v}(r) \) vector valued function, displacement or velocity field.

\( \text{Grad} \vec{v}(r) \) is a tensor field \( \nabla \vec{V} \)

\[
d\vec{V} = \vec{V}(r + d\vec{r}) - \vec{V}(r) = (\nabla \vec{V})d\vec{r}
\]

\( (\nabla \vec{V})_{ij} = e_i.(\nabla \vec{V})\hat{e}_j \)

\[
= e_i \frac{\partial \vec{V}}{\partial x_j} = \frac{\partial}{\partial x_j} (e_i.\vec{V}) = \frac{\partial v_i}{\partial x_j} = v_{i,j}
\]

\[
[\nabla \vec{V}] = \begin{bmatrix}
\frac{\partial v_1}{\partial x_1} & \frac{\partial v_1}{\partial x_2} & \frac{\partial v_1}{\partial x_3} \\
\frac{\partial v_2}{\partial x_1} & \frac{\partial v_2}{\partial x_2} & \frac{\partial v_2}{\partial x_3} \\
\frac{\partial v_3}{\partial x_1} & \frac{\partial v_3}{\partial x_2} & \frac{\partial v_3}{\partial x_3}
\end{bmatrix} = [v_{i,j}]
\]

Define \( \frac{\partial v_i}{\partial x_j} = v_{i,j} \)

Note that \((\_)_j \) indicates differentiation in the \( x_{j}^{th} \) direction.

Also, \( \frac{\partial x_i}{\partial x_j} = x_{i,j} = \delta_{ij} \)

Next: 2C4 Divergence of a Vector Field: -Scalar Field
2C4(a) **Divergence of a Vector Field: - Scalar Field.**

Previous: 2C3 Vector Field, Gradient of a Vector Field

Let \( \mathbf{v}(r) \) be a vector field.

\[
\text{div} \mathbf{v}(r) = tr(\nabla \mathbf{V})
\]

Recall diagonal element,

\[
\frac{\partial v_1}{\partial x_1}, \frac{\partial v_2}{\partial x_2}, \frac{\partial v_3}{\partial x_3}
\]

\[
\text{div}(\mathbf{v}) = \frac{\partial v_1}{\partial x_1} + \frac{\partial v_2}{\partial x_2} + \frac{\partial v_3}{\partial x_3} = \frac{\partial v_m}{\partial x_m} = v_{m,m}
\]

Next: 2C4 Divergence of a Tensor Field
Let $\tilde{T}(\tilde{r})$ be a tensor field. $Div[\tilde{T}(\tilde{r})]$ is a vector field. For any vector $\tilde{a}$,

$$div(T_{im}e_m) - 0 = \frac{\partial T_{im}}{\partial x_m} = T_{im,m}$$

If $\tilde{b} = \alpha \tilde{a}$

$$div\tilde{b} = \frac{\partial \tilde{b}_i}{\partial x_i} = \alpha \frac{\partial a_i}{\partial x_i} + \alpha \frac{\partial \alpha}{\partial x_i} a_i$$

$$= \alpha.div(\tilde{a}) + (\nabla \alpha) \tilde{a}$$

Next: 2C5 Curl of a Vector Field: Vector Field
Let \( \mathbf{v}(\mathbf{r}) \) be a vector field. \( \tilde{t}^A \) is a dual vector of \((\nabla \mathbf{v})^A\)

\[
\text{Curl of } \mathbf{v} = \nabla \times \mathbf{v} = 2\tilde{t}^A.
\]

\[
[\nabla \tilde{V}]^A = \begin{bmatrix}
0 & -\frac{1}{2} \left( \frac{\partial v_1}{\partial x_2} - \frac{\partial v_2}{\partial x_1} \right) & \frac{1}{2} \left( \frac{\partial v_1}{\partial x_3} - \frac{\partial v_3}{\partial x_1} \right) \\
-\frac{1}{2} (v_{1,2} - v_{2,1}) & 0 & -\frac{1}{2} (v_{2,3} - v_{3,2}) \\
-\frac{1}{2} (v_{1,3} - v_{3,1}) & -\frac{1}{2} (v_{2,3} - v_{3,2}) & 0
\end{bmatrix}
\]

\[
\nabla \times \mathbf{v} = 2\tilde{t}^A
\]

\[
= (v_{3,2} - v_{2,3})\hat{e}_1 + (v_{1,3} - v_{3,1})\hat{e}_2 + (v_{2,1} - v_{1,2})\hat{e}_3
\]

Next: Problem 2C4
Problem 2C4

Consider a temperature field given by \( \theta = 3xy \).

(a) Find the heat flux at the point \( A(1,1,1) \) if 
\[ q = -k \nabla \theta. \]

(b) Find the heat flux at the same point as part (a) if 
\[ q = -K \nabla \theta, \]
where
\[ [K] = \begin{bmatrix} k & 0 & 0 \\ 0 & 2k & 0 \\ 0 & 0 & 3k \end{bmatrix} \]

Solution: \( \nabla \theta = 3y e_1 + 3x e_2 \rightarrow (\nabla \theta)_A = 3e_1 + 3e_2 \)

(a) \[ q = -3k(e_1 + e_2) \]

(b) \[ [q] = - \begin{bmatrix} k & 0 & 0 \\ 0 & 2k & 0 \\ 0 & 0 & 3k \end{bmatrix} \begin{bmatrix} 3 \\ 3k \\ 0 \end{bmatrix} = - \begin{bmatrix} 3k \\ 6k \\ 0 \end{bmatrix} \rightarrow q = -3ke_1 - 6ke_2 \]
Chapter 3
Kinematics of a Continuum

Kinematics refers to the motion of a particle without regard to what causes the motion. Particles in a continuum refer to the infinitesimal volume of a body.

Particle P at t=0 moves to P’ at t=t. The position vector,

\[ \mathbf{r}(t) = x(t)\mathbf{e}_1 + y(t)\mathbf{e}_2 + z(t)\mathbf{e}_3 \]

Every particle moves according

\[ x_i = x_i(X_1, X_2, X_3, t) \quad i = 1, 2, 3 \]

\[ \mathbf{x} = \mathbf{x}(\mathbf{X}, t) \rightarrow \text{Pathline} \]

Position of particle \( X_i = \mathbf{x}(\mathbf{X}, 0) \)

\( X_1, X_2, X_3 \rightarrow \text{Position of particles (material coordinates)} \)

\( x_1, x_2, x_3 \rightarrow \text{Spatial coordinates} \)
In a reference configuration, we may use original particle description leading to a Lagrangian description. If we refer to current coordinates at $t=t$, then it is called Eulerian description.

Next: 3-1 Example
3.1 Example:

Given: \( \vec{x} = \vec{X} + ktX_2 \hat{e}_1 \)

In terms of components

\[
\begin{align*}
x_1 &= X_1 + ktX_2 \\
x_2 &= x_2 \\
x_3 &= x_3
\end{align*}
\]

At:

\[
\begin{align*}
t &= 0 & t &= t \\
O(0,0,0) & O(0,0,0) \\
C(0,1,0) & C(kt,1,0) \\
B(1,10) & B(1 + kt,10)
\end{align*}
\]

Note that the motion is simply a shearing process.
3.2 Material and Spatial Description

Consider the variables

Scalar $\theta$ Temperature
Vector $\mathbf{v}$ Velocity
Tensor $\mathbf{T}$ Stress

Then $\theta = \theta_1(X_1, X_2, X_3, t)$  
$\mathbf{v}_1 = \mathbf{v}_1(X_1, X_2, X_3, t)$

$\mathbf{T} = T_1(X_1, X_2, X_3, t) \iff$ Lagrangian

If $\theta = \theta_1(x_1, x_2, x_3, t)$  
$\mathbf{v}_1 = \mathbf{v}_1(x_1, x_2, x_3, t)$

$\mathbf{T} = \mathbf{T}_1(x_1, x_2, x_3, t) \iff$ Spatial or Eulerian

Next: 3.3 Material Derivative
3.3 Material Derivative.

Material derivative is defined as the time rate of change of quantity $(\theta)$ for a fixed particle.

(i) If $\theta = \theta_1(X_1, X_2, X_3, t) \rightarrow$ Material description

$$\frac{D\theta}{Dt} = \left( \frac{\partial \theta_1}{\partial t} \right)_{X_i,\text{fixed}}$$

(ii) If $\theta = \theta_2(x_1, x_2, x_3, t) \rightarrow$ Spatial description

$$\frac{D\theta}{Dt} = \left( \frac{\partial \theta_2}{\partial t} \right)_{X_i,\text{fixed}}$$

$$= \frac{\partial \theta_2}{\partial t} + \frac{\partial \theta_2}{\partial x_1} \frac{\partial x_1}{\partial t} + \frac{\partial \theta_2}{\partial x_2} \frac{\partial x_2}{\partial t} + \frac{\partial \theta_2}{\partial x_3} \frac{\partial x_3}{\partial t} + \left( \frac{\partial \theta_2}{\partial t} \right)_{X_i,\text{fixed}}$$

Thus, in spatial description, we have an additional term $\vec{v} \cdot \nabla \theta$ indicating the effect of velocity at that location $x_i$.

Next: 3.4 Acceleration of a Particle
3.4 Acceleration of a Particle

It is the material derivative of velocity. Thus in material description:

\[
\bar{x} = \bar{x}(\bar{X}, t)
\]

\[
\bar{v} = \left( \frac{\partial x}{\partial t} \right)_{X_i \text{ fixed}} \equiv \frac{D\bar{X}}{Dt}
\]

\[
\bar{a} = \left( \frac{\partial v}{\partial t} \right)_{X_i \text{ fixed}} \equiv \frac{D\bar{v}}{Dt}
\]

In terms of spatial description

\[
\bar{a} = \frac{\partial \bar{v}}{\partial t} + (\bar{\nabla} \bar{v}) \bar{v}
\]

If \( \bar{v} = \bar{v}(x_1, x_2, x_3, t) \),

Then \( a_i = \frac{\partial v_i}{\partial t} + v_j \frac{\partial v_i}{\partial x_j} \)

Next: 3.5 Displacement Field
3.5 Displacement Field

Previous: 3.4 Acceleration of a Particle

\[ \vec{u} = \vec{r}_i - \vec{r}_o \]

\[ = \vec{x}(\vec{X}, t) - \vec{X} \]

Example:

\[ x_1 = \frac{X_1}{2} \quad x_2 = X_2 \quad x_3 = X_3 \]

\[ u_1 = \frac{X_1}{2} - X_1 = -\frac{X_1}{2} \]

\[ u_2 = u_3 = 0 \]

Next: 3.6 Rigid Body Motion
3.6 Rigid Body Motion

(a) Rigid Body Translation
\[ \vec{x} = \vec{X} + c(t) \]
\[ \vec{u} = c(t) \text{ No relative displacement} \]

(b) Rigid Body Rotation
\[ \vec{x} - \vec{b} = \tilde{R}(t)(\vec{X} - \vec{b}) \]

where \( \tilde{R}(t) \) is proper orthogonal (rotation) tensor.
\[ \tilde{R}(0) = I \text{ Rotation about particle } \vec{b} \]

Rotation about origin \( \vec{x} = \tilde{R}(t)\vec{X} \)

\[ \begin{array}{c}
\text{Rotation about origin } \vec{x} = \tilde{R}(t)\vec{X} \\
\end{array} \]
\[
\begin{align*}
\overline{PQ} &= \overline{X}_p - \overline{X}_q \\
\overline{x}_p - \overline{b} &= \tilde{R}(t)(\overline{X}_p - \overline{b}) \\
\overline{x}_q - \overline{b} &= \tilde{R}(t)(\overline{X}_q - \overline{b}) \\
\overline{x}_p - \overline{x}_q &= \tilde{R}(t)(\overline{X}_p - \overline{X}_q) \\
\text{or} \quad \Delta x &= \tilde{R}(t)\Delta \overline{X} \\

\Delta \overline{x} - \Delta \overline{X} &= (\tilde{R}(t)\Delta \overline{X})\cdot \tilde{R}(t)\Delta \overline{X} \\
(\Delta x)^2 &= (\Delta \overline{X})^2 \cdot R \cdot R^T \\
(\Delta x)^2 &= (\Delta \overline{X})^2
\end{align*}
\]

Results indicate that there is no change in length.
3.6(c) General Rigid Body Motion

\[ \bar{x} = \tilde{R}(t) \left( \bar{X} - \bar{b} \right) + c(t) \]

\[ \bar{v} = \dot{R}(t) \left( \bar{X} - \bar{b} \right) + \dot{c}(t) \]

\[ \left( \bar{X} - \bar{b} \right) = \tilde{R}^T (\bar{x} - \bar{c}) \]

Substituting \( \bar{v} = \dot{R} \tilde{R} (\bar{x} - \bar{c}) + \dot{c}(t) \)

However

\[ R \cdot R^T = I \]

\[ \dot{R} \cdot R^T - \dot{R} \cdot R^T = 0 \]

\[ (\dot{R} \cdot R^T)^T = (\dot{R} \cdot R)^T = -\dot{R} \cdot R^T \]

However \( \dot{R} \cdot R^T \) is anti symmetric with \( \bar{w} \) as a dual vector. Using this concept

\[ \bar{w} \times (\bar{x} - \bar{c}) + \dot{c} \]

\[ \bar{v} = \bar{w} \times (\bar{x} - \bar{c}) + \dot{c} \]

Let \( \bar{r} = \bar{x} - \bar{c} \) then

\[ \bar{v} = \bar{w} \times \bar{r} + \dot{c} \]

Next: 3.7 Infinitessimal Deformation
3.7 Infinitesimal Deformations

Previous: 3.6(c) General Rigid Body Motion

Let $P(\vec{X}), Q(\vec{X} + d\vec{X})$ deform to $P' (\vec{x}) + Q' (\vec{x} + d\vec{x})$ with

$$\begin{align*}
\bar{u}_P (\vec{X}) &= \vec{x} - \vec{X} \\
\bar{u}_Q (\vec{X} + d\vec{X}) &= (\vec{x} + d\vec{x}) - (\vec{X} + d\vec{X}) \\
d\vec{x} &= P' + Q' \\
&= \left[ (\vec{x} + d\vec{x}) - (\vec{X} + d\vec{X}) \right] - (\vec{x} - \vec{X}) + d\vec{X} \\
d\vec{x} &= \bar{u} (\vec{X} + d\vec{X}) - \bar{u}(\vec{X}) + d\vec{X}
\end{align*}$$

Define displacement gradient

$$\nabla u = \frac{du}{d\vec{a}}$$

Then
\[ d\vec{x} = d\vec{X} + (\nabla \vec{u}).d\vec{X} \]

\[ \nabla \vec{u} = \begin{bmatrix} \frac{\partial u_1}{\partial a_1} & \frac{\partial u_1}{\partial a_2} & \frac{\partial u_1}{\partial a_3} \\ \frac{\partial u_2}{\partial a_1} & \frac{\partial u_2}{\partial a_2} & \frac{\partial u_2}{\partial a_3} \\ \frac{\partial u_3}{\partial a_1} & \frac{\partial u_3}{\partial a_2} & \frac{\partial u_3}{\partial a_3} \end{bmatrix} \]

Next: 3.7 Example Problem
3.7 Example Problem

Previous: 3.7 Infinitessimal Deformation

Particle A:

\[ u_1 = ka_2^2 \]
\[ u_2 = u_3 = 0 \]
\[ x_1 = u + a_1 = a_1 + ka_2^2 \]
\[ x_2 = a_2; x_3 = a_3 \]

Line OB
\[ x_1 = \text{Constant} = 0 \]
\[ x_1 = ka_2^2 \text{ (Parabolic)} \]

Next: 3.7 Example Prob. (Cont’d - B)
Particle B

We use \( a_1, a_2, a_3 \) instead of \( x_1, x_2, x_3 \) for convenience

\[
\begin{align*}
(a_1, a_2, a_3) &= (0, 1, 0) \\
(x_1, x_2, x_3) &= (k, 1, 0)
\end{align*}
\]

\[
\nabla \vec{u} = \begin{bmatrix} 0 & 2ka_2 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}
\]

Now

\[
\begin{align*}
d\vec{x} &= d\vec{a} + (\nabla \vec{u}) d\vec{a} \\
d\vec{a} &= d\vec{a} \left( I + \nabla \vec{u} \right)
\end{align*}
\]
3.7 (Cont’d – Deformation Gradient)

Let \( \widetilde{F} = \) deformation gradient \( = (I + \nabla \mathbf{u}) \), then
\[
F_{ij} = u_{i,j} + \delta_{ij}
\]
\[
d\mathbf{x} = \widetilde{F} \cdot da
\]
\[
dx_i = F_{ij} da_j
\]

Consider a length of \( da \cdot da \) then

\[
\begin{align*}
(d\mathbf{x})^2 &= (\widetilde{F} da) \cdot (\widetilde{F} da) \\
(ds)^2 &= da \cdot (\widetilde{F}^T \widetilde{F} da)
\end{align*}
\]

If \( \widetilde{F} \) is orthogonal, then \( \widetilde{F}^T \widetilde{F} = I \), then \( (dx)^2 = (da)^2 \) i.e. no change in length

Thus rigid body rotation does not produce any change in length.

Consider
\[
\overline{F}^T \overline{F} = (I + \nabla \mathbf{u})^T (I + \nabla \mathbf{u})
\]
\[
= I + \nabla \mathbf{u} + (\nabla \mathbf{u})^T + \nabla \mathbf{u}^T \cdot \nabla \mathbf{u}
\]

If \( \nabla \mathbf{u}^T \cdot \nabla \mathbf{u} \) is small compared to \( \nabla \mathbf{u} \) then
\[
\bar{F}^T \bar{F} = \bar{I} + 2\bar{E}
\]

Where \( \bar{E} = \frac{1}{2} \left( \nabla u + \nabla u^T \right) \) infinitesimal strain tensor

\[
E_{ij} = \frac{1}{2} \left( u_{i,j} + u_{j,i} \right)
\]

Next: 3.8 Geometrical Meaning of \( E_{ij} \)
3.8 Geometrical Meaning of $E_{ij}$

Consider two material elements

\[
\begin{align*}
&dx^{(1)} = \tilde{F} d\bar{a}^{(1)} \\
&dx^{(2)} = \tilde{F} d\bar{a}^{(2)} \\
&dx^{(1)} dx^{(2)} = \tilde{F}^T \tilde{F} d\bar{a}^{(1)} d\bar{a}^{(2)}
\end{align*}
\]

Diagonal terms

Case (i)

Let

\[
\begin{align*}
&d\bar{a}^{(1)} = d\bar{a}^{(2)} = d\bar{a} = ds \hat{n} \\
&dS = \left| d\bar{a}^{(1)} \right| = n \cdot ds
\end{align*}
\]

Let $ds$ be the deformed length, then
\[(ds)^2 = (dS)^2 + \tilde{F}^T \tilde{F} \left( \tilde{d\bar{a}}^{(1)} \right) \]

\[\tilde{F}^T \tilde{F} = \tilde{I} + 2\tilde{E} \]

\[(ds)^2 - (dS)^2 = \tilde{d\bar{a}}^{(1)} \tilde{F}^T \tilde{F} d\bar{a}^{(1)} \]

\[= \hat{n}\tilde{E}\hat{n} \cdot 2(ds)^2 \]

\[(ds)^2 - (dS)^2 = (ds - dS)(ds + dS) \]

\[\approx 2dS(ds - dS) \]

\[\therefore \frac{ds - dS}{dS} = \hat{n}\tilde{E}\hat{n} = E_{nn} \text{ (no sum)} \]

\[E_{11} \text{ Elongation in 1 direction, } E_{22} \text{ in 2 and } E_{33} \text{ in 3 hence engineering strain } \frac{ds - dS}{dS} \text{ is the diagonal term of E in three directions.} \]
3.8 Cont’d (b)  
Previous: 3.8 Geometrical Meaning of E_{ij}

Off Diagonal Terms

$d\vec{a}_1^1 d\vec{s}_1 \hat{m} = d\vec{a}_2^2 = d\vec{s}_2 \hat{n}$

$d\vec{a}_1^{(1)} d\vec{a}_2^{(2)} = 2(d\vec{s}_1)(d\vec{s}_2) \hat{m}.\hat{E}.\hat{n}$

$= (d\vec{s}_1)(d\vec{s}_2) \cos \theta$

$\gamma = \frac{\pi}{2} - \theta = \text{Shear strain}$

$\cos \left( \frac{\pi}{2} - \gamma \right) = \sin \gamma \approx \gamma$ for small $\gamma$

$\therefore \gamma = 2 \hat{m}.\hat{E}.\hat{n}$
Thus $2E_{12}$ gives angle between lines lying along $x_1$ and $x_2$ directions.

$2E_{12} \Rightarrow$ Shear strain at $P$ of $x_1 - x_2$. 
Given:

\[ u_1 = k \left( 2a_1 + a_2^2 \right) \]
\[ u_2 = k \left( a_1^2 - a_2^2 \right) \]
\[ u_3 = 0 \]

Assume k is small i.e order of \(10^{-4}\)

Find: Particle \( P \left( X = \hat{e}_1 - \hat{e}_2 \right) \), find the unit elongation and change in angle. (normal and shear strain)

Solution:

\[
[\nabla u] = k \begin{bmatrix}
2 & 2a_2 & 0 \\
2a_1 & -2a_2 & 0 \\
0 & 0 & 0
\end{bmatrix}
\]

\[
[\nabla u] = k \begin{bmatrix}
2 & -2 & 0 \\
2 & 2 & 0 \\
0 & 0 & 0
\end{bmatrix}_{at(1,-1,0)}
\]

\[
[E] = \frac{1}{2} [\nabla u + \nabla u^T] = k \begin{bmatrix}
2 & 0 & 0 \\
0 & 2 & 0 \\
0 & 0 & 0
\end{bmatrix}
\]

\[ E_{11} = E_{22} = 2k \]
\[ \Rightarrow da_1 \text{ and } da_2 \text{ stretch by 2k times} \]
\[ dx_1 - da_1 = 2k \quad dx_2 - da_2 = 2k \]

If \( k = 2 \times 10^{-4} \) then

\[ dx_1 = 2 \times 10^{-4} da_1 \]
\[ dx_2 = 2 \times 10^{-4} da_1 \]

\( E_{12} = 0 \Rightarrow \text{no shear strain} \)

Since \( x_i = a_i + u_i \)

\[ \vec{d}x^{(1)} = [\nabla \vec{u}]da^{(1)} + da_1 \]

\[ \begin{bmatrix} \vec{d}a^{(1)} \\ 0 \\ 0 \end{bmatrix} + k \begin{bmatrix} 2 & -2 & 0 \\ 2 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} da_1 \\ 0 \\ 0 \end{bmatrix} = da_1 \begin{bmatrix} 1 + 2k \\ 2k \\ 0 \end{bmatrix} \]

Line OA \( \alpha = \tan^{-1} \frac{2kd}{da_1} \approx da_1 \)

Similarly, Line OB \( \Rightarrow \beta = 2k \quad \text{(No shear strain)} \)
3.9 Principal Strain

Previous: Example Problem 3.8.2

Since \( \tilde{E} \) is real and symmetric, there are three mutually perpendicular directions \( \hat{n}_1, \hat{n}_2 \) and \( \hat{n}_3 \) for which is diagonal

\[
[E]_{n_i} = \begin{bmatrix}
E_1 & 0 & 0 \\
0 & E_2 & 0 \\
0 & 0 & E_3 \\
\end{bmatrix}
\]

\( \hat{n}_1, \hat{n}_2 \) and \( \hat{n}_3 \) are the principal directions of strain

\( E_1, E_2, E_3 \) are the three Principal strains (maximum and minimum strains)

Characteristic equation for \( \tilde{E} \) is

\[
\lambda^3 - I_1 \lambda^2 + I_2 \lambda - I_3 = 0
\]

\( \lambda \)'s are the scalar invariants

\( e \) = change in unit volume = \( \Delta V/V \)

\[
e = E_{ii} = E_{11} + E_{22} + E_{33} = \frac{\partial u_i}{\partial a_j} = div \vec{u}
\]

Next: 3.10 Dilataion
3.10 Dilatation

When a body deforms from a volume $dV$ to $dv$ where $V, v$ are volumes in $\{X\}$ and $\{x\}$, then

$$
dv = ds_1 \bullet ds_2 \bullet ds_3
$$

$$
= dS_1 (1 + E_1) dS_2 (1 + E_2) dS_3 (1 + E_3)
$$

$$
= dS_1 dS_2 dS_3 (1 + E_1 + E_2 + E_3 + H.O.T)
$$

$$
= dV (1 + E_1 + E_2 + E_3)
$$

Change in Volume $= dv - dV$

Volumetric Strain $= \frac{dv - dV}{dV}$

$= E_1 + E_2 + E_3$

$= E_{ii} = \text{first scalar invariant of } \widetilde{E}$

Next: 3.11 Rotation Tensor
3.11 Rotation Tensor

Previous: 3.10 Dilataion

\[
d\vec{x} = d\vec{a} + (\nabla \vec{u}) d\vec{a} \\
\frac{1}{2} \left[ (\nabla \vec{u}) + \nabla \vec{u}^T \right] = \tilde{E}
\]

\[
d\vec{x} = d\vec{a} + (\tilde{E} + \tilde{\Omega}) d\vec{a} \\
\frac{1}{2} \left[ (\nabla \vec{u}) - \nabla \vec{u}^T \right] = \tilde{\Omega}
\]

\(d\vec{a}\) is stretched and rotated to result in \(d\vec{x}\). Stretching comes purely from \(\vec{E}\), whereas the rotation comes both from \(\vec{E}\) and \(\tilde{\Omega}\). However, if \(d\vec{a}\) is along the eigen vector direction of \(\vec{E}\), then change (rotation) in \(d\vec{x}\) comes purely from \(\vec{E}\). Thus \(\tilde{\Omega}\) denotes the rotation of eigen vector of \(\vec{E}\). If \(\vec{t}^A\) is dual vector of \(\tilde{\Omega}\), then

\[
\vec{t}^A \times d\vec{a} = \tilde{\Omega} d\vec{a}
\]

and \(\vec{t}^A = \tilde{\Omega}_{32} \hat{e}_1 + \tilde{\Omega}_{13} \hat{e}_2 + \tilde{\Omega}_{21} \hat{e}_3\)

where \(\hat{e}_1, \hat{e}_2, \hat{e}_3\) lie along the principal direction of \(\vec{E}\).

Next: 3.12 Time Rate of Change of a Material Element
3.12 Time Rate of Change of a Material Element

We need to compute the rate of change $d\vec{a}$ at time $t$. Let us use spatial coordinates $(x_1, x_2, x_3)$ and denote $\frac{D}{Dt}d\vec{x} = (\nabla \vec{v})d\vec{x}$.

Velocity gradient $= \nabla \vec{v} = v_{i,j}$ where $(i, j)$ with respect to spatial coordinates

Thus $\nabla \vec{v} = \begin{bmatrix}
\frac{\partial v_1}{\partial x_1} & \frac{\partial v_1}{\partial x_2} & \frac{\partial v_1}{\partial x_3} \\
\frac{\partial v_2}{\partial x_1} & \frac{\partial v_2}{\partial x_2} & \frac{\partial v_2}{\partial x_3} \\
\frac{\partial v_3}{\partial x_1} & \frac{\partial v_3}{\partial x_2} & \frac{\partial v_3}{\partial x_3}
\end{bmatrix}$

Next: 3.13 Rate of Deformation Tensor.
3.13 Rate of Deformation Tensor.

Previous: 3.12 Time Rate of Change of a Material Element

Let \( \nabla \vec{v} = D + W \)

Where \( D \) = symmetric part of velocity gradient.

\[
D = \frac{1}{2} \left[ (\nabla \vec{v}) + (\nabla \vec{v})^T \right] = \text{Rate of deformation tensor}
\]

\( W \) = anti-Symmetric part of velocity gradient

\[
W = \frac{1}{2} \left[ (\nabla \vec{v}) - (\nabla \vec{v})^T \right]
\]

= Spin tensor

Thus

\[
D_{ij} = \frac{1}{2} \left( v_{i,j} + v_{j,i} \right)
\]

\[
W_{ij} = \frac{1}{2} \left( v_{i,j} - v_{j,i} \right)
\]

\( \vec{D} \) describes the rate of change of length \( d\vec{x} \)

\( \vec{W} \) describes the rate of rotation of \( d\vec{x} \)

Next: 3.14 Spin tensor and Angular velocity vector
3.14 Spin tensor and Angular velocity vector

Previous: 3.13 Rate of Deformation Tensor.

Since \( \mathbf{\tilde{W}} \) is anti symmetric for any vector \( \mathbf{a} \)

\[
\mathbf{\tilde{W}} \mathbf{a} = \mathbf{\omega} \times \mathbf{a}
\]

Where \( \mathbf{\omega} = -(W_{23} \hat{e}_1 + W_{31} \hat{e}_2 + W_{12} \hat{e}_3) \)

Consider a vector \( d\mathbf{x} \), then

\[
\mathbf{\tilde{W}} d\mathbf{x} = \mathbf{\omega} \times d\mathbf{x}
\]

\[
\frac{D}{Dt}(d\mathbf{x}) = (\nabla \mathbf{v}) d\mathbf{x} = (\mathbf{\tilde{D}} + \mathbf{\tilde{W}}) d\mathbf{x}
\]

\[
= \mathbf{\tilde{D}} d\mathbf{x} + \mathbf{\tilde{W}} d\mathbf{x}
\]

\[
= \mathbf{\tilde{D}} d\mathbf{x} + \mathbf{\tilde{W}} \times d\mathbf{x}
\]

(simply rotate, no length change)

\( 2\mathbf{\tilde{W}} \Rightarrow \) vorticity tensor

Next: 3.15 Conservation of Mass
3.15 Conservation of Mass

Previous: 3.14 Spin tensor and Angular velocity vector

Since $\rho dV$ is the mass of an element with volume $dV$,

$$\frac{D}{Dt}(\rho dV) = 0 \Rightarrow \rho \frac{D}{Dt}(dV) + dV \frac{D\rho}{Dt} = 0$$

now, dividing by $dV$

$$\Rightarrow \rho \frac{\partial v_i}{\partial x_i} + \frac{D\rho}{Dt} = 0$$

Conservation of Mass / Continuity Equation

In spatial coordinates

$$\rho \text{div}(\vec{V}) + \frac{D\rho}{Dt} = 0$$

In Cartesian coordinates

$$\frac{D\rho}{Dt} = \frac{\partial \rho}{\partial t} + \vec{v} \cdot \nabla \rho$$

In Cartesian coordinates

$$\frac{D\rho}{Dt} = \frac{\partial \rho}{\partial t} + \vec{v} \cdot \nabla \rho \text{ and } \rho \text{div}(\vec{V}) + \frac{D\rho}{Dt} = 0$$

Thus

$$\rho \left( \frac{\partial v_1}{\partial x_1} + \frac{\partial v_2}{\partial x_2} + \frac{\partial v_3}{\partial x_3} \right) + \frac{\partial \rho}{\partial t} + v_1 \frac{\partial \rho}{\partial x_1} + v_2 \frac{\partial \rho}{\partial x_2} + v_3 \frac{\partial \rho}{\partial x_3} = 0$$

Note, for incompressible material

$$\text{div}\vec{V} = 0$$

or

$$\frac{\partial v_1}{\partial x_1} + \frac{\partial v_2}{\partial x_2} + \frac{\partial v_3}{\partial x_3} = 0$$

Next: 3.16 Compatibility Conditions
3.16 Compatibility conditions

Recall \( \overline{u} \equiv (u_1, u_2, u_3) \) has the three displacement components, but \( \overline{E} \) strain tensor has six components \( E_{11}, E_{22}, E_{33}, E_{12}, E_{13}, E_{23} \). If \( \overline{u} \) is known and continuous \( \overline{E} \) exists and is unique. However, given a set of \( \overline{E} \), displacement field \( \overline{u} \) need not exist, reason being that there are 6 equations and 3 unknowns.

Consider \( E_{11} = a_2 \) all other \( E_{ij} = 0 \)

\[
\frac{\partial u_1}{\partial a_1} = a_2^2
\]

\( u_1 = a_1 a_2 + f(a_2, a_3) \)

\( E_{22} = 0 \Rightarrow u_2 = g(a_1, a_3) \)

\( E_{23} = 0 \Rightarrow \frac{\partial u_1}{\partial a_2} + \frac{\partial u_2}{\partial a_1} = 0 \)

\( 2a_1 a_2 + \frac{\partial f(a_2, a_3)}{\partial a_2} + \frac{\partial g(a_1, a_3)}{\partial a_1} = 0 \)

\( \Rightarrow f_1(a_3) + g(a_3) \neq 0 \)
Hence such a displacement field cannot exist
There are thus for $E_{ij} \left( a_1, a_2, a_3 \right)$ are continuous with continuous second partial derivative, for them to have single valued continuous solutions $u_1, u_2$ and $u_3$. The six equations are
3.16 Compatibility Conditions

Recall \( \vec{u} \equiv (u_1, u_2, u_3) \) has the three displacement components, but \( \tilde{E} \) strain tensor has six components \( E_{11}, E_{22}, E_{33}, E_{12}, E_{13}, E_{23} \). If \( \vec{u} \) is known and continuous \( \tilde{E} \) exists and is unique. However, given a set of \( \tilde{E} \), displacement field \( \vec{u} \) need not exist, reason being that there are 6 equations and 3 unknowns.

Consider \( E_{11} = a_2^2 \) all other \( E_{ij} = 0 \)
\[
\frac{\partial u_1}{\partial a_1} = a_2^2
\]

\[u_1 = a_1a_2^2 + f(a_2, a_3)\]

\[E_{22} = 0 \Rightarrow u_2 = g(a_1, a_3)\]

\[E_{23} = 0 \Rightarrow \frac{\partial u_1}{\partial a_2} + \frac{\partial u_2}{\partial a_1} = 0\]

\[2a_1a_2 + \frac{\partial f(a_2, a_3)}{\partial a_2} + \frac{\partial g(a_1, a_3)}{\partial a_1} = 0\]

\[\Rightarrow f_1(a_3) + g(a_3) \neq 0\]

Hence such a displacement field cannot exist

Next: 3.16 (b) Continued
There are thus for $E_{ij}(a_1, a_2, a_3)$ are continuous with continuous second partial derivative, for them to have single valued continuous solutions $u_1$, $u_2$, and $u_3$. The six equations are:

$$
\begin{align*}
\frac{\partial^2 E_{11}}{\partial a_2^2} + \frac{\partial^2 E_{22}}{\partial a_1^2} &= 2 \frac{\partial^2 E_{12}}{\partial a_1 \partial a_2} \\
\frac{\partial^2 E_{33}}{\partial a_2^2} + \frac{\partial^2 E_{11}}{\partial a_3^2} &= 2 \frac{\partial^2 E_{31}}{\partial a_3 \partial a_1} \\
\frac{\partial^2 E_{33}}{\partial a_2^2} + \frac{\partial^2 E_{22}}{\partial a_3^2} &= 2 \frac{\partial^2 E_{23}}{\partial a_2 \partial a_3} \\
\frac{\partial^2 E_{11}}{\partial a_2 \partial a_3} &= \frac{\partial}{\partial a_1} \left( -\frac{\partial E_{23}}{\partial a_1} + \frac{\partial E_{31}}{\partial a_2} + \frac{\partial E_{12}}{\partial a_3} \right) \\
\frac{\partial^2 E_{22}}{\partial a_3 \partial a_1} &= \frac{\partial}{\partial a_2} \left( -\frac{\partial E_{31}}{\partial a_2} + \frac{\partial E_{12}}{\partial a_3} + \frac{\partial E_{23}}{\partial a_1} \right) \\
\frac{\partial^2 E_{33}}{\partial a_1 \partial a_2} &= \frac{\partial}{\partial a_3} \left( -\frac{\partial E_{12}}{\partial a_3} + \frac{\partial E_{23}}{\partial a_1} + \frac{\partial E_{31}}{\partial a_2} \right)
\end{align*}
$$

These equations are called compatibility equations.

Note:

1) If $\bar{u}$ field is given, compatibility is automatically satisfied.

2) If $E_{ij}$ are linear in $a_1$, $a_2$, and $a_3$, then all second partial derivative vanish and compatibility is satisfied.
Problem 3.56

Previous: 3.16 (b) Compatibility Conditions

\[
[E] = k \begin{bmatrix}
    a_1^2 & a_2^2 + a_3^2 & a_1 a_3 \\
    a_2^2 + a_3^2 & 0 & a_1 \\
    a_1 a_3 & a_1 & a_2^2
\end{bmatrix}
\]

\[
\frac{\partial^2 E_{11}}{\partial a_2^2} + \frac{\partial^2 E_{22}}{\partial a_1^2} = 0 + 0 = 0
\]

\[
\frac{\partial^2 E_{22}}{\partial a_3^2} + \frac{\partial^2 E_{33}}{\partial a_2^2} = 0 + 2 \neq 0
\]

Not satisfied, Hence incompatible field.

The same problem arises when rate of deformation \( D_{ij} \) is specified in fluids problems. However, when velocity field \( v_1, v_2, v_3 \) are specified, compatibility is automatically satisfied.

For strain-rate related problems, when using compatibility equations, replace \( \bar{v} \) by \( \bar{u} \) and \( \bar{E} \) by \( \bar{D} \)

Next: 3.18 Deformation Gradient
In spatial coordinates \( \mathbf{x} = \mathbf{x}(\mathbf{a}, t) \)

\[
d\mathbf{x} = \mathbf{x}(\mathbf{a} + d\mathbf{a}, t) - \mathbf{x}(\mathbf{a}, t) = (\nabla \mathbf{x}) d\mathbf{a}
\]

Thus \( d\mathbf{x} = \tilde{F} d\mathbf{a} \)

\( \tilde{F} \) is the deformation gradient at \( \mathbf{a} \).

In indicial notations, \( dx_i = F_{ij} da_j \)

\( x_i = a_i + u_i \)

Since \( dx_i = \left( \delta_{ij} + u_{i,j} \right) da_j \)

Thus \( F_{ij} = u_{i,j} + \delta_{ij} \) or \( \tilde{F} = \nabla \mathbf{u} + \mathbf{I} \)
Example 3.6.1

Previous: 3.18 Deformation Gradient

\[
\begin{align*}
   x_1 &= 3a_3 \\
   x_2 &= -a_1 \\
   x_3 &= -2a_2
\end{align*}
\]

\[
\tilde{F} = \left[ F_{ij} \right] =
\begin{bmatrix}
   \frac{\partial x_1}{\partial a_1} & \frac{\partial x_1}{\partial a_2} & \frac{\partial x_1}{\partial a_3} \\
   \frac{\partial x_2}{\partial a_1} & \frac{\partial x_2}{\partial a_2} & \frac{\partial x_2}{\partial a_3} \\
   \frac{\partial x_3}{\partial a_1} & \frac{\partial x_3}{\partial a_2} & \frac{\partial x_3}{\partial a_3}
\end{bmatrix}
\]

=\begin{bmatrix}
   0 & 0 & 3 \\
   -1 & 0 & 0 \\
   0 & -2 & 0
\end{bmatrix}

If \( \tilde{F}\tilde{F}^T = I \), then \( \tilde{F} \) is a rotation tensor \( \tilde{I} \). Then at that location, rigid body motion occurs. Recall that \( \tilde{F}(a_1, a_2, a_3) \), i.e., it can vary from point to point.

Next: 3.20 Finite Deformation
3.20 Finite Deformation

Recall that $d\bar{x} = \tilde{F}d\bar{a}$ or $dx_j = F_{ij}da_j$, where $da_j$ is the original vector between two points, and $d\bar{x}$ the vector after deformation. If $\tilde{F}$ is a pure $R$, then $d\bar{x}$ and $d\bar{a}$ have same length, i.e., no strain but rigid body motion. It is likely that in a body some parts be straining and in others put rotation but no strain.

If $\tilde{F}$ is symmetric, let $\tilde{F}$ be designated as $\tilde{U}$, then

$$d\bar{x} = \tilde{U}d\bar{a}$$

In the neighborhood of $d\bar{a}$, there is only pure stretching. Since $\tilde{U}$ is real and symmetric, $\tilde{U}$ can be made diagonal with specific $\hat{n}_1, \hat{n}_2, \hat{n}_3$ corresponding to $\lambda_1, \lambda_2, \lambda_3$, \ldots
Let stretch \( \lambda = \frac{|\mathbf{d}x|}{|\mathbf{d}a|} \)

Thus eigen values of \( \tilde{U} \) are the principal stretches (minimum, intermediate, maximum)

Next: 3.20 Example Problem
3.20 Example Problem
Previous: 3.20 Finite Deformation

\[ x_1 = 3a_1, \ x_2 = 4a_2, \ x_3 = a_3 \]

\[
\begin{bmatrix}
3 & 0 & 0 \\
0 & 4 & 0 \\
0 & 0 & 1
\end{bmatrix}
\]

\[ \lambda = 2.41 \]

\[ \lambda' = \frac{5}{1.414} \]

Rotation of OB to OB'
3.21 Polar Decomposition Theorem
Previous: 3.20 Example Problem

Let \( \mathbf{F} \) Deformation gradient tensor

If \( \mathbf{F} \) is proper orthogonal, then \( \mathbf{F} = \mathbf{R} \), rigid body displacements.
If \( \mathbf{F} \) is symmetric, then \( \mathbf{F} = \mathbf{\tilde{U}} \), pure stretch deformation tensor.
If \( \mathbf{F} \) is real with \( \det \mathbf{F} \neq 0 \), i.e., \( \mathbf{F}^{-1} \) exists,
then, \( \mathbf{F} = \mathbf{R} \cdot \mathbf{\tilde{U}} \) or \( \mathbf{F} = \mathbf{\tilde{V}} \cdot \mathbf{\tilde{R}} \)
\( \mathbf{\tilde{U}} \) and \( \mathbf{\tilde{V}} \) are positive definite symmetric tensors.
\( \mathbf{\tilde{R}} \) is a proper orthogonal \( \iff \) polar decomposition.
Therefore, \( \mathbf{\tilde{R}} \), \( \mathbf{\tilde{U}} \), \( \mathbf{\tilde{V}} \) are unique.

\[
d\mathbf{x} = \mathbf{\tilde{R}}d\mathbf{x}' = \mathbf{\tilde{R}}\mathbf{\tilde{U}}d\mathbf{\tilde{a}} = \mathbf{\tilde{F}}d\mathbf{\tilde{a}}
\]
We see that \( d\mathbf{x} = \mathbf{\tilde{F}}d\mathbf{\tilde{a}} = \mathbf{\tilde{R}}\mathbf{\tilde{U}}d\mathbf{\tilde{a}} \)

Since \( \mathbf{\tilde{F}} = \mathbf{\tilde{R}}\mathbf{\tilde{U}} = \mathbf{\tilde{V}}\mathbf{\tilde{R}} \)

\[
\mathbf{\tilde{R}}^T \mathbf{\tilde{R}} \mathbf{\tilde{U}} = \mathbf{\tilde{R}}^T \mathbf{\tilde{V}} \mathbf{\tilde{R}} \quad \text{or} \quad \mathbf{\tilde{U}} = \mathbf{\tilde{R}}^T \mathbf{\tilde{V}} \mathbf{\tilde{R}}
\]

\[
d\mathbf{x} = Rd\mathbf{x}' = \mathbf{\tilde{R}}\mathbf{\tilde{U}}d\mathbf{\tilde{a}} = \mathbf{\tilde{F}}d\mathbf{\tilde{a}}
\]

Next: 3.22 Stretch Tensors
3.22 Stretch Tensors

Previous: 3.21 Polar Decomposition Theorem

\[ \tilde{F} = \tilde{R} \tilde{U} \], Dropping (~)

a) \[ F^T F = (RU)^T RU = U^T R^T RU = U^T U = U^2 \]

Note that U is symmetric, i.e., \( U = U^T \)

\[ U = \sqrt{F^T F} \]

b) Since \( F = RU \), \( R = FU^{-1} \)

c) \( F = VR \), \( V = FR^T = RUR^T \)

Next: 3.25 Left Cauchy-Green Deformation Tensor
3.23 Right Cauchy-Green Deformation Tensor

We know that $F = \frac{dx}{d\bar{a}}$ provides the deformation gradient. One of the strain measures was infinitesimal strain $\tilde{E}$, stretch tensor $\lambda_i, i = 1, 2, 3$. We have a new measure $C$ called the Green's deformation tensor. We know that $\tilde{U}$ is the right stretch tensor. Define

$$\tilde{C} = \tilde{U}^2 = \tilde{F}^T \tilde{F}$$

Note that if $\tilde{F}$ were to be a pure rotation tensor (proper orthogonal), then $\tilde{C} = \tilde{I}$

Next: 3.22 (b) Components of $\tilde{C}$: (1. Diagonal Element)
3.23 (b) Components of $\tilde{C}$: (1. Diagonal Element)

Consider two differential elements $dx^{(1)}$ and $dx^{(2)}$ deformed from $da^{(1)}$ and $da^{(2)}$.

Since

$$dx^{(1)} dx^{(2)} = \tilde{F} \ da^{(1)} \cdot \tilde{F} \ da^{(1)} = da^{(1)} \tilde{F}^T \ F \ da^{(2)}$$

$$= da^{(1)} \ C \ da^{(2)}$$

For example, if the original element is $d\vec{a} = dS \ \hat{e_i}$ deformed to $d\vec{x} = ds \ \hat{n}$ then

$$\bigg( ds \bigg)^2 = \bigg( dS \bigg)^2 \ \hat{e_i} \tilde{C} \hat{e_i}$$

Thus $C_{ii} = \left( \frac{ds}{dS} \right)^2$ represents the ratio of the square of the deformed element with respect to an element originally lying along the x-axis.

Next: 3.23 (c) Components of $\tilde{C}$: (2. Off-Diagonal Element)
3.23 (c) Components of $\tilde{C}$: (2. Off-Diagonal Element)

Previous: 3.23 (b) Components of $\tilde{C}$: (1. Diagonal Element)

If $da^{(1)} = dS_1 \hat{e}_1$ and $da^{(2)} = dS_2 \hat{e}_2$ which deform into $dx^{(1)} = ds_1 \hat{m}$ and $dx^{(2)} = ds_2 \hat{n}$, then

$$ds_1 ds_2 \cos \beta = dS_1 dS_2 \hat{e}_1 \tilde{C} \hat{e}_2$$

which leads to

$$C_{12} = \frac{ds_1 ds_2}{dS_1 dS_2} \cos(dx^{(1)}, dx^{(2)})$$

Similar expressions can be derived for $C_{23}$ and $C_{31}$.

Next: 3.24 Lagrange Strain Tensor
3.24 **Lagrange Strain Tensor**

Previous: 3.23 (c) Components of $\tilde{C}$: (2. Off-Diagonal Element)
This is one of the most important of the finite strain measures. This tensor $\tilde{E}^*$ needs to be used whenever the strain is not infinitesimal and when there is rigid body rotation during the deformation. As defined in the previous section, if $\tilde{C}$ is the Green deformation tensor then,

$$\tilde{E}^* = \frac{1}{2}(\tilde{C} - \tilde{I})$$

where $\tilde{E}^*$ is the Lagrangian Finite Strain tensor. If there is no deformation (or rigid body rotation), then $\tilde{C} = \tilde{I}$ leading to $\tilde{E}^* = 0$.

Note that $\tilde{E}^*$ can also be expressed as follows:

$$\tilde{E}^* = \frac{1}{2} \left[ \nabla \bar{u} + \left( \nabla \bar{u} \right)^T \right] + \frac{1}{2} \left( \nabla \bar{u} \right)^T \nabla \bar{u}$$

$$E_{ij} = \frac{1}{2} \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) + \frac{1}{2} \frac{\partial u_k}{\partial x_i} \frac{\partial u_k}{\partial x_j}$$

Next: 3.24 (b) Components of $\tilde{E}$ (1. Diagonal Element)
Consider two differential elements $dx^{(1)}$ and $dx^{(2)}$ deformed from $da^{(1)}$ and $da^{(2)}$. Since

$$dx^{(1)}dx^{(2)} - da^{(1)}da^{(2)} = da^{(1)} \cdot (\tilde{C} - I) da^{(1)}$$

$$= 2 \, da^{(1)} \, \tilde{E}^* \, da^{(2)}$$

For example, if the original element is $d\vec{a} = dS \, \hat{e}_1$ deformed to $d\vec{x} = ds \, \hat{n}$ then

$$e_1 \tilde{E}^* e_2 = \frac{(dS)^2 - (ds)^2}{2ds^2}$$

Thus $E_{11} = \frac{ds^2 - dS^2}{ds^2}$ represents the ratio of the difference in the squared length to that of original length, when the element originally is aligned along the x-axis. Similar interpretation can be made for other components $E_{22}$ and $E_{33}$.

Next: 3.24 (c) Components of $\tilde{E}$ (2. Off-Diagonal Element)
3.24 (c) **Components of \( \tilde{E} \) (2. Off-Diagonal Element)**

Previous: 3.24 (b) Components of \( \tilde{E} \) (1. Diagonal Element)

If \( da^{(1)} = dS_1 \hat{e}_1 \) and \( da^{(2)} = dS_2 \hat{e}_2 \) which deform into \( dx^{(1)} = ds_1 \hat{m} \) and \( dx^{(2)} = ds_2 \hat{n} \), then

\[
2E_{12} = \frac{ds_1 ds_2}{dS_1 dS_2} \cos(\hat{n}, \hat{m})
\]

Similar expressions can be derived for \( E_{23} \) and \( E_{31} \).

Next: 3.25 Left Cauchy-Green Deformation Tensor
3.25 Left Cauchy-Green Deformation Tensor

We know that

\[ F = RV = VR, \quad V \text{ is left stretch tensor} \]

Define \( B \equiv V^2 \)

B is left Green's tensor or Finger deformation tensor.

Note \( F = VR \)

\[ FF^T = VR \cdot R^T \cdot V^T = VV^T = V^2 \quad \text{since } V \text{ is symmetric} \]

Thus \( B = FF^T \quad \text{Note} \quad C = FF^T \)

Relationship between \( B \) and \( C \)

\[ F = RU \quad \text{or} \quad FF^T = R \cdot U \cdot U^T R^T = RU^2 T = RCT^T \]

Thus \[ B = RCR^T \]

Again, \( R^T \cdot B \cdot R = R^T R \cdot CR^T \cdot R = C \)

or \[ C = R^T BR \]

Note that if \( \hat{n} \) is eigenvector of \( \tilde{C} \) with eigenvalue \( \lambda \), then \( R\hat{n} \) is eigenvector of \( \tilde{B} \) with the same eigenvalue \( \lambda \)

Similarly, relationship exists between \( \tilde{U} \) and \( \tilde{V} \), i.e., \( \tilde{B} \) and \( \tilde{V} \) are obtained by pure rotation of \( \tilde{C} \) and \( \tilde{U} \).

Next: 3.25 (b) Meaning of \( \tilde{B} \)
3.25 (b) Meaning of $\tilde{B}$

Previous: 3.25 Left Cauchy-Green Deformation Tensor

Consider $d\bar{a} = da \cdot \hat{n}$ and $\hat{n} = R^T \cdot \hat{e}_1$

Where $\tilde{R}$ is rotation associated with $\tilde{F}, \left( \tilde{F} = \tilde{R} \tilde{U} = \tilde{V} \tilde{R} \right)$

$$ds^2 = da^2 \cdot \hat{n} \cdot \tilde{C} \hat{n}$$
$$= da^2 \cdot R^T \hat{e}_1 \cdot \tilde{C} \cdot R^T \hat{e}_1$$
$$= da^2 \cdot \hat{e}_1 \cdot \left(CR^T\right)^T \cdot R^T \cdot \hat{e}_1$$
$$= da^2 \cdot \hat{e}_1 \cdot R^T CR^T \cdot \hat{e}_1$$
$$= da^2 \cdot \hat{e}_1 \cdot B \cdot \hat{e}_1$$

$$ds^2 = da^2 \cdot \hat{e}_1 \cdot B \cdot \hat{e}_1 \quad \text{with} \quad d\bar{a} = da \left(R^T \hat{e}_1\right)$$

$$B_{11} = \left(\frac{ds}{da}\right)^2 \quad \text{for element} \quad d\bar{a} = da \left(R^T \hat{e}_1\right)$$

In component form,

$$B_{ij} = \delta_{ij} + \left(\frac{\partial u_i}{\partial a_j} + \frac{\partial u_j}{\partial a_i}\right) + \left(\frac{\partial u_i}{\partial a_m} \frac{\partial u_j}{\partial a_m}\right)$$

Next: 3.26 Eulerian Strain Tensor
3.26 Eulerian Strain Tensor

Previous: 3.25 (b) Meaning of $\tilde{B}$

$$\tilde{e}^* \equiv \frac{1}{2}(\tilde{I} - \tilde{B}^{-1})$$

We note the $\tilde{e}^*$ is defined with respect to current coordinates, $x_1, x_2, x_3$ and not $a_1, a_2, a_3$

Note $\delta x = \tilde{F}\delta a$ or $\delta a = F^{-1}\delta x$

$$da_i = F_{ij}^{-1}dx_j \Rightarrow F_{ij}^{-1} = \frac{\partial a_i}{\partial x_j}$$

Note that $x_i = x_i(a_1, a_2, a_3, t) \Rightarrow F_{ij} = \frac{\partial x_i}{\partial a_j}$

$$a_i = a_i(x_1, x_2, x_3, t) \Rightarrow F_{ij}^{-1} = \frac{\partial a_i}{\partial x_j}$$

$$\begin{bmatrix}
\frac{\partial a_1}{\partial x_1} & \frac{\partial a_1}{\partial x_2} & \frac{\partial a_1}{\partial x_3} \\
\frac{\partial a_2}{\partial x_1} & \frac{\partial a_2}{\partial x_2} & \frac{\partial a_2}{\partial x_3} \\
\frac{\partial a_3}{\partial x_1} & \frac{\partial a_3}{\partial x_2} & \frac{\partial a_3}{\partial x_3}
\end{bmatrix}^{-1}$$

Also

$$\delta a^{(1)}\delta a^{(2)} = \delta x^{(1)} \cdot \tilde{B}^{-1} \delta x^{(2)}$$

Note that

$$\delta x^{(1)}\delta x^{(2)} = \delta a^{(1)}\tilde{C}\delta a^{(2)}$$

Since $B_{11}^{-1} = \frac{d\alpha^2}{ds^2}$, $e_{11}^* = \frac{d\alpha^2 - d\alpha^2}{ds^2}$

Definitions $e^* = 1_2(\nabla_x \bar{u} + \nabla_x \bar{u}^T) - \frac{\nabla_x \bar{u}^T \nabla_x \bar{u}}{2}$

In terms of indicial rotations,
\[ e_{ij}^* = \frac{1}{2} \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) - \frac{1}{2} \left( \frac{\partial u_k}{\partial x_i} \frac{\partial u_k}{\partial x_j} \right) \]

\[ e_{11}^* = \frac{\partial u_1}{\partial x_1} - \frac{1}{2} \left[ \left( \frac{\partial u_1}{\partial x_1} \right)^2 + \left( \frac{\partial u_2}{\partial x_1} \right)^2 + \left( \frac{\partial u_3}{\partial x_1} \right)^2 \right] \]

\[ e_{12}^* = \frac{1}{2} \left( \frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1} \right) - \frac{1}{2} \left[ \left( \frac{\partial u_1}{\partial x_1} \right) \left( \frac{\partial u_1}{\partial x_2} \right) + \left( \frac{\partial u_2}{\partial x_1} \right) \left( \frac{\partial u_2}{\partial x_2} \right) + \left( \frac{\partial u_3}{\partial x_1} \right) \left( \frac{\partial u_3}{\partial x_2} \right) \right] \]

Next: 3.28 Change of Area due to Deformation
3.28 Change of Area due to Deformation

Previous: 3.26 Eulerian Strain Tensor

Let
\[ d\vec{a}^{(1)} = da_1 \hat{e}_1 \]
\[ d\vec{a}^{(2)} = da_2 \hat{e}_2 \] at \( \vec{a} \) at time \( t_0 \)

The area formed by and \( d\vec{a}^{(2)} \), it is given by,
\[ dA_0 = d\vec{a}^{(1)} \times d\vec{a}^{(2)} = da_1 da_2 \hat{e}_3 = dA_0 \hat{e}_3 \]
d\( A_0 \) … undeformed area

Deformed area \( d\vec{A} = Fd\vec{a}^{(1)} \times Fd\vec{a}^{(2)} \)
\[ = da_1 da_2 \cdot \vec{F}\hat{e}_1 \times \vec{F}\hat{e}_2 \]
\[ = dA_0 \vec{F}\hat{e}_1 \times \vec{F}\hat{e}_2 \]

Let \( d\vec{A} = dA \hat{n} \) where \( \hat{n} \) is the unit outward normal to \( dA \).
Thus
\[ dA \hat{n} = dA_0 \cdot (\vec{F}\hat{e}_1 \times \vec{F}\hat{e}_2) \]
\[ \vec{F}\hat{e}_1 \cdot dA \cdot \hat{n} = \vec{F}\hat{e}_2 \cdot dA \cdot \hat{n} = 0 \]
\[ \vec{F}\hat{e}_3 \cdot dA = dA_0 \cdot (\vec{F}\hat{e}_3 \cdot \vec{F}\hat{e}_1 \times \vec{F}\hat{e}_2) \]

Since \( \vec{F}\hat{e}_1, \vec{F}\hat{e}_2 \) and \( \hat{n} \) are mutually perpendicular.
\( \vec{a} \cdot \vec{b} \times \vec{c} \) is the determinant of rows with vectors \( \vec{a}, \vec{b}, \vec{c} \).
Thus
\[ \vec{F}\hat{e}_3 \cdot \vec{F}\hat{e}_1 \times \vec{F}\hat{e}_2 = \det \vec{F} \]
Thus
\[ \vec{F}\hat{e}_3 \cdot dA \cdot \hat{n} = dA_0 \det F \] (B)
\[ \hat{e}_1 \cdot \vec{F}^T \cdot \hat{n} = \hat{e}_2 \cdot \vec{F}^T \cdot \hat{n} = 0 \] (perpendicular to each other)
Or
\[ \hat{e}_3 \cdot \tilde{F} \cdot \hat{n} = \left( \frac{dA_0}{dA} \right) \det \tilde{F} \Rightarrow \text{scalar} \]

Thus \( \tilde{F}^T \hat{n} \) and \( \hat{e}_3 \) are in the same direction, or

\[ \tilde{F}^T \hat{n} = \left( \frac{dA_0}{dA} \right) \det \tilde{F} \cdot \hat{e}_3 \]

\[ \therefore dA\hat{n} = \left( \frac{dA_0}{dA} \right) \cdot (\det \tilde{F}) \left( F^{-1} \right)^T \hat{e}_3 \]

In general

\[ dA\hat{n} = dA_0 \cdot (\det \tilde{F}) \left( F^{-1} \right)^T \hat{n}_0 \]

\( dA_0 \) is the magnitude of initial area with normal \( \hat{n}_0 \)
\( \tilde{F} \) is the deformation gradient
\( dA \) and \( \hat{n} \) are the magnitudes and direction in terms of original configuration.

Next: 3.29 Volume Change due to Deformation
3.29 Volume Change due to Deformation

Previous: 3.28 Change of Area due to Deformation

At the location, \( \mathbf{a} \), let

\[
\begin{align*}
\mathbf{da}^{(1)} &= \mathbf{da}_1 \hat{e}_1 \\
\mathbf{da}^{(2)} &= \mathbf{da}_2 \hat{e}_2 \\
\mathbf{da}^{(3)} &= \mathbf{da}_3 \hat{e}_3
\end{align*}
\]

At time \( t_0 \)

\[
\mathbf{dV}_0 = \mathbf{da}_1 \mathbf{da}_2 \mathbf{da}_3
\]

\[
\mathbf{dV} = \tilde{\mathbf{F}} \mathbf{da}^{(1)} \cdot \tilde{\mathbf{F}} \mathbf{da}^{(2)} \cdot \tilde{\mathbf{F}} \mathbf{da}^{(3)}
\]

\[
= \mathbf{da}^{(1)} \cdot \mathbf{da}^{(2)} \cdot \mathbf{da}^{(3)} \cdot \left( \tilde{\mathbf{F}} \hat{e}_1 \cdot \tilde{\mathbf{F}} \hat{e}_2 \times \tilde{\mathbf{F}} \hat{e}_3 \right)
\]

\[
\mathbf{dV} = \det \tilde{\mathbf{F}} \cdot \mathbf{dV}_0
\]

Thus

\[
\frac{\mathbf{dV}}{\mathbf{dV}_0} = \det \tilde{\mathbf{F}}
\]

Recall

\[
\tilde{\mathbf{C}} = \mathbf{F}^T \mathbf{F} \quad \text{det} \mathbf{C} = \text{det} \mathbf{B}
\]

\[
\tilde{\mathbf{B}} = \mathbf{FF}^T = \left( \text{det} \mathbf{F} \right)^2
\]

\[
\mathbf{dV} = \sqrt{\text{det} \mathbf{C}} \ \mathbf{dV}_0
\]

or

\[
= \sqrt{\text{det} \mathbf{B}} \ \mathbf{dV}_0
\]

\[
\rho_o \mathbf{dV}_0 = \rho \mathbf{dV}
\]

In general,

\[
\frac{\mathbf{dV}}{\mathbf{dV}_0} = \det \mathbf{F}
\]

\[
\rho_o = \rho \det \mathbf{F}
\]

For incompressible materials

\[
\det \mathbf{F} = \det \mathbf{B} = \det \mathbf{C} = 1 \quad \text{or} \quad \rho = \rho_o
\]

Next: Example 3.61
Example 3.61 (page 170)

Given $x_1 = 3a_3$, $x_2 = -a_1$, $x_3 = -2a_2$

(a) $[F] = \begin{bmatrix} 0 & 0 & 3 \\ -1 & 0 & 0 \\ 0 & -2 & 0 \end{bmatrix}$

(b) $\tilde{C} = [F^T][F] = \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & -2 \\ 3 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 3 \\ -1 & 0 & 0 \\ 0 & -2 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 9 \end{bmatrix}$

(c) $\tilde{B} = [F][F]^T = \begin{bmatrix} 0 & 0 & 3 \\ -1 & 0 & 0 \\ 0 & -2 & 0 \end{bmatrix} \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & -2 \\ 3 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 9 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 4 \end{bmatrix}$

(d) $\sqrt{F^T F} = \sqrt{C} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$, $U^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{3} \end{bmatrix}$

(e) $[R] = [F][U^{-1}] = \begin{bmatrix} 0 & 0 & 3 \\ -1 & 0 & 0 \\ 0 & -2 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{3} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix}$

(f) $[E^*] = \frac{1}{2}[\tilde{C} - \tilde{I}] = \frac{1}{2}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 6 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \frac{3}{2} & 0 \\ 0 & 0 & 2 \end{bmatrix}$

(g) $[e^*] = \frac{1}{2}[\tilde{I} - B^{-1}] = \frac{1}{2}\begin{bmatrix} 1 & -\frac{1}{9} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 - \frac{1}{4} \end{bmatrix} = \begin{bmatrix} \frac{4}{9} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \frac{3}{8} \end{bmatrix}$
(h) \( \frac{\Delta V}{\Delta V_0} = \sqrt{\det B} = \sqrt{9 \cdot 1.4} = 6 \)

(i) \( \tilde{dA} = dA_0 \cdot (\det \tilde{F}) \cdot \tilde{F}^{-1} \cdot \hat{n}_0 \)

\[
dA_0 = 1 \\
\det \tilde{F} = 6 \\
\hat{n}_0 = \hat{e}_2 \\
F^{-1} = \frac{1}{6} \begin{bmatrix}
0 & -6 & 0 \\
0 & 0 & -3 \\
2 & 0 & 0
\end{bmatrix}
\]

\[
\begin{bmatrix}
0 & -6 & 0 \\
0 & 0 & -3 \\
2 & 0 & 0
\end{bmatrix} \begin{bmatrix}
-6 \\
0 \\
0
\end{bmatrix} \Rightarrow dA = -6\hat{e}_1
4 Stress

Body Forces and Surface Forces

Force/moments determine the kinetics of motion. Forces may be classified as external forces acting external to the body or internal forces acting between two parts of the same body.

External Forces

Body forces acting based on volume or mass.
- e.g. magnetic per unit volume
- gravitational per unit mass

Surface Forces
- e.g. At free surfaces, concentrated distributed pressure reaction
- Contact forces between two bodies

Internal Forces

Inertial forces
- Friction; forces set to be in dynamic equilibrium
- Can be considered as body forces in continuum mechanics.

Concept of Stress
Body B is subjected to forces $F$. Section $\Delta A$ has a unit normal $\hat{n}$, $\Delta F$ is the force acting on $\Delta A$. Then stress vector $\bar{t}_n$ is defined as

$$\bar{t}_n = \lim_{\Delta A \to 0} \frac{\Delta F}{\Delta A}$$

This Cauchy stress vector $\bar{t}_n$ is the function of location $\bar{x}_i$, unit normal $\hat{n}$ to the surface $\Delta S$ at $\hat{n}$ and time $t$

$$\bar{t} = \bar{t} (\bar{x}, t, \hat{n})$$
Let $\tilde{T}$ be a tensor at point $P$. The three faces have unit normal $(1,0,0),(0,1,0)$ and $(0,1,0)$ with unit area.

Let $\vec{t}_{n_1}$ be the stress vector for $\hat{n}(1,0,0)$, and
$\vec{t}_{n_2}$ be the stress vector for $\hat{n}(0,1,0)$, and
$\vec{t}_{n_3}$ be the stress vector for $\hat{n}(0,0,1)$, then
\[ \mathbf{t}_n = t_1 \mathbf{e}_1 + t_2 \mathbf{e}_2 + t_3 \mathbf{e}_3 \]

\[ t_1 = T_{11} \quad \mathbf{t}_n = \mathbf{T} \cdot \mathbf{n}_1 \]

and \( t_2 = T_{12} \) or \( \mathbf{t}_{n_2} = \mathbf{T} \cdot \mathbf{n}_2 \)

\[ t_3 = T_{13} \quad \mathbf{t}_{n_3} = \mathbf{T} \cdot \mathbf{n}_3 \]

Thus \( \mathbf{t}_n = \mathbf{T} \cdot \mathbf{n} \)

\( \mathbf{T} \) is the stress tensor with the first subscript in “\( T_{ij} \)”, “\( i \)” indicates the plane and “\( j \)” indicates the direction.
4.3 Components of Stress Tensor

If $\hat{e}_1, \hat{e}_2, \hat{e}_3$ are three unit vectors representing three planes at a point, then

$$\bar{t}_{e_i} = \tilde{T}\hat{e}_i$$

or

$$\bar{t}_{e_1} = \tilde{T}\hat{e}_1$$ $$\bar{t}_{e_2} = \tilde{T}\hat{e}_2$$ $$\bar{t}_{e_3} = \tilde{T}\hat{e}_3$$


At a point P, let $\hat{n}$ be the unit normal, then $\bar{t}_n$

Let $\hat{n} = \hat{e}_1$, then
\[ \bar{\tau} = T_{11}\hat{e}_1 + T_{21}\hat{e}_2 + T_{31}\hat{e}_3 \]

\( T_{11} \) is the normal stress component. Then

\[ \tau = \text{shear stress} = T_{21}\hat{e}_2 + T_{31}\hat{e}_3 \]

\[ |\tau_1| = \sqrt{T_{21}^2 + T_{31}^2} \]

\[ \bar{\tau} = \bar{T}\hat{n} \quad \text{or} \quad t_i = T_{ij}n_j \quad \text{or} \quad [t] = [T][n] \]
Taking moments about $A$

$$\sum M_A = 0 = T_{12} \cdot \left(\frac{\Delta x_1}{2}\right) \cdot \Delta x_2 \Delta x_3 + (T_{12} + \Delta T_{12}) \cdot \left(\frac{\Delta x_1}{2}\right) \cdot \Delta x_2 \Delta x_3$$

$$-T_{21} \cdot \left(\frac{\Delta x_2}{2}\right) \cdot \Delta x_1 \Delta x_3 - (T_{21} + \Delta T_{21}) \cdot \left(\frac{\Delta x_2}{2}\right) \cdot \Delta x_1 \Delta x_3 = 0$$

Dividing by $\Delta x_1 \Delta x_2 \Delta x_3$

$$T_{21} = T_{12}$$

Similarly, $T_{23} = T_{32}$ and $T_{31} = T_{13}$

Thus, $\tilde{T}$ is symmetric. Thus $\tilde{T}$ has 6 independent components.
4.5 Principal Stresses

Since \([T]\) is a symmetric tensor, the eigenvectors of \(\overline{T}\) give principal directions and eigenvalues give the maximum, intermediate or minimum values. Note that three invariants

\[ I_1, I_2 \text{ and } I_3 \]

\[ I_1 = T_{11} + T_{22} + T_{33} \]

\[ I_2 = \begin{vmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{vmatrix} + \begin{vmatrix} T_{11} & T_{13} \\ T_{31} & T_{33} \end{vmatrix} + \begin{vmatrix} T_{22} & T_{23} \\ T_{32} & T_{33} \end{vmatrix} \]

\[ I_3 = \det[T] \]

Next: 4.6 Maximum Shear Stress
4.6 Maximum Shear Stress

Previous: 4.5 Principal Stresses

\[ \hat{n} = n_1 \hat{e}_1 + n_2 \hat{e}_2 + n_3 \hat{e}_3 \]

Let $\tilde{T}$ be the tensor at $P$. Let $T_1, T_2, T_3$ be the principal stresses, then

\[
\begin{bmatrix}
T_1 & 0 & 0 \\
0 & T_2 & 0 \\
0 & 0 & T_3
\end{bmatrix}
\]

is the tensor $\tilde{T}$ in eigendirections set

\[
\bar{t} = [T] \{\hat{n}\} = \begin{bmatrix}
n_1 & T_1 \\
n_2 & T_2 \\
n_3 & T_3
\end{bmatrix}
\]

\[= n_1 T_1 \hat{e}_1 + n_2 T_2 \hat{e}_2 + n_3 T_3 \hat{e}_3\]

Next: 4.6(b) Maximum Shear Stress (Ct’d)
4.6(b) Maximum Shear Stress (Ct’d)

Normal stress
\[ T_n = \hat{n} \cdot \hat{t} \]
\[ = n_1^2 T_1 + n_2^2 T_2 + n_3^2 T_3 \]

Shear stress
\[ T_s = |t|^2 - T_n^2 \]
\[ = T_1^2 n_1^2 + T_2^2 n_2^2 + T_3^2 n_3^2 - \left( T_1^2 n_1^2 - T_2^2 n_2^2 - T_3^2 n_3^2 \right)^2 \]
\[ = f(n_1, n_2, n_3) \]
But \[ n_1^2 + n_2^2 + n_3^2 = 1 \]

Next: 4.6(c) Maximum Shear Stress (Ct’d)
4.6(c) Maximum Shear Stress (Ct’d)

It can be shown that if $T_1, T_2, T_3$ are the three principal stress quantities at $\hat{e}_1, \hat{e}_2$ and $\hat{e}_3$ respectively, then

At $\hat{n}_1 = \frac{1}{\sqrt{2}}(\hat{e}_1 + \hat{e}_2)$ we have shear $T_{s3} = \left| \frac{T_1 - T_2}{2} \right|$

$\hat{n}_2 = \frac{1}{\sqrt{2}}(\hat{e}_2 + \hat{e}_3)$

$\hat{n}_3 = \frac{1}{\sqrt{2}}(\hat{e}_3 + \hat{e}_1)$

$T_{s1} = \frac{|T_2 - T_3|}{2}$

$T_{s2} = \frac{|T_3 - T_1|}{2}$

If $(T_n)_{\text{max}}$ and $(T_n)_{\text{min}}$ are the maximum and minimum principal stresses, then

$\left( T_s \right)_{\text{max}} = \frac{(T_n)_{\text{max}} - (T_n)_{\text{min}}}{2}$

$\hat{n} = \frac{T_1 + T_2}{2} \left( \hat{n}_1 + \hat{n}_2 \right) \frac{1}{n_1^2 + n_2^2}$

Next: 4.6(d) Maximum Shear Stress (2-D Mohr’s Circle)
2-D Mohr’s Circle

Same stress state at P
Mohr’s circle gives all possible “θ” for \( \hat{n} = \cos \theta \) with x-axis.
\( T_1, T_2 \) …Principle stress (with zero shear stress).

Next: Problem 4.1
Problem 4.1

Previous: 4.6(d) Maximum Shear Stress (2-D Mohr’s Circle)

Given:

\[
\begin{bmatrix}
1 & 2 & 3 \\
2 & 4 & 5 \\
3 & 5 & 0
\end{bmatrix}
\]

\(\mathbf{T}\) at specific point

\(\mathbf{T}\) is written in terms of \(\hat{e}_1, \hat{e}_2,\) and \(\hat{e}_3\)

Find: Normal and shear stress on planes with \(\hat{n} = \hat{e}_1, \hat{e}_2,\) and \(\hat{e}_3\)

(a) Normal stress on \(\hat{e}_1\) plane \(\Rightarrow\) 1 MPa
    \(\hat{e}_2\) plane \(\Rightarrow\) 4 MPa
    \(\hat{e}_3\) plane \(\Rightarrow\) 0 MPa

(b) Shear stress on \(\hat{e}_1\) plane

\[
\tau_{e_1} = \sqrt{2^2 + 3^2} = \sqrt{13} \text{ MPa at } \tan^{-1} \frac{3}{2} = \theta
\]

\(\hat{e}_2\) plane

\[
\tau_{e_2} = \sqrt{5^2 + 2^2} = \sqrt{29} \text{ MPa}
\]

\(\hat{e}_3\) plane

\[
\tau_{e_3} = \sqrt{5^2 + 3^2} = \sqrt{34} \text{ MPa}
\]

Next: Problem 4.3
Problem 4.3

Given \( \mathbf{T} = \begin{bmatrix} 2 & -1 & 3 \\ -1 & 4 & 0 \\ 3 & 0 & -1 \end{bmatrix} \) Mpa

Find:
Stress vector through the point P and parallel to plane
\( x_1 - 2x_2 + 3x_3 = 4 \)

Solution:
Normal to the plane \( \mathbf{N} = \hat{e}_1 - 2\hat{e}_2 + 3\hat{e}_3 \)
Unit normal \( \Rightarrow \frac{1}{\sqrt{14}}(\hat{e}_1 - 2\hat{e}_2 + 3\hat{e}_3) \)
\( \mathbf{t} = \mathbf{T}\mathbf{n} \)

Stress vector \( \mathbf{t} = \frac{1}{\sqrt{14}} \begin{bmatrix} 2 & -1 & 3 \\ -1 & 4 & 0 \\ 3 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix} = \frac{1}{\sqrt{14}}(13\hat{e}_1 - 9\hat{e}_2) \)
Problem 4.8

Given:
\[
\tilde{T} = \begin{bmatrix}
\alpha x_2^2 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}
\]

Find:

a) Stress distribution on x₁=0 plane (y-z plane)

Plot the stress vector distribution on the plane Section PP′

(b)

(c) Find the total force F
\[
\bar{F} = \int_{-1}^{1} \int_{-1}^{1} (\alpha x_2^2) \, dx_2 \, dx_3 \quad \hat{e}_1 = 2 \left( \frac{\alpha x_2^3}{3} \right)_{-1}^{1}
\]
\[
= \frac{4}{3} \alpha \hat{e}_1
\]
\[
\bar{M} = \vec{r} \times \bar{F}
\]
\[
M = \int_{-1}^{1} \int_{-1}^{1} (x_2 \hat{e}_2 + x_3 \hat{e}_3) \times (\alpha x_2^2 \hat{e}_1) \, dx_2 \, dx_3
\]
\[
= 0
\]
Problem 4.12

We define $\tilde{S} =$ Deviatoric Stress Tensor

$$S_{ij} = T_{ij} - \frac{T_{kk}}{3} \delta_{ij}$$

$$S_{11} = T_{11} - \frac{1}{3} (T_{11} + T_{22} + T_{33})$$
$$S_{22} = T_{22} - \frac{1}{3} (T_{11} + T_{22} + T_{33})$$
$$S_{33} = T_{33} - \frac{1}{3} (T_{11} + T_{22} + T_{33})$$

(a) First invariant $S_{11} + S_{22} + S_{33}$

$$S_{11} + S_{22} + S_{33} = 0$$

(b)

$$\tilde{T} = \begin{bmatrix} 60 & 50 & -20 \\ 50 & 80 & 40 \\ -20 & 40 & 90 \end{bmatrix} \text{ MPa } \frac{T_{kk}}{3} = \frac{180}{3} = 60$$

$$\tilde{S} = \begin{bmatrix} 0 & 50 & -20 \\ 50 & -30 & 40 \\ -20 & 40 & 30 \end{bmatrix} \text{ MPa } S_{11} + S_{22} + S_{33} = 0$$

(c) Let $\hat{n}$ be eigenvector of $\tilde{T}$

$$\tilde{T}\hat{n} = \lambda \hat{n}$$

$$\tilde{S}\hat{n} = \left[ \tilde{T} - \left( \frac{1}{3} T_{kk} \right) \tilde{I} \right] \hat{n}$$

$$= (\lambda \hat{n} - \frac{1}{3} T_{kk}) \hat{n}$$

$$= \lambda' \hat{n}$$
Hence, eigenvectors remain the same.

However, the principal values are \( \lambda_i - \frac{1}{3} T_{kk} \)
Problem 4.23

\[
\vec{T} = \begin{bmatrix}
300 & 0 & 0 \\
0 & -200 & 0 \\
0 & 0 & 400
\end{bmatrix} \text{kPa}
\]

(a) Find the shear stress on the plane with \( \vec{N} = 2\hat{e}_1 + 2\hat{e}_2 + \hat{e}_3 \)

\[
\hat{n} = \frac{1}{3} \left( 2\hat{e}_1 + 2\hat{e}_2 + \hat{e}_3 \right)
\]

\[
\bar{t}_n = \frac{1}{3} \begin{bmatrix}
300 & 0 & 0 \\
0 & -200 & 0 \\
0 & 0 & 400
\end{bmatrix} \begin{bmatrix}
2 \\
2 \\
1
\end{bmatrix}
\]

\[
= \frac{100}{3} \begin{bmatrix}
6 \\
-4 \\
4
\end{bmatrix}
\]

\[
T_n = \hat{n} \cdot \bar{t}_n
\]

\[
= \frac{100}{3} \begin{bmatrix}
2 & 2 & -1
\end{bmatrix} \begin{bmatrix}
6 \\
-4 \\
4
\end{bmatrix} = 88.9\text{kPa}
\]

\[
T_s^2 = |\bar{t}_n|^2 - T_n^2
\]

\[
= 67700 \text{ or } T_s = 260\text{kPa}
\]

(b) Maximum Shear stress
\[
(T_s)_{\text{max}} = \frac{400 - (-200)}{2} = 300\text{kPa}
\]

at \[\frac{\hat{e}_2 + \hat{e}_3}{\sqrt{2}}\]
Consider the state of stress given by
\[
\mathbf{T} = \begin{bmatrix}
T_{11} & T_{12} & T_{13} \\
T_{21} & T_{22} & T_{23} \\
T_{31} & T_{32} & T_{33}
\end{bmatrix}_{\{\hat{e}_1, \hat{e}_2, \hat{e}_3\}}
\]

\(\mathbf{T}\) is symmetric. When we determine the principal value, let \(T'^{I}, T'^{II}\) and \(T'^{III}\) be the principal value. Let \(T'^{I} \geq T'^{II} \geq T'^{III}\) be the principal values.

Let \(T'^{I}\) be associated \(n^{(1)} = \{n_{1}^{(1)}, n_{2}^{(1)}, n_{3}^{(1)}\}\),
\[
\hat{n}^{(2)} = \{n_{1}^{(2)}, n_{2}^{(2)}, n_{3}^{(2)}\}
\]
for \(T'^{II}\)
\[
\hat{n}^{(3)} = \{n_{1}^{(3)}, n_{2}^{(3)}, n_{3}^{(3)}\}
\]
for \(T'^{III}\)

Note that \(n, \hat{n}^{(1)}, \hat{n}^{(2)}, \hat{n}^{(3)}\) form a rectangular Cartesian coordinate system. If \([Q]\) represents the transformation from \(\{e_{1}, e_{2}, e_{3}\}\) to \(\{\hat{n}^{(1)}, \hat{n}^{(2)}, \hat{n}^{(3)}\}\), then
\[
\begin{bmatrix}
T'^{I} & 0 & 0 \\
0 & T'^{II} & 0 \\
0 & 0 & T'^{III}
\end{bmatrix} = [Q]^{T} [T] [Q]
\]

X-axis represents the eigen value with coordinate system \(n^{(1)}, n^{(2)}, n^{(3)}\). As we rotate the radius with arbitrary angles \(\alpha, \beta, \gamma\) with old system (eigen vector system) with \(\alpha^{2} + \beta^{2} + \gamma^{2} = 1\)
Thus $[T'] = [Q]^T[T]_{eigin}[Q]$

\[
\left( \frac{T^I + T^{III}}{2}, \frac{T^I - T^{III}}{2} \right)
\]
4.7 Equations of Motion

NOTES

- On -ve face, -ve direction, stress component is positive
- There is a gradient of of stress component from –ve to +ve face. Stress field is continuously varying.

Note that center of cube is origin P

Let \( \vec{T} \) be the stress tensor at a point P. Let \( \vec{X} \) be the body force/volume at that point.

Note : \( \vec{B} \) be the body force/mass. \( \rho \vec{B} = \vec{X} \)

Next: 4.7 Equations of Motion (Cnt’d)
4.7 (b) Equations of Motion (Cnt’d)

Previous: Equations of Motion

Summing all the forces in the $x_1$ direction

\[
\left( T_{11} + \frac{\partial T_{11}}{\partial x_1} \right) dx_2 dx_3 - T_{11} dx_2 dx_3 \\
+ \left( T_{21} + \frac{\partial T_{21}}{\partial x_2} \right) dx_1 dx_3 - T_{21} dx_1 dx_3 \\
+ \left( T_{31} + \frac{\partial T_{31}}{\partial x_3} \right) dx_2 dx_1 - T_{31} dx_2 dx_1 \\
+ X_i dx_1 dx_2 dx_3 = 0
\]

Dividing by $dx_1 dx_2 dx_3$ we have

\[
\frac{\partial T_{11}}{\partial x_1} + \frac{\partial T_{21}}{\partial x_2} + \frac{\partial T_{31}}{\partial x_3} + X_1 = 0
\]

or

\[
\frac{\partial T_{ji}}{\partial x_i} + X_i = 0 \quad \text{or} \quad T_{ji,j} + X_i = 0
\]

Since we already know that

\[
T_{ij} = T_{ji} \quad \text{for the moment equilibrium}
\]

\[
T_{ij,j} + X_i = 0 \quad \text{is the equation of the equilibrium}
\]

3 equations (3+1) terms per equation

Next: 4.9 Boundary Conditions
4.9 Boundary Conditions

Previous: Equations of Motion (Cnt’d)

If \( \hat{n} \) is the unit outward normal at a given point in surface then,

\[
\hat{t} = \hat{T}\hat{n} \quad (1)
\]

where \( \hat{t} \) is the force vector or traction vector per unit area. Equation (1) represents the stress or traction boundary condition.
4.10 First P-K Stress

Previous: 4.9 Boundary Conditions

We define
\( T_{ij} \) = Cauchy stress tensor (Force/area in the deformed geometry)

We know for equilibrium equation
\[
\frac{\partial T_{ji}}{\partial x_i} + X_i = 0
\]

Define two other stress tensors, first and second Piola-Kirchoff Stress tensors

Three different stress measures

\[
\tilde{T} = \text{Cauchy Stress} \Rightarrow \tilde{T}
\]
\[
\tilde{T}_0 = \text{I Piola Kirchoff Stress} \Rightarrow \tilde{\sigma}
\]
\[
\tilde{T} = \text{II Piola Kirchoff Stress} \Rightarrow \tilde{S}
\]

We will denote \( \tilde{\sigma} \) as first P-K and \( \tilde{S} \) as II P-K stress

Note

<table>
<thead>
<tr>
<th>Stress</th>
<th>Book</th>
<th>Ours</th>
<th>Configuration</th>
</tr>
</thead>
<tbody>
<tr>
<td>Cauchy</td>
<td>( \tilde{T} )</td>
<td>( \tilde{T} )</td>
<td>Current force</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>Current area</td>
</tr>
<tr>
<td>I P-K</td>
<td>( \tilde{T}_0 )</td>
<td>( \tilde{\sigma} )</td>
<td>Current force</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>Original area</td>
</tr>
<tr>
<td>II P-K</td>
<td>( \tilde{T} )</td>
<td>( \tilde{S} )</td>
<td>Fictitious force</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>Original area</td>
</tr>
</tbody>
</table>

Previous: 4.10 First P-K Stress
4.10 (a) First P-K Stress

Previous: 4.10 First P-K Stress

Let

\[ ds_o \rightarrow \text{Undeformed area with } \vec{N} \]
\[ ds \rightarrow \text{Deformed area with } \hat{n} \]
\[ d\vec{P} \rightarrow \text{Same force vector with same orientation (in deformed and undeformed)} \]
Resolution of forces in 2-D

We can develop two basis for stress relative to the undeformed area.
Define the first p-k stress tensor $\sigma$ such that it gives the actual force $d\bar{P}$ on the deformed surface $dS$ but measured w.r.t undeformed surface $dS_o$. 

Next: 4.10(c) First P-K contd
4.10(c) First P-K contd

Previous: 4.10(b) First P-K contd

In 1-D case $\sigma_{11} = \frac{dP_1}{dS_0}$ where $dP_1$ is the component of $dP$ in the $x_1$ direction.

Let $dP_i$ be the component of $d\bar{P}$ in the $i^{th}$ direction.

Recall.

\[
t_i = T_{ji} n_j \quad \text{(in the deformed)}
\]

\[
dP_i = n_j T_{ji} dS \quad \text{(in the deformed)}... (1)
\]

\[
\frac{dP_i}{dS_o} = N_j \sigma_{ji} \quad \text{(in the original)}...... (2)
\]

Thus from (1) and (2)

\[
N_j \sigma_{ji} dS_o = dP_i = n_j T_{ji} dS
\]

Next: 4.10(d) First P-K contd
Now, we know that

$$\rho n_i dS = \rho_o N_j \frac{\partial a_j}{\partial x_i} dS_0$$

Substituting

$$N_j \sigma_{ij} dS_o = T_{ji} \left[ \frac{\rho_o}{\rho} N_k \frac{\partial a_k}{\partial x_j} dS_o \right]$$

or

$$N_j dS_o \left[ \sigma_{ji} - \frac{\rho_o}{\rho} T_{ki} \frac{\partial a_j}{\partial x_k} \right] = 0$$

Finally,

$$T_{ij} = \frac{\rho}{\rho_o} \frac{\partial x_i}{\partial a_k} \sigma_{kj}$$

Cauchy \hspace{2cm} Det F \hspace{2cm} def gradient

First P-k Stress

Note, the tensor is not symmetric \textit{i.e.} $\sigma_{ij} \neq \sigma_{ji}$ First P-K stress is not symmetric

Next: 4.10(e) First P-K contd
(2) II P-K Stress $S_{ij}$

$S_{ij}$ is defined in terms of fictitious force $d\tilde{P}$ applied to undeformed surface $dS_o$. This force is related to the real force $d\bar{P}$ (applied to $dS$) in the same way that a material vector $d\tilde{a}$ at $\tilde{a}$ is related by the deformation to the corresponding spatial vector $d\tilde{x}$ at $\tilde{x}$. The angle $\theta$ is the same.

$d\tilde{P}$ on $dS$ and $d\tilde{P}$ on $dS_o$

$$d\tilde{P}_i = \frac{\partial a_i}{\partial x_j} dP_j$$

$$= \left( \delta_{ij} - \frac{\partial u_i}{\partial x_j} \right) dP_j$$

Or

$$N_j S_{ji} dS_o = d\tilde{P}_i = \frac{\partial a_i}{\partial x_j} d\tilde{P}_j$$

$$d\tilde{P}_i = \frac{\partial a_i}{\partial x_j} \left( n_k T_{kj} dS_o \right) \ \ \ \ \ (1)$$
4.10(f) First P-K contd

We know that

\[ \rho n_i dS = \rho o N_j \frac{\partial a_i}{\partial x_j} dS_o \] .......(2)

Substituting,

\[ S_{ij} = \frac{\rho o}{\rho} \frac{\partial a_i}{\partial x_j} \frac{\partial a_i}{\partial x_j} T_{kl} \]

Note the P-K stress is symmetric \( S_{ij} = S_{ji} \)

Next: 4.11 Equation of Motion in Terms of Undeformed configuration
4.11 EQUATION OF MOTION IN TERMS OF UNDEFORMED CONFIGURATION

We defined the Cauchy’s stress $\tilde{T}$ in terms of current forces and current geometry. $\tilde{\sigma}$, the first Piola-Kirchoff stress was defined using the current force and original configuration. $\sigma$ is similar to the engineering stress but is unsymmetric due to the rotated forces in the current configuration. We know that in terms of cauchy’s stress,

$$\frac{\partial T_{im}}{\partial x_m} + \rho B_i = \rho a_i$$

(1)

where $\tilde{B}$ is the body force /mass and $\rho$ is current density and $\tilde{a}$ is the acceleration.

The above equation can also be written as

$$\text{div}\tilde{T} + \rho \tilde{B} = \rho \tilde{a}$$

We can easily show that (1) can also be written as

$$\frac{\partial \sigma_{im}}{\partial a_m} + \rho_o B_i = \rho_o a_i$$

all in terms of original geometry and first P-K stress. Thus in terms of material coordinates,

$$\text{div}\tilde{\sigma} + \rho_o \tilde{B} = \rho_o \tilde{a}$$
In general we have:

- Conservation of Energy (I law of thermodynamics)
- Principle of Entropy (II law)
- Law of Heat Conduction
- Conservation of linear/angular momentum (Newton’s law)
4.13 **Heat Conduction Law**

Previous: Governing Field Equations

\[ \bar{q} \text{ (per unit area)} \]

\[ Q_1 \]

\[ Q_2 \]

\[ \bar{q} = \text{Heat flow / area.} \]

\[ Q_c = \text{Net heat flow} = Q_2 - Q_1 \]

Consider the X direction

\[ \left\{ q_1_{\mid x_1 + dx_1, x_2, x_3} - q_1_{\mid x_1, x_2, x_3} \right\} dx_2 dx_3 \]

\[ = - \frac{\partial q_1}{\partial x_1} dx_1 dx_2 dx_3 \]

The equation is similar in the other direction. Thus

\[ Q_c = - \left( \frac{\partial q_1}{\partial x_1} + \frac{\partial q_2}{\partial x_2} + \frac{\partial q_3}{\partial x_3} \right) dV \]

\[ = -(\text{div.} \bar{q}) dV \]

or

\[ \bar{q} = -k \nabla \theta \]

where

- k - thermal conductivity
- \( \nabla \theta \) - temperature gradient
4.14 Conservation of Energy

\[ Q_s \text{ (Radiation)} \]
\[ Q_c \text{ (conduction)} \]
\[ A(\vec{x},t) \]

\[ U \text{ = Internal Energy} \]
\[ KE \text{ = Kinetic Energy} \]
\[ Q_c \text{ = Net heat due to conduction} \]
\[ Q_s \text{ = Net heat due to radiation} \]
\[ P \text{ = Rate of work done (mechanical)} \]

Next: 4.14(b) Conservation of Energy (Cnt’d)
4.14(b) Conservation of Energy (Cnt’d)

Conservation of Energy,

$$\frac{D}{Dt}(U + KE) = P + Q_c + Q_s$$

But

$$P = \frac{D}{Dt}(KE) + T_{ij} \frac{\partial U_i}{\partial x_j} dV$$

Define $u = \text{internal energy per unit mass.}$

$$\frac{DU}{Dt} = T_{ij} \frac{\partial v_i}{\partial x_j} dV - \frac{\partial q_i}{\partial x_i} dV + Q_s$$

conservation of mass $\Rightarrow \frac{D}{Dt}(\rho dV) = 0$

combining,

$$\rho \frac{DU}{Dt} = T_{ij} \frac{\partial u_i}{\partial x_j} - \frac{\partial q_i}{\partial x_i} + \rho q_s$$

Next: 4.15 Entropy Production (II law)
4.15 **Entropy Production (II law)**

Previous: 4.14(b) Conservation of Energy (Cnt’d)

\[ \eta(x, t) = \text{Entropy/mass} \]

Rate of increase of entropy of a volume = \( \frac{D}{Dt}(\rho \eta dV) \)

\[ = \frac{D}{Dt}(\rho dV) + \rho dV \frac{D\eta}{Dt} \]

According to the second law,

\[ \rho \frac{D\eta}{Dt} \geq \text{div} \frac{q}{\theta} + \rho \frac{q_s}{\theta} \]

\( \theta = \text{Temperature} \)
\( \bar{q} = \text{Heat flux} \)
\( q_s = \text{Heat added} \)
CHAPTER 5: The Elastic Solid and Elastic Boundary Value Problems

Constitutive equation is the relation between kinetic (stress, stress-rate) quantities and kinematic (strain, strain-rate) quantities for a specific material. It is a mathematical description of the actual behavior of a material. The same material may exhibit different behavior at different temperatures, rates of loading and duration of loading time. Though researchers always attempt to widen the range of temperature, strain rate and time, every model has a given range of applicability.

Constitutive equations distinguish between solids and liquids; and between different solids. In solids, we have: Metals, polymers, wood, ceramics, composites, concrete, soils…In fluids we have: Water, oil air, reactive and inert gases
5.1 The Elastic Solid and Elastic Boundary Value Problems

Relationship between kinetics (stress, stress rate) and kinematics (strain, strain-rate) determines constitutive properties of materials. Internal constitution describes the material's response to external thermo-mechanical conditions. This is what distinguishes between fluids and solids, and between solids wood from platinum and plastics from ceramics.

Elastic Solid

Uniaxial test: The test often used to get the mechanical properties

\[
\sigma = \frac{P}{A_0} = \text{engineering stress}
\]

\[
\varepsilon = \frac{\Delta l}{l_0} = \text{engineering strain}
\]

\[
E = \frac{\sigma}{\varepsilon}
\]
\[ \varepsilon_d = \text{diametral strain} \]
\[ \varepsilon_a = \text{axial strain} \]

\[ \nu = \text{Poisson's Ratio} = -\frac{\varepsilon_d}{\varepsilon_a} \]

Load-displacement response

Table 5.1

<table>
<thead>
<tr>
<th>Material</th>
<th>Tensile modulus ((E)) (GPa)</th>
<th>Tensile strength ((\sigma_s)) (GPa)</th>
<th>Density ((\rho)) (g/cm(^3))</th>
<th>Specific modulus ((E/\rho))</th>
<th>Specific strength ((\sigma_s/\rho))</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Fibers</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>E-glass</td>
<td>72.4</td>
<td>3.5(^a)</td>
<td>2.54</td>
<td>28.5</td>
<td>1.38</td>
</tr>
<tr>
<td>S-glass</td>
<td>85.5</td>
<td>4.6(^a)</td>
<td>2.48</td>
<td>34.5</td>
<td>1.85</td>
</tr>
<tr>
<td>Graphite (high modulus)</td>
<td>390.0</td>
<td>2.1</td>
<td>1.90</td>
<td>205.0</td>
<td>1.1</td>
</tr>
<tr>
<td>Graphite (high tensile strength)</td>
<td>240.0</td>
<td>2.5</td>
<td>1.90</td>
<td>126.0</td>
<td>1.3</td>
</tr>
<tr>
<td>Boron</td>
<td>385.0</td>
<td>2.8</td>
<td>2.63</td>
<td>146.0</td>
<td>1.1</td>
</tr>
<tr>
<td>Silica</td>
<td>72.4</td>
<td>5.8</td>
<td>2.19</td>
<td>33.0</td>
<td>2.65</td>
</tr>
<tr>
<td>Tungsten</td>
<td>414.0</td>
<td>4.2</td>
<td>19.30</td>
<td>21.0</td>
<td>0.22</td>
</tr>
<tr>
<td>Beryllium</td>
<td>240.0</td>
<td>1.3</td>
<td>1.83</td>
<td>131.0</td>
<td>0.71</td>
</tr>
<tr>
<td>Kevlar 49 (aramid polymer)</td>
<td>130.0</td>
<td>2.8</td>
<td>1.50</td>
<td>87.0</td>
<td>1.87</td>
</tr>
<tr>
<td><strong>Conventional materials</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Steel</td>
<td>210.0</td>
<td>0.34-2.1</td>
<td>7.8</td>
<td>26.9</td>
<td>0.043-0.27</td>
</tr>
<tr>
<td>Aluminum alloys</td>
<td>70.0</td>
<td>0.14-0.62</td>
<td>2.7</td>
<td>25.9</td>
<td>0.052-0.23</td>
</tr>
<tr>
<td>Glass</td>
<td>70.0</td>
<td>0.7-2.1</td>
<td>2.5</td>
<td>28.0</td>
<td>0.28-0.84</td>
</tr>
<tr>
<td>Tungsten</td>
<td>350.0</td>
<td>1.1-4.1</td>
<td>19.30</td>
<td>18.1</td>
<td>0.057-0.21</td>
</tr>
<tr>
<td>Beryllium</td>
<td>300.0</td>
<td>0.7</td>
<td>1.83</td>
<td>164.0</td>
<td>0.38</td>
</tr>
</tbody>
</table>

\(^a\)Virgin strength values. Actual strength values prior to incorporation into composite are approximately 2.1 (GPa).
Table 5.2
Properties of some aluminum-alloy matrix materials

<table>
<thead>
<tr>
<th>Alloy</th>
<th>Modulus (Gpa)</th>
<th>Yield Stress (0.2% offset) (Mpa)</th>
<th>Ultimate Tensile Strength (Mpa)</th>
<th>Strain to Failure %</th>
</tr>
</thead>
<tbody>
<tr>
<td>1100</td>
<td>63</td>
<td>43</td>
<td>86</td>
<td>20</td>
</tr>
<tr>
<td>2024</td>
<td>71</td>
<td>128</td>
<td>240</td>
<td>13</td>
</tr>
<tr>
<td>5052</td>
<td>68</td>
<td>135</td>
<td>265</td>
<td>13</td>
</tr>
<tr>
<td>6061</td>
<td>70</td>
<td>77</td>
<td>136</td>
<td>16</td>
</tr>
<tr>
<td>Al-7Si</td>
<td>72</td>
<td>65</td>
<td>120</td>
<td>23</td>
</tr>
</tbody>
</table>

Note that the modulus does not change (if any), but there is wide range of variation in yield and ultimate tensile strength and failure strain.
5.2 Linear Elastic Solid

If $T_{ij}$ is Cauchy tensor and $E_{ij}$ is small strain tensor, then in general,

$$E_{ij} = C_{ijkl} T_{kl}$$

where $C_{ijkl}$ is the fourth order elasticity tensor, since $C_{ijkl}$ is a tensor,

$$C_{ijkl} = Q_{mi} Q_{nj} Q_{rk} Q_{sl} C_{mnrs}$$

However, we know that $E_{kl} = E_{lk}$ and $T_{ij} = T_{ji}$, then

$$C_{ijkl} = C_{jikl} = C_{iklj}$$

We have $[C]_{4 \times 4}$ symmetric matrix with 36 constants, If elasticity is a unique scalar function of stress and strain, strain energy is given by

$$dU = T_{ij} dE_{kl} \text{ or } U = T_{ij} E_{ij}$$

then $T_{ij} = \frac{\partial U}{\partial E_{ij}}$

$$\Rightarrow C_{ijkl} = C_{klij}$$

$$\Rightarrow \text{Number of independent constants} = 21$$

Now consider that there is one plane of symmetry (monoclinic) material, then

One plane of symmetry $\Rightarrow$ 13

If there are 3 planes of symmetry, it is called an ORTHOTROPIC material, then

Orthotropy $\Rightarrow$ 3 planes of symmetry $\Rightarrow$ 9

Where there is isotropy in a single plane, then

Planar isotropy (Planar isotropic) $\Rightarrow$ 5

<table>
<thead>
<tr>
<th>Crystal structure</th>
<th>Rotational symmetry</th>
<th>Number of independent elastic constants</th>
</tr>
</thead>
<tbody>
<tr>
<td>Triclinic</td>
<td>None</td>
<td>21</td>
</tr>
<tr>
<td>Monoclinic</td>
<td>1 twofold rotation</td>
<td>13</td>
</tr>
<tr>
<td>Crystal System</td>
<td>Rotational Symmetry</td>
<td>Number</td>
</tr>
<tr>
<td>----------------</td>
<td>---------------------------------</td>
<td>--------</td>
</tr>
<tr>
<td>Orthorhombic</td>
<td>2 perpendicular twofold rotations</td>
<td>9</td>
</tr>
<tr>
<td>Tetragonal</td>
<td>1 fourfold rotation</td>
<td>6</td>
</tr>
<tr>
<td>Hexagonal</td>
<td>1 sixfold rotation</td>
<td>5</td>
</tr>
<tr>
<td>Cubic</td>
<td>4 threefold rotations</td>
<td>3</td>
</tr>
<tr>
<td>Isotropic</td>
<td></td>
<td>2</td>
</tr>
</tbody>
</table>
# Elastic Solid

(General 5.1 and 5.2)

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<td>Problems</td>
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<td>Linear elastic isotropic for small strain</td>
<td>#5.1, #5.2, #5.3</td>
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<thead>
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</tr>
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<td>5.32 Change of frame</td>
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<td>5.33 Constitutive Equation for an Elastic Medium under Large Deformation</td>
<td>5.100</td>
</tr>
<tr>
<td>5.34 Constitutive Equation for an Isotropic Elastic Medium</td>
<td></td>
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<tr>
<td>5.35 Simple Extension of an Incompressible Isotropic Elastic Rectangular Block</td>
<td></td>
</tr>
<tr>
<td>5.36 Simple Shear</td>
<td></td>
</tr>
</tbody>
</table>
5.3 Linear isotropic Solid

A material is isotropic if its mechanical properties are independent of direction

\[ T_{ij} = C_{ijkl} E_{kl} \quad \text{with } \hat{e}_i \text{ basis} \]

\[ T_{ij} ' = C_{ijkl} ' E_{kl} ' \quad \text{with } \hat{e}_i ' \text{ basis} \]

\[ C_{ijkl} = C_{ijkl} ' \]

Note that the isotropy of a tensor is equivalent to the isotropy of a material defined by the tensor.

Most general form of \( C_{ijkl} \) (Fourth order) is a function

\[ \alpha \delta_{ij} \delta_{kl} + \beta \delta_{ik} \delta_{jl} + \gamma \delta_{il} \delta_{jk} \]

Thus for isotropic material

\[ T_{ij} = C_{ijkl} E_{kl} = \lambda \epsilon_{kk} \delta_{ij} + 2 \mu E_{ij} \]

For \( i \neq j \) \( T_{ij} = 2 \mu E_{ij} \)

\( \lambda \) and \( \mu \) are called Lame's constants. \( \mu \) is also the shear modulus of the material (sometimes designated as \( G \)).

Next: 5.4 Relationship between Youngs Modulus \( E_y \), Poissons Ratio \( \gamma \)
5.4 Relationship between Youngs Modulus $E_y$, Poissons Ratio $\gamma$

Shear modulus $\mu = G$ and Bulk modulus $k$

We know that

$$T_{ij} = \lambda \delta_{ij} \varepsilon_{kk} + 2\mu E_{ij}$$

(Let $e \equiv E_{kk} = E_{11} + E_{22} + E_{33}$)

Inverting, $T_{kk} = (3\lambda + 2\mu)\varepsilon_{kk}$ or $\varepsilon_{kk} = \frac{1}{(3\lambda + 2\mu)} T_{kk}$

$$E_{ij} = \frac{1}{2\mu} \left[ T_{ij} - \frac{\lambda}{3\lambda + 2\mu} T_{kk} \delta_{ij} \right]$$

Consider uniaxial stress $T_{11} \neq 0$ and all other $T_{ij} = 0$

Thus $E_{11} = \frac{\lambda + \mu}{\mu(3\lambda + 2\mu)} T_{11}$

(a) Define $E_y = \frac{T_{11}}{E_{11}}$ or $E_y = \frac{\mu(3\lambda + 2\mu)}{\lambda + \mu}$ ...........(1)

(b) Define $\gamma = -\frac{E_{22}}{E_{11}} = \frac{E_{33}}{E_{11}} \gamma = \frac{\lambda}{2(\lambda + \mu)}$ .................(2)

From (1) and (2), $\mu = \frac{E_y}{2(1+\gamma)}$
Consider a simple shear state

\[ T_{12} = T_{21} = \tau \Rightarrow E_{12} = E_{21} = \frac{\tau}{2\mu} \]

\[ G \equiv \frac{\tau}{2E_{12}} = \mu \]

Consider hydrostatic stress state

\[ T_{ij} = \sigma_H \delta_{ij} \quad \text{when} \quad \sigma_H = -p \Rightarrow \text{pressure} \]

\[ E_{kk} = \frac{3\sigma_H}{2\mu + 3\lambda} \]

\[ k = \frac{\text{Hydrostress}}{\text{Volume change}} = \frac{\sigma_H}{E_{kk}} = \frac{2\mu + 3\lambda}{3} \]

\[ k = \lambda + \frac{2}{3} \mu \]
For interrelationship, see table (below)

<table>
<thead>
<tr>
<th>λ, μ</th>
<th>$\frac{\nu E_Y}{(1+\nu)(1-2\nu)}$</th>
<th>$\frac{2\mu\nu}{1-2\nu}$</th>
<th>$\frac{\mu(E_Y-2\mu)}{3\mu-E_Y}$</th>
<th>$\frac{3k\nu}{1+\nu}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\mu$</td>
<td>$\frac{E_Y}{2(1+\nu)}$</td>
<td>$\mu$</td>
<td>$\frac{3k(1-2\nu)}{2(1+\nu)}$</td>
<td>$\mu$</td>
</tr>
<tr>
<td>$k$</td>
<td>$\frac{\lambda+\frac{2}{3}\mu}{3(1-2\nu)}$</td>
<td>$\frac{2\mu(1+\nu)}{3(1-2\nu)}$</td>
<td>$\frac{\mu E_Y}{3(3\mu-E_Y)}$</td>
<td>$k$</td>
</tr>
<tr>
<td>$E_Y$</td>
<td>$\frac{\mu(3\lambda+2\mu)}{\lambda+\mu}$</td>
<td>$E_Y$</td>
<td>$2\mu(1+\nu)$</td>
<td>$E_Y$</td>
</tr>
<tr>
<td>$\nu$</td>
<td>$\frac{\nu}{2(\lambda+\mu)}$</td>
<td>$\nu$</td>
<td>$\nu$</td>
<td>$\frac{E_Y-1}{2\mu}$</td>
</tr>
</tbody>
</table>

Note for isotropic materials, there are only 2 constants $E$ and $\gamma$ (engineering) or Lame ($\lambda$ and $\mu$)
5.5 EQUATIONS OF INFINITESIMAL THEORY OF ELASTICITY

Boundary Value Problems

Given a body subjected to surface and body forces, with specified boundary conditions, to determine the displacement field knowing the material constitutive equations. In this case, we assume that the strain is small and there is no rigid body rotation. Further we assume that the material is governed by linear elastic isotropic material model.

Field Equations

(1) Strain Displacement Relationships
\[ E_{ij} = \frac{1}{2} \left( u_{i,j} + u_{j,i} \right) \] (1)

(2) Stress Strain Relations
\[ T_{ij} = \lambda E_{kk} \delta_{ij} + 2 \mu E_{ij} \] (2)

(3) Cauchy Traction Conditions (Cauchy Formula)
\[ t_i = T_{j,i} n_j \]

(4) Equilibrium Equation
\[ T_{ji,j} + X_j = 0 \rightarrow \text{For Statics} \]
\[ T_{ji,j} + \rho B_i = 0 \]
\[ T_{ji,j} + \rho B_i = \rho a_i \rightarrow \text{For Dynamics} \]

In general, we know that
\[ \frac{\partial T_{ij}}{\partial x_j} + \rho B_i = \rho a_i \]

\( B_i \) is the body force/mass
\( \rho B_i \) is the body force/volume = \( X_i \)
\( a_i \) is the acceleration

For small displacement \( x_i = a_i \)
Thus \( \nu_i = \frac{Dx_i}{Dt} = \frac{\partial u_i}{\partial t} \bigg|_{x_i \text{ fixed}} + \nu_j \frac{\partial u_i}{\partial x_j} \) 

Assume \( v << 1 \), then

\[ \nu_i = \frac{\partial u_i}{\partial t} \bigg|_{x_i \text{ fixed}} \]

\[ a_i = \frac{\partial \nu_i}{\partial t} = \frac{\partial^2 u_i}{\partial t^2} \]

Since \( dV = dV_0 (1 + E_{kk}) \)

\[ \rho = \frac{1}{1 + E_{kk}} \rho_o = (1 + E_{kk})^{-1} \rho_o \]

\[ \approx \left(1 - E_{kk}\right) \rho_o \]

For small strain, \( \rho \approx \rho_o \)

Thus for small displacement/rotation problem

\[
\frac{\partial T_{ij}}{\partial x_j} + \rho B_i = \rho \frac{\partial^2 u_i}{\partial t^2}
\]

\[ T_{ij} + \rho B_i = \rho \ddot{u}_i \]

or

Consider a Hookean elastic solid, then
\[ T_{ij} = \lambda E_{kk} \delta_{ij} + 2 \mu E_{ij} \]
\[ = \lambda u_{k,k} \delta_{ij} + \mu \left( u_{i,j} + u_{j,i} \right) \]
\[ T_{ij,j} = \lambda u_{k,kj} \delta_{ij} + \mu \left( u_{i,ij} + u_{j,ij} \right) \]

Thus, equation of equilibrium becomes
\[
\rho_o \frac{\partial^2 u_i}{\partial t^2} = \rho_o B_i + \left( \lambda + \mu \right) \frac{\partial E_{kk}}{\partial x_i} + \mu \frac{\partial^2 u_i}{\partial x_i \partial x_j}
\]

For static equilibrium \(\frac{\partial^2 u_i}{\partial t^2} = 0\) Then
\[
\left( \lambda + \mu \right) \frac{\partial E_{kk}}{\partial x_1} + \mu \left( \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2} \right) u_1 + \rho_o B_1 = 0
\]
\[
\left( \lambda + \mu \right) \frac{\partial E_{kk}}{\partial x_2} + \mu \left( \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2} \right) u_2 + \rho_o B_2 = 0
\]
\[
\left( \lambda + \mu \right) \frac{\partial E_{kk}}{\partial x_3} + \mu \left( \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2} \right) u_3 + \rho_o B_3 = 0
\]

The above equations are called Navier’s equations of motion.
In terms of displacement components
\[
\left( \lambda + \mu \right) \nabla E_{kk} + \mu \text{div} \nabla \vec{u} + \rho_o \vec{B} = \rho_o \frac{\partial^2 \vec{u}}{\partial t^2}
\]
5.7 PRINCIPLE OF SUPERPOSITION

Let \( u_i^{(1)} \) be solution to problem (1) and \( u_i^{(2)} \) be solution to problem (2). Then for infinitesimal field the solution of problem (3) \( u_i^{(3)} \) is

\[
   u_i^{(3)} = u_i^{(1)} + u_i^{(2)}
\]

This is called principle of superposition. Hence complex problem can be split into a number of simple problem with known solution and the solutions added.
Problem 5.4

Given

\[ \lambda = 119.2 \text{GPa} \left(17.3 \times 10^6 \ psi\right) \]
\[ \mu = 79.2 \text{GPa} \left(11.5 \times 10^6 \ psi\right) \]

Find the \( E_y, \gamma \) and \( k \)

From the table,

\[ E_y = \frac{\mu(3\lambda + 2\mu)}{\lambda + \mu} = 206 \text{GPa} \]
\[ \gamma = \frac{\lambda}{2(\lambda + \mu)} = 0.3 \]
\[ k = \lambda + \frac{2}{3} \mu = 172 \text{GPa} \left(25 \times 10^6\right) \]
Problem 5.8

If \( \tilde{E} = \begin{bmatrix} 100 & -100 & 0 \\ -100 & -50 & 0 \\ 0 & 0 & 200 \end{bmatrix} \times 10^{-6} \)

Find the stress components, given \( \lambda = 119.2 \text{ Gpa} \ (17.3 \times 10^6 \text{ psi}) \)
And \( \mu = 79.2 \text{ Gpa} \ (11.5 \times 10^6 \text{ psi}) \)

Solution:

\[
T_{ij} = \lambda E_{kk} \delta_{ij} + 2\mu \varepsilon_{ij}
\]

\( \tilde{T} = \lambda e\tilde{I} + 2\mu \tilde{E} \) where \( e = E_{kk} = E_{11} + E_{22} + E_{33} \)

\[
e = (100 - 50 - 200)10^{-6} = 250 \times 10^{-6}
\]

\[
\tilde{T} = 119.2 \times 250 \times 10^{-6} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + 
\]

\[
\begin{bmatrix} -100 & -100 & 0 \\ -100 & -50 & 0 \\ 0 & 0 & 200 \end{bmatrix} \times 10^{-6} \text{ Gpa}
\]

\[
= \begin{bmatrix} 45.6 & -15.8 & 0 \\ -15.8 & 21.9 & 0 \\ 0 & 0 & 61.5 \end{bmatrix} \times 10^3 \text{ Mpa}
\]
Problem 5.9

Given

\[
\begin{bmatrix}
100 & 42 & 6 \\
42 & -2 & 0 \\
6 & 0 & 15
\end{bmatrix}
\] MPa

\(E_y = 207\) GPa
\(\mu = 79.2\) GPa

(1) Find \([E]\)

\[\left[ E \right] = \frac{1}{2\mu} \left[ T \right] - \frac{\gamma}{E_y} \left( T_{kk} \right) \left[ I \right] \]

\[\gamma = \frac{E_y}{2\mu} - 1 = 0.3\]

\[= \frac{10^{-3}}{2(79.2)} \begin{bmatrix} 100 & 42 & 6 \\ 42 & -2 & 0 \\ 6 & 0 & 15 \end{bmatrix} - \frac{0.3}{207} \times 113 \times 10^{-3} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\]

\[= \begin{bmatrix} 4.67 & 2.65 & 0.379 \\ 2.65 & -1.76 & 0 \\ 0.379 & 0 & -0.691 \end{bmatrix} \times 10^{-4}\]

(2) Volume change for 5cm cube.

\[e = E_{11} + E_{22} + E_{33}\]

\[= 2.22 \times 10^{-4}\]

\[\Delta V = eV = 2.22 \times 10^{-4} \left( 5^3 \right)\]

\[= 2.77 \times 10^{-4} \text{ cm}^3\]
Problem 5.19

Elastic waves

Consider \( u_1 = \varepsilon [\sin \beta (x_3 - ct) + \alpha \sin \beta (x_3 + ct)] \), \( u_2 = u_3 = 0 \)

(1) Transverse wave in \( x_3 \) plane

(2) Strain components / Stress

\[
E_{11} = E_{22} = E_{33} = 0 \quad \text{Also} \quad E_{12} = E_{23} = 0
\]

\[
E_{13} = E_{31} = \frac{1}{2} \frac{\partial u_1}{\partial x_3} = \frac{1}{2} \varepsilon [\cos \beta (x_3 - ct) + \alpha \cos \beta (x_3 + ct)]
\]

\[
T_{13} = T_{31} = \mu \frac{\partial u_1}{\partial x_3}
\]

(3) Equation of motion with zero body force

\[
\frac{\partial T_{13}}{\partial x_3} = \rho_o \frac{\partial^2 u_1}{\partial t^2}
\]

\[
\mu \beta^2 \varepsilon [-\sin \beta (x_3 - ct) - \alpha \sin \beta (x_3 + ct)]
\]

\[
= \rho_o (\beta c)^2 \varepsilon [-\sin \beta (x_3 - ct) - \alpha \sin \beta (x_3 + ct)]
\]

\[
\mu = \rho_o c^2
\]

\[
x_3 = 0 \rightarrow \text{Stress Vector}
\]

(4)
Problem 5.14

Given
\[ u_1 = k a_3 a_2 \]
\[ u_2 = k a_3 a_1 \quad k = 10^{-4} \]
\[ u_3 = k (a_1^2 - a_2^2) \]

Find: (a) Stress components, (assume no rotations)
(b) In the absence of body force, is it an equilibrium stress field?

Solution

\[ \nabla \ddot{u} = \begin{bmatrix} 0 & k a_3 & k a_1 \\ k a_3 & 0 & k a_1 \\ 2 k a_1 & -2 k a_2 & 0 \end{bmatrix} \]

(a)
\[ E = \frac{1}{2} \begin{bmatrix} 0 & k a_3 & \frac{k(2 a_1 + a_2)}{2} \\ k a_3 & 0 & \frac{k(a_1 - 2 a_2)}{2} \\ \frac{k(2 a_1 + a_2)}{2} & \frac{k(a_1 - 2 a_2)}{2} & 0 \end{bmatrix} \]

Constitutive equation is
\[ T_{ij} = \lambda E_{kk} \delta_{ij} + 2 \mu E_{ij} \]
\[ E_{kk} = 0 \Rightarrow T_{ij} = 2 \mu E_{ij} \]

Thus
\[ [T] = 2 \mu k \begin{bmatrix} 0 & a_3 & \frac{(2 a_1 + a_2)}{2} \\ a_3 & 0 & \frac{(a_1 - 2 a_2)}{2} \\ \frac{(2 a_1 + a_2)}{2} & \frac{(a_1 - 2 a_2)}{2} & 0 \end{bmatrix} \]

Since \[ k = 10^{-4} \] and no rotations \[ x \approx a_i \]

\[ [T] = 2 \mu k \begin{bmatrix} 0 & x_3 & \frac{(2 x_1 + x_2)}{2} \\ x_3 & 0 & \frac{(x_1 - 2 x_2)}{2} \\ \frac{(2 x_1 + x_2)}{2} & \frac{(x_1 - 2 x_2)}{2} & 0 \end{bmatrix} \]
(b) No body force. $T_{ji,j} = 0$ (verify)

\[ T_{11,1} + T_{12,2} + T_{13,3} = 0 \]

\[ T_{21,1} + T_{22,2} + T_{23,3} = 0 \]

\[ T_{31,1} + T_{32,2} + T_{33,3} = 0 = 2\mu k - 2\mu k = 0 \]

Hence a valid field.
## CLASSIFICATION OF IMPACT PHENOMENON

<table>
<thead>
<tr>
<th></th>
<th>Low Velocity</th>
<th>High Velocity</th>
</tr>
</thead>
<tbody>
<tr>
<td>Extent of Deformation</td>
<td>Global</td>
<td>Local</td>
</tr>
<tr>
<td>Modal Response</td>
<td>Low Frequency</td>
<td>High Frequency</td>
</tr>
<tr>
<td>Loading/Response time</td>
<td>Millie Sec- Seconds</td>
<td>Sub mille seconds</td>
</tr>
<tr>
<td>Strains</td>
<td>0.5-10 %</td>
<td>&gt; 60%</td>
</tr>
<tr>
<td>Strain Rates</td>
<td>$10^{-2} - 10^1$</td>
<td>$&gt;10^5$</td>
</tr>
<tr>
<td>Pressures</td>
<td>Order of $\sigma_y$</td>
<td>10 –100 $\sigma_y$</td>
</tr>
<tr>
<td>Failure Mode</td>
<td>Large Plastic Flow</td>
<td>Mat’l Separation</td>
</tr>
</tbody>
</table>
TYPES OF IMPACT PROBLEMS

HIGH RATE TENSION TEST

ROD IMPACT

PLATE IMPACT
PROJECTILE IMPACT

PROJECTILE IMPACT ON COMPOSITES
HYPERVELOCITY IMPACT

BEFORE IMPACT

AFTER IMPACT

\[ V \]

\[ V - \Delta V \]
MATERIAL BEHAVIOR UNDER HYPERVELOCITY IMPACT

MATERIAL “DISINTEGRATION”
HIGH ENERGY DENSITY
INTENSE RADIATION → IMPACT FLASH
MELTING
VAPORIZATION

IMPACT VELOCITIES REQUIRED TO PRODUCE MELTING AND VAPORIZATION OF THREE METALS

<table>
<thead>
<tr>
<th></th>
<th>IRON</th>
<th>ALUMINUM</th>
<th>LEAD</th>
</tr>
</thead>
<tbody>
<tr>
<td>Incipient melting (Km/s)</td>
<td>6.6 – 9.2</td>
<td>5.2 – 7.3</td>
<td>1.3 –1.8</td>
</tr>
<tr>
<td>Complete melting (Km/s)</td>
<td>7.1 – 10</td>
<td>6.6 – 9.2</td>
<td>1.6 – 2.2</td>
</tr>
<tr>
<td>Incipient vaporization (Km/s)</td>
<td>10 – 14</td>
<td>11 –15</td>
<td>3.3 – 4.6</td>
</tr>
<tr>
<td>Complete vaporization (Km/s)</td>
<td>19 – 26</td>
<td>24 – 33</td>
<td>6.8 – 9.5</td>
</tr>
</tbody>
</table>

DYNAMICS OF MOTION

The area is called elastodynamics. Very important in the study of stress wave propagation

- Crash analysis
- Earth quakes
- Armor/Anti-armor
- Low – medium – high impact studies

\[
\left(\text{few } \frac{mm}{Sec} \rightarrow \frac{m}{Sec} \rightarrow \frac{2 - 5Km}{Sec} \text{ velocities}\right)
\]
In very high impact studies, simple Hooke’s law is not valid. There are two types of waves

**TRANSVERSE (SHEAR) WAVES**

**LONGITUDINAL (DILATIONAL) WAVES**
5.8 PLANE IRROTATIONAL WAVES

Consider the particle displacement

\[ u_1 = \varepsilon \sin \frac{2\pi}{l} (x_1 - c_L t) \]
\[ u_2 = u_3 = 0 \quad \varepsilon \ll 1 \]

Note that the motion along \( x_1 \) direction and the wave (phase) velocity is \( c_L \) in the +ve \( x_1 \) direction. \( l \) is the wavelength. Particle velocity at a given \( x = x_1 \) is \( \frac{du_1}{dt} \)

Strain

\[ E_{11} = \varepsilon \frac{2\pi}{l} \cos \frac{2\pi}{l} (x_1 - c_L t) \]
\[ E_{23} = E_{12} = E_{13} = E_{33} = 0 \]
\[ E_{kk} = e = E_{11} \]

Stress

\[ T_{11} = (\lambda + 2\mu) E_{11} \quad (\lambda \varepsilon_{kk} \delta_{ij} + 2\mu \varepsilon_{ij} = T_{ij}) \]
\[ T_{22} = T_{33} = \lambda E_{11} \]
\[ T_{12} = T_{23} = T_{31} = 0 \]

In the absence of body forces, equation of equilibrium is

\[ \rho \frac{\partial^2 u_i}{\partial t^2} = \frac{\partial T_{ij}}{\partial x_j} \]

or,

\[ \frac{\partial^2 u_1}{\partial t^2} = \frac{\partial T_{11}}{\partial x_1} = (\lambda + 2\mu) \frac{\partial^2 u_1}{\partial x_1^2} \]
Substituting $\frac{\partial^2 u_1}{\partial t^2}$ and $\frac{\partial^2 u_1}{\partial x_1^2}$ and using $u_1 = \varepsilon \sin \frac{2\pi}{l} (x_1 - c_L t)$, we have

$$\rho_o c_L^2 = \lambda + 2\mu \quad \text{or}$$

$$c_L = \sqrt{\frac{\lambda + 2\mu}{\rho_o}}$$

Thus the longitudinal wave travels at a velocity of $c_L$ and is only a function of $\lambda, \mu$ (elastic constants) and density.
5.8 (b)

IRROTATIONAL WAVE

\[ \frac{\partial u_1}{\partial x_2} = \frac{\partial u_2}{\partial x_1} = \frac{\partial u_3}{\partial x_2} = \frac{\partial u_2}{\partial x_3} = \frac{\partial u_1}{\partial x_3} = \frac{\partial u_3}{\partial x_1} = 0 \]

Thus \[ \frac{1}{2} \left( \frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i} \right) = 0 \]

Hence there is no rotation of particle. Planes remain same.

Volume change due to the wave

\[ e = E_{11} = \varepsilon \frac{2\pi}{l} (x_1 - c_t t) \]

Volume changes also harmonically at a given location \( x_1 \), thus it is called dilatational wave.
Example: Problem 5.22

Given:
\[ u_3 = \sin \beta (x_3 - ct) + \alpha \sin \beta (x_3 + ct) \]
\[ u_1 = u_2 = 0 \]

(a) Nature of Wave
Longitudinal in \( \hat{e}_3 \) direction.

(b) Strain
Only non-zero
\[ E_{33} = \frac{\partial u_3}{\partial x_3} = \beta \left[ \cos \beta (x_3 - ct) + \alpha \cos \beta (x_3 + ct) \right] \]

Stress components
\[ T_{11} = T_{22} = \lambda \frac{\partial u_3}{\partial x_3} \]
\[ T_{33} = (\lambda + 2\mu) \frac{\partial u_3}{\partial x_3} \]

(c) Using equilibrium equation, determines \( c \).
\[ \frac{\partial T_{33}}{\partial x_3} = \rho_o \frac{\partial^2 u_3}{\partial t^2} \]
\[ - (\lambda + 2\mu) \beta^2 \left[ \sin \beta (x_3 - ct) + \sin \beta (x_3 + ct) \right] \]
\[ = \rho \beta^2 c^2 \left[ \sin \beta (x_3 - ct) + \sin \beta (x_3 + ct) \right] \]

Thus
\[ \lambda + 2\mu = \rho_o c^2 \]

or \[ c = \sqrt{\frac{\lambda + 2\mu}{\rho_o}} \]
Problem 5.19

Elastic waves

Consider \( u_1 = \varepsilon [\sin \beta (x_3 - ct) + \alpha \sin \beta (x_3 + ct)] \), \( u_2 = u_3 = 0 \)

(1) Transverse wave in \( x_3 \) plane: wave propagates in the \( x_1 \) direction. Can be understood from inspection.

(2) Strain components / Stress

\[
E_{11} = E_{22} = E_{33} = 0 \quad \text{Also} \quad E_{12} = E_{23} = 0
\]

\[
E_{13} = E_{31} = \frac{1}{2} \frac{\partial u_1}{\partial x_3} = \frac{1}{2} \varepsilon \left[ \cos \beta (x_3 - ct) + \alpha \cos \beta (x_3 + ct) \right]
\]

\[
T_{13} = T_{31} = \mu \frac{\partial u_1}{\partial x_3}
\]

(3) Equation of motion with zero body force

\[
\frac{\partial T_{13}}{\partial x_3} = \rho_o \frac{\partial^2 u_1}{\partial t^2}
\]

\[
\mu \beta^2 \varepsilon \left[ -\sin \beta (x_3 - ct) - \alpha \sin \beta (x_3 + ct) \right]
\]

\[
= \rho_o \left( \beta c \right)^2 \varepsilon \left[ -\sin \beta (x_3 - ct) - \alpha \sin \beta (x_3 + ct) \right]
\]

\[
\mu = \rho_o c^2
\]

\( x_3 = 0 \rightarrow \text{Stress Vector} \)
5.9 Transverse (Shear) waves

This is also called as distortional or equal volume or equivoluminal wave. The material volume does not change, but only distortion occurs.

General motion

\[ u_1 = 0 \]
\[ u_2 = \varepsilon \sin \frac{2\pi}{l} \left( x_1 - c_T t \right) \]
\[ u_3 = 0 \]

Strain Components

\[ E_{11} = E_{22} = E_{23} = E_{13} = 0 \]
\[ E_{12} = \frac{\varepsilon}{2} \left( \frac{2\pi}{l} \right) \cos \frac{2\pi}{l} \left( x_1 - c_T t \right) \]
Stress Components

\[ T_{12} = 2\mu E_{12} = \mu \varepsilon \frac{2\pi}{l} \cos \frac{2\pi}{l}(x_1 - c_T t) \]

Since equation of motion is

\[ \frac{\partial T_{12}}{\partial x_2} = \rho \frac{\partial^2 u_2}{\partial t^2} \]

We get, \( \mu = \rho c_T^2 \) or \( c_T = \sqrt{\frac{\mu}{\rho}} \)

Recall that \( c_L = \sqrt{\frac{\lambda + 2\mu}{\rho}} \)

Thus

\[ \frac{c_L}{c_T} = \sqrt{\frac{\lambda + 2\mu}{\mu}} = \sqrt{\frac{2 - 2\nu}{1 - 2\nu}} \]

Thus the ratio depends only on the Poisson’s ratio.

When \( \gamma = 0.3 \quad \frac{c_L}{c_T} = 1.87 \)

For \( \gamma < 0.5 \) (most metals and alloys) \( c_L \) is always greater than \( c_T \).
Example 5.9.3

If an infinite train of harmonic plane waves propagates on a plane with unit direction $\hat{e}_n$, find the form of longitudinal and transverse wave

Soln:
Let P be a generic point on the plane. Since the wave is restricted to the plane (normal to $\hat{e}_n$). Motion of particle P should be identical to all the points on the plane.

Thus the form will be

$$\text{Longitudinal Wave:}$$
$$\vec{u} = \varepsilon \text{Sin} \left[ \frac{2\pi}{l} \left( \vec{x} \cdot \hat{e}_n - ct - \eta \right) \right] \hat{e}_n$$

$$\text{Transverse Wave:}$$
$$\vec{u} \text{ Should be parallel to the plane (perpendicular to } \hat{e}_n \text{) }$$
\[ \bar{u} = \varepsilon \sin \left[ \frac{2\pi}{l} \left( \bar{x} \hat{e}_n - c_r t - \eta \right) \right] \hat{e}_i \]
Example 5.9.4
Polarized waves in $x_1 - x_2$ plane

Transverse wave along $\hat{e}_{n_1}$
Amplitude $\varepsilon_1$ and wavelength $l_1$

\[
\hat{e}_{n_1} = Sin\alpha e_1 - Cos\alpha e_2
\]

\[
\vec{x} \hat{e}_{n_1} = x_1 \sin \alpha_1 - x_2 \cos \alpha_1
\]

\[
\hat{e}_{l_1} = \pm(Cos\alpha e_1 + Sin\alpha e_2)
\]

Displacement Field

\[
u_1 = \cos \alpha_1 \varepsilon_1 \sin \left(\frac{2\pi}{l_1} (x_1 \sin \alpha_1 - x_2 \cos \alpha_1 - c_T t - \eta_1)\right)
\]

\[
u_2 = \sin \alpha_1 \varepsilon_1 \sin \left(\frac{2\pi}{l_1} (x_1 \sin \alpha_1 - x_2 \cos \alpha_1 - c_T t - \eta_1)\right)
\]

\[
u_3 = 0
\]

Transverse Wave Along $\hat{e}_{n_2}$
Amplitude $\varepsilon_2$
Wavelength $l_2$

\[
\hat{e}_{n_2} = \sin \alpha_2 \hat{e}_1 + \cos \alpha_2 \hat{e}_2
\]

\[
\vec{x} \hat{e}_{n_2} = x_1 \sin \alpha_2 + x_2 \cos \alpha_2
\]

\[
\hat{e}_{t_2} = \pm \left( \cos \alpha_2 \hat{e}_1 - \sin \alpha_2 \hat{e}_2 \right)
\]

Displacement Field

\[
u_1 = \cos \alpha_2 \varepsilon_2 \sin \left[ \frac{2\pi}{l_2} \left( x_1 \sin \alpha_2 + x_2 \cos \alpha_2 - c_T t - \eta_2 \right) \right]
\]

\[
u_2 = \sin \alpha_2 \varepsilon_2 \cos \left[ \frac{2\pi}{l_2} \left( x_1 \sin \alpha_2 + x_2 \cos \alpha_2 - c_T t - \eta_2 \right) \right]
\]

\[
u_3 = 0
\]

Longitudinal Wave along $\hat{e}_{n_3}$

Amplitude $\varepsilon_3$
Wave Length $l_3$

\[
\hat{e}_{n_3} = \sin \alpha_3 \hat{e}_1 + \cos \alpha_3 \hat{e}_2
\]

\[
\vec{x} \hat{e}_{n_3} = x_1 \sin \alpha_3 + x_2 \cos \alpha_3
\]

Direction along $\hat{e}_{n_3}$

\[
u_1 = \sin \alpha_3 \varepsilon_3 \sin \left[ \frac{2\pi}{l_3} \left( x_1 \sin \alpha_3 + x_2 \cos \alpha_3 - c_L t - \eta_3 \right) \right]
\]

\[
u_2 = \cos \alpha_3 \varepsilon_3 \sin \left[ \frac{2\pi}{l_3} \left( x_1 \sin \alpha_3 + x_2 \cos \alpha_3 - c_L t - \eta_3 \right) \right]
\]

\[
u_3 = 0
\]
5.10 Reflection of Plane Elastic Waves.

If a single transverse wave is incident on a free surface, it produces two waves as shown below.

\[
\begin{align*}
\text{Incident} & \quad \text{Transverse} & \quad \text{Reflected} & \quad \text{Transverse} & \quad \text{Reflected Longitudinal} \\
\alpha_1 & \quad \alpha_2 & \quad \alpha_3 & \quad x_1 & \quad x_2
\end{align*}
\]

With all the three waves we have

\[
\begin{align*}
u_1 &= \cos \alpha_1 \varepsilon_1 \sin \phi_1 + \cos \alpha_2 \varepsilon_2 \sin \phi_2 + \cos \alpha_3 \varepsilon_3 \sin \phi_3 \\
u_1 &= \cos \alpha_1 \varepsilon_1 \sin \phi_1 - \cos \alpha_2 \varepsilon_2 \sin \phi_2 + \cos \alpha_3 \varepsilon_3 \sin \phi_3 \\
u_3 &= 0
\end{align*}
\]

where

\[
\begin{align*}
\phi_1 &= \frac{2\pi}{l_1} (x_1 \sin \alpha_1 - x_2 \cos \alpha_1 - c_T t - \eta_1) \\
\phi_2 &= \frac{2\pi}{l_2} (x_1 \sin \alpha_2 + x_2 \cos \alpha_2 - c_T t - \eta_2) \\
\phi_3 &= \frac{2\pi}{l_3} (x_1 \sin \alpha_3 - x_2 \cos \alpha_3 - c_T t - \eta_3)
\end{align*}
\]

From traction free boundary conditions,

\[\widetilde{T} \hat{e}_2 = 0\]

and the equilibrium conditions we can show that
$$\frac{\sin \alpha_1}{l_1} = \frac{\sin \alpha_2}{l_2} = \frac{\sin \alpha_3}{l_3}$$

and $$\frac{c_T}{l_1} = \frac{c_T}{l_2} = \frac{c_L}{l_3}$$