1.5 Degrees of Freedom and Constraints

- need 3 coordinates to specify position of particle
- \( \mathbf{F} = m\mathbf{a} \) is a vector equation which can be split into 3 component equations
• many problems include particles constrained to move on a surface or curve

• Then don’t need to use all 3 coordinates
• There is usually some functional relationship between some of the coordinates - called constraint equations

• have a constraint force associated with each constraint - normal to constraint
• Degree of freedom is the number of independent coordinates necessary to describe the configuration of the system.
Each constraint reduces the number of degrees of freedom by one.

- \( \text{DOF} = \text{Number of coordinates} - \text{Number of constraints} \)
Example

For the system below, find the constraint equations
\[ y_1 = h \]
\[ z_1 = 0 \]
\[ z_2 = 0 \]
\[ (x_2 - x_1)^2 + (y_2 - y_1)^2 - l^2 = 0 \]
1.6 Impulse and Momentum

- Newton’s law is \( F = \frac{dp}{dt} \)
- integrate this with respect to time from \( t_1 \) to \( t_2 \)
\[ \int_{t_1}^{t_2} F \, dt = \int_{t_1}^{t_2} \frac{dp}{dt} \, dt \]

\[ = \int_{p(t_1)}^{p(t_2)} dp \]

\[ = p(t_2) - p(t_1) \]

\[ = m \mathbf{v}(t_2) - m \mathbf{v}(t_1) \]
• this is called the **Impulse Momentum Theorem**

\[ \int F \, dt = \text{impulse of force } F \]
• For more than 1 particle, momentum of system is sum of all momenta

• When $F = 0$ over $t_1$ to $t_2$ we get

$$p(t_1) = p(t_2)$$

• conservation of linear momentum
if time duration under consideration is very short, we can get impulsive forces with very large magnitude

\[ \hat{F}(t) = \int_{t_1}^{t_1 + \epsilon} F \, dt \]

as \( \epsilon \to 0 \), \( F \to \infty \) so that the integral remains finite
• Impulsive forces result in instantaneous changes in velocity

• $\int F \, dt$ for nonimpulsive forces is small compared to the term for impulsive forces
Angular Momentum
Angular momentum about $O$ is

$$\mathbf{H}_O = \mathbf{r} \times m\mathbf{v}$$
Angular momentum is a relative quantity since it is calculated about a specific point.

Usually pick $O$ fixed, at center of gravity, or where we can simplify $H_O$. 
Differentiate with respect to time

\[
\frac{dH_O}{dt} = \frac{dr}{dt} \times m\mathbf{v} + \mathbf{r} \times \frac{dm\mathbf{v}}{dt} = \mathbf{r} \times ma = \mathbf{r} \times \mathbf{F} = M_O
\]
Integrate with respect to time

\[ \int_{t_1}^{t_2} M_O dt = \int_{H_1}^{H_2} dH_O dt \]

\[ = H_O(t_2) - H_O(t_1) \]
Example: Central Force Problems

Consider a particle moving such that force $\mathbf{F}$ always acts through the fixed origin $O$. 
\[ F = -F_\text{e}_r \]

- Moment about \( O \) is 0
- Then angular momentum about \( O \) is conserved.
\[ H_O = \mathbf{r} \times m \mathbf{v} = r \mathbf{e}_r \times m(\dot{r} \mathbf{e}_r + r \dot{\theta} \mathbf{e}_\theta) = m r^2 \dot{\theta} \mathbf{k} = \text{constant} \]

- Let \( h = r^2 \dot{\theta} = \text{angular momentum per unit mass} \)
Example

A 5-kg sphere is mounted on a rigid rod of negligible mass and length $L = 5$ m and is rotating freely about a vertical shaft as shown. The vertical shaft is rotating at a speed of 120 rpm when $\theta = 60^\circ$. Determine the angular speed when $\theta = 45^\circ$. 
\[ m \quad \text{mg} \quad \frac{\text{m}}{\text{L}} \text{O} \quad \text{x} \quad \text{y} \quad \theta \quad L \quad m \quad \omega \quad mg \]
Solution

- The moment about $O$ is

$$L(\sin \theta \mathbf{i} - \cos \theta \mathbf{j}) \times (-mg\mathbf{j})$$

$$= -mgL \sin \theta \mathbf{k}$$

- Hence there is no moment about the $x$ or $y$ axes.
Angular momentum about $O$ is given by

\[ H_O = \mathbf{r} \times m\mathbf{v} \]

\[ = L(\sin \theta \mathbf{i} - \cos \theta \mathbf{j}) \]

\[ \times m((L\dot{\theta}(\cos \theta \mathbf{i} + \sin \theta \mathbf{j}) \]

\[ - \omega L \sin \theta \mathbf{k}) \]

\[ = mL^2\dot{\theta} \mathbf{k} + mL^2\omega \sin^2 \theta \mathbf{j} \]

\[ + mL^2\omega \cos \theta \sin \theta \mathbf{i} \]
Now the pin supporting the bar will provide a moment about the $\mathbf{i}$ vector, so that we have conservation of angular momentum about the $\mathbf{j}$ vector (or $y$ axis).
Hence

\[ \omega \sin^2 \theta = \text{constant} \]

and

\[ \omega_2 = \omega_1 \frac{\sin^2 \theta_1}{\sin^2 \theta_2} \]

\[ = 120 \times \frac{\sin^2(60)}{\sin^2(45)} = 180 \text{ rpm} \]