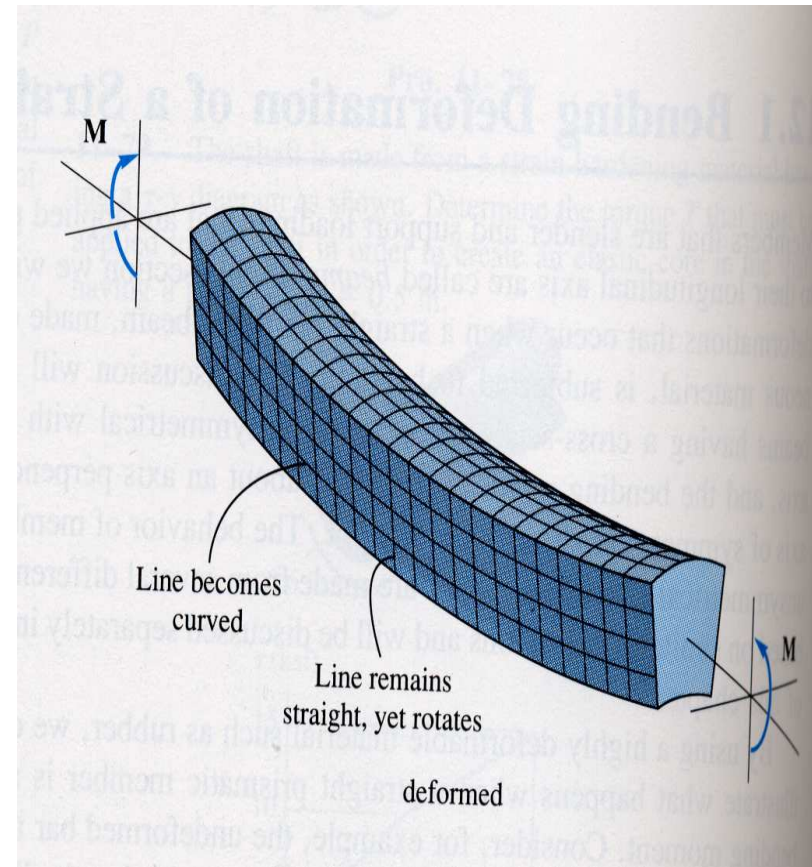
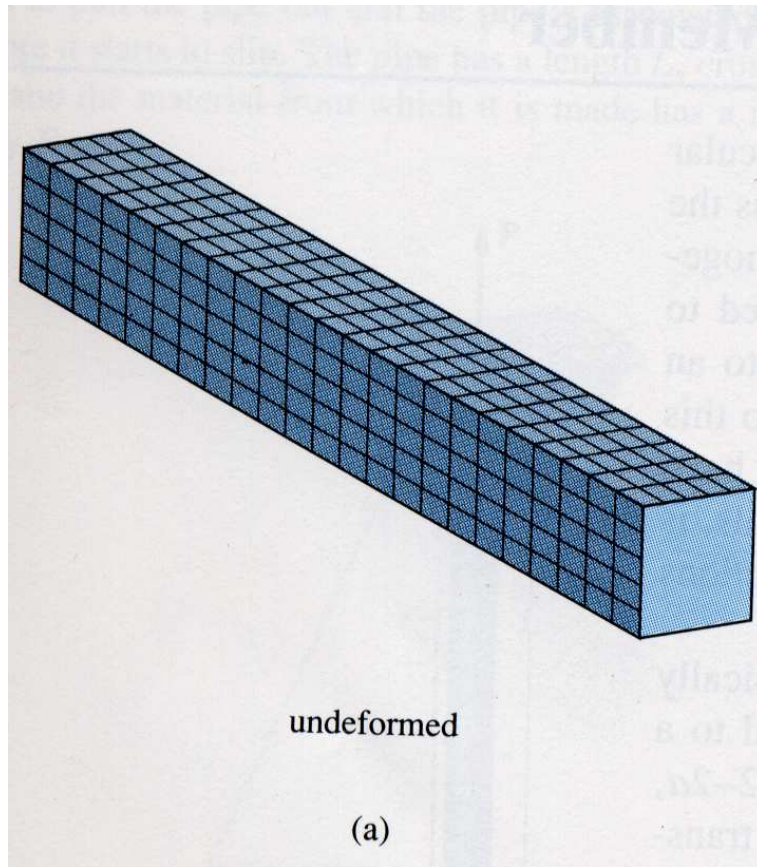


Mechanics & Materials 1

Chapter 12

Bending

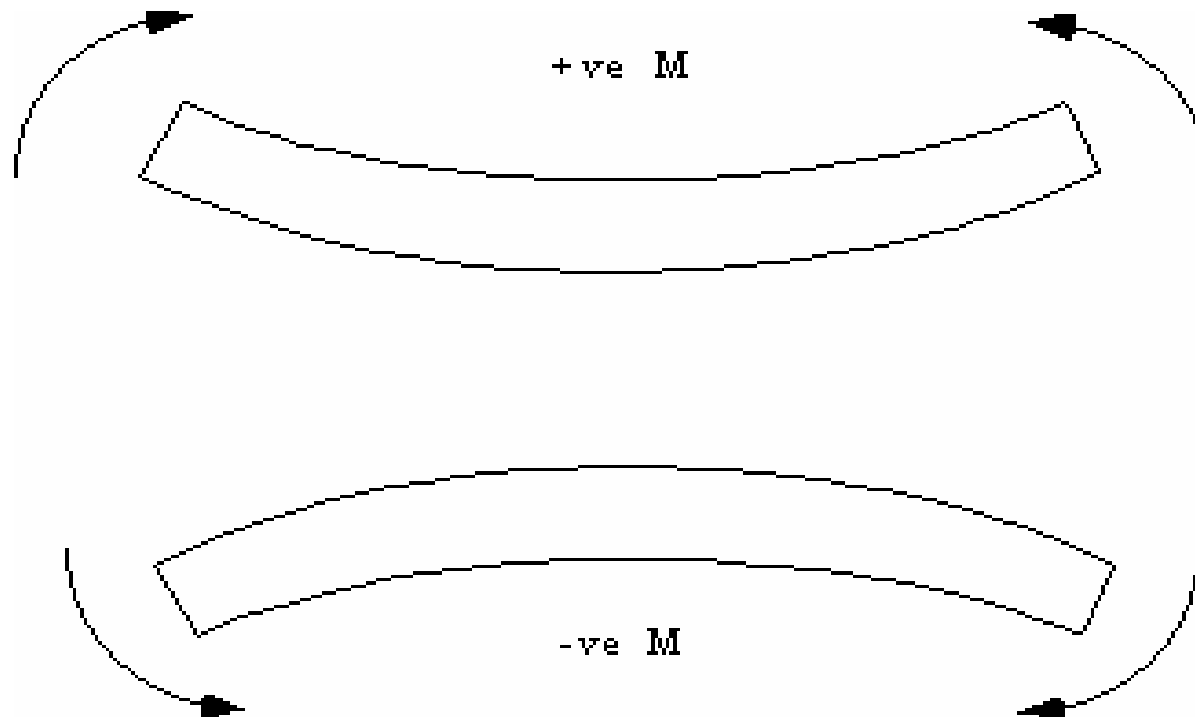
Bending



Assumptions for Analysis

- a) Transverse planes before bending remain transverse after bending, i.e. No warping.
- b) Beam material is homogeneous and isotropic and obeys Hooke's law with E the same in tension or compression.
- c) The beam is straight and has constant or slightly tapered cross section.
- d) Loads do not cause twisting or buckling. This is satisfied if the loading plane coincides with the section's symmetry axis.
- e) Applied load is pure bending moment.
- f) The definition for beams with applied positive and negative bending moments are as follows:

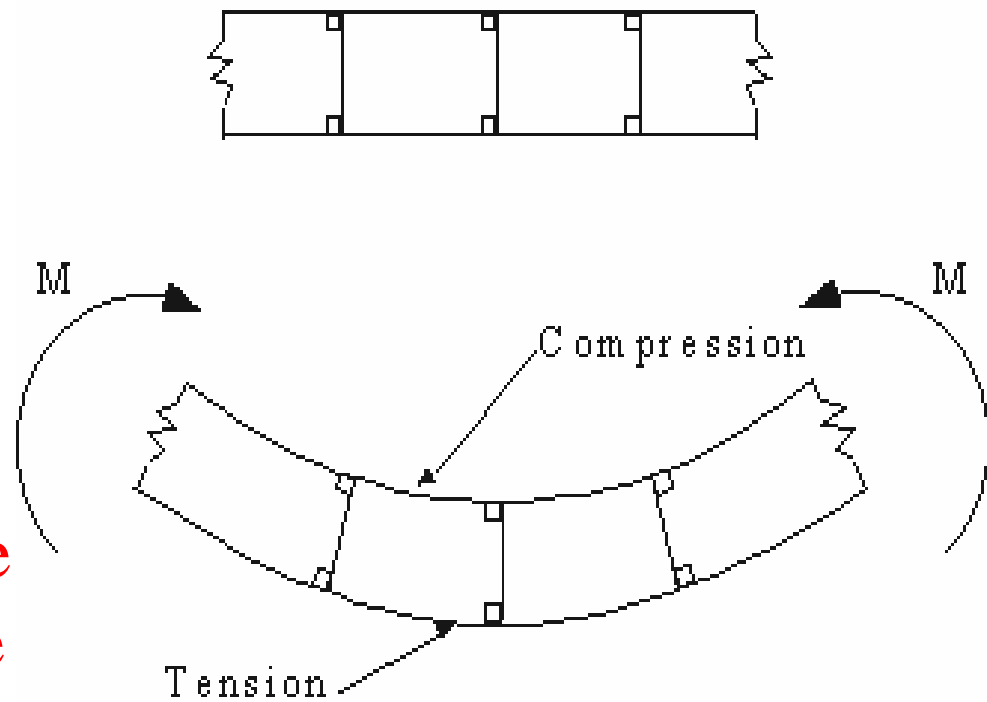
Positive and Negative Moments



Diagrams showing beams experiencing positive (top) and negative (bottom) bending moments

Beams Deformation Geometry

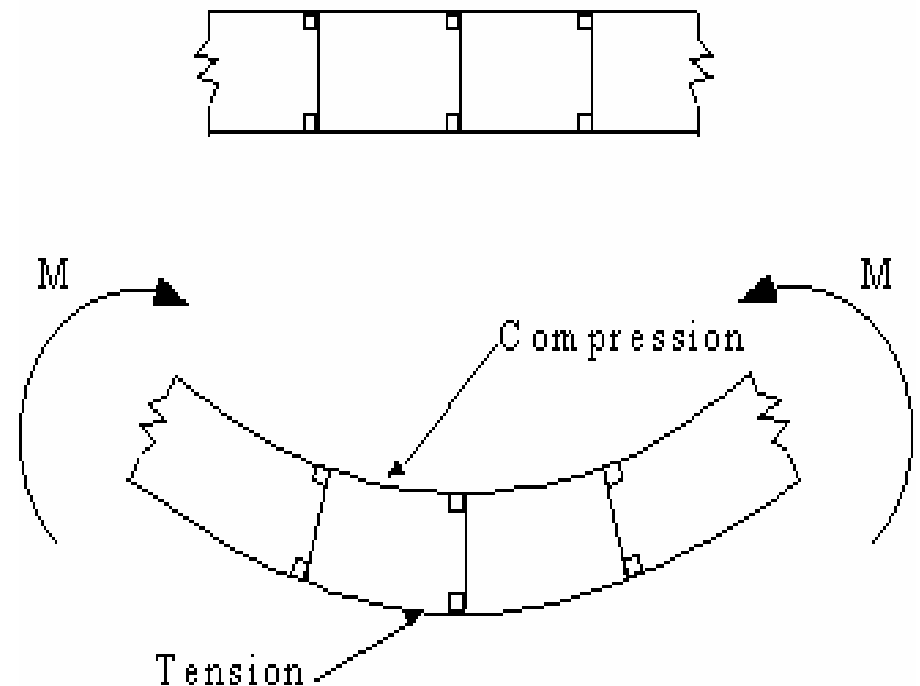
- When a beam is subjected to a pure bending moment, it will deform into a curved shape and this shape is the **arc of a circle with a very large radius compared to the size of the beam.**



Beam before and after a positive bending moment is applied indicating regions of +ve and -ve stress

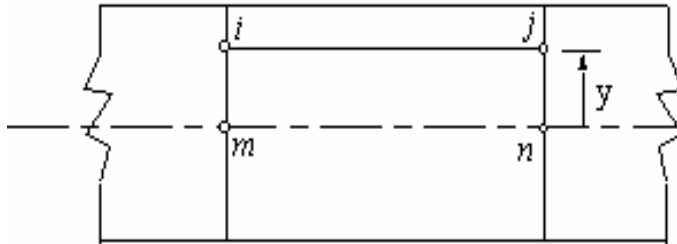
Beams Deformation Geometry

- The fibers on the top surface are experiencing a compressive stresses, and those on the bottom a tensile stress.
- Thus, at some point between these two surfaces, there must be a plane where the stresses and strains are ZERO. This is the 'Neutral Plane' (NP) or **Neutral Axis** (NA).



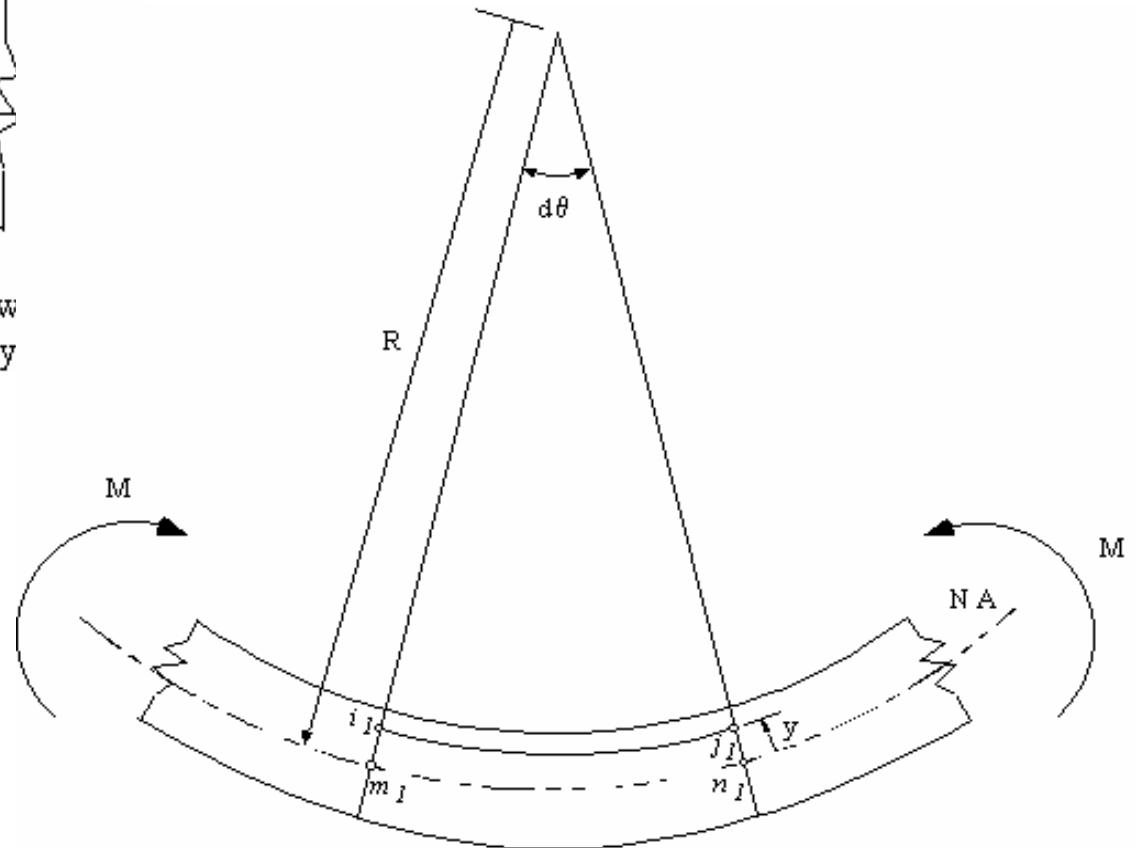
Beam before and after a positive bending moment is applied indicating regions of +ve and -ve stress

Beams Deformation Geometry



Undeflected segment of beam show points on the N.A. and two points a distance y

Mark a section a distance y from the Neutral Axis as ij , and another section on the Neutral Axis as mn . These sections are of equal length as they define the length between two transverse planes.



Beam deformed by positive bending moment showing Radius of curvature and deformed segment ij and mn

Beam Deformation Geometry

- The applied bending moment causes the segment ij and mn to deform into concentric arcs i_1j_1 and m_1n_1 with an angle $d\theta$ between the segments i_1m_1 and j_1n_1 . The distance between these two arcs is still y .
- The strain of segment i_1j_1 is defined as length i_1j_1 minus the original length ij over the original length, such that:

$$\epsilon_{xx} = \frac{i_1j_1 - ij}{ij}$$

- Now length mn and ij are defined as:

$$mn = ij = R d\theta$$

- The length i_1j_1 is defined as:

$$i_1j_1 = (R - y) d\theta$$

Stress -Strain in Bending

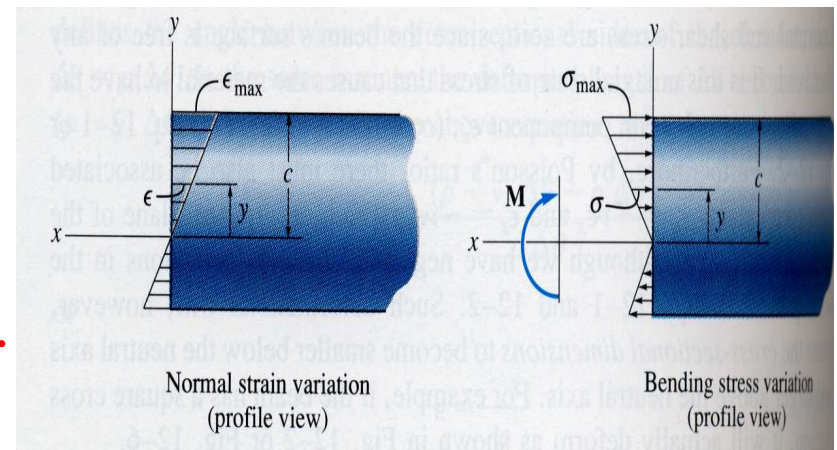
- So the strain becomes

$$\epsilon_{xx} = \frac{(R - y)d\theta - Rd\theta}{Rd\theta} = -\frac{y}{R}$$

- Which indicates that **the strain is linearly varying with y** . Note that this equation is relative to the neutral axis. It also doesn't relate to the loading. And since stress is strain times Young's Modulus then the stress can be defined by the following equation

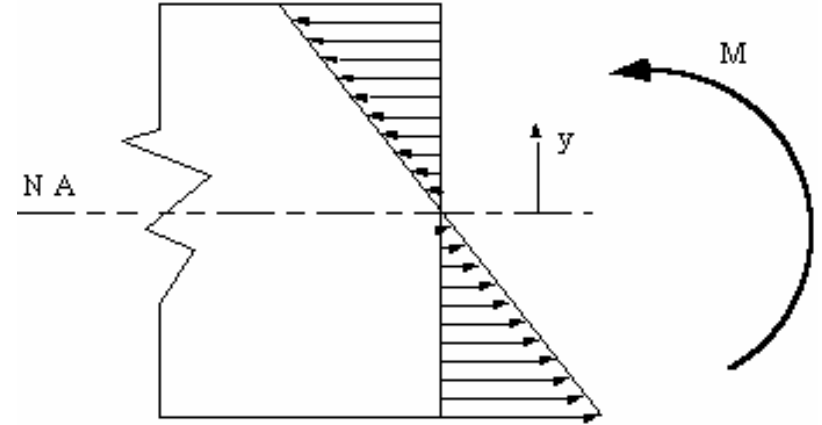
$$\sigma = -E \frac{y}{R}$$

and is also linearly varying with y .



Stress -Strain in Bending

- The drawing of the right hand end of the beam showing the stress distribution and applied bending moment.



Right hand end of beam showing applied bending moment M and normal stress distribution

- Let $\sigma_{xx} dA$ be the component of force acting on the element of area dA .

Force Equilibrium

- We now use equilibrium conditions on the stresses generated on the right hand side of the beam:

$$\begin{aligned}\sum F_x &= \int_A \sigma_{xx} dA = 0 \\ - \int_A \frac{Ey}{R} dA &= 0 \Rightarrow \frac{E}{R} \int_A y dA = 0\end{aligned}$$

- For materials with E constant the condition

$$0 = \int_A y dA$$

gives the origin for the 'y-axis' on the centroid of the section, i.e.. the location of the Neutral Plane.

Moment Equilibrium

- The applied moment **M** must be equal to the moment generated internally by the stress caused by the external moment, such that:

$$\int_A y \sigma_x dA = M$$

using the stress formula

$$M = \int_A \frac{-Ey^2}{R} dA = \frac{-E}{R} \int_A y^2 dA$$

Second Moment of Area: Moment of Inertia

- We now define the term

$$\int_A y^2 dA = I$$

as the **Second Moment of Area or Moment of Inertia** of the beam **about the Neutral Axis**

It is a measure of the stiffness of the cross sectional shape from a geometric point of view, without considering the material properties.

Bending (Flexure) Formula

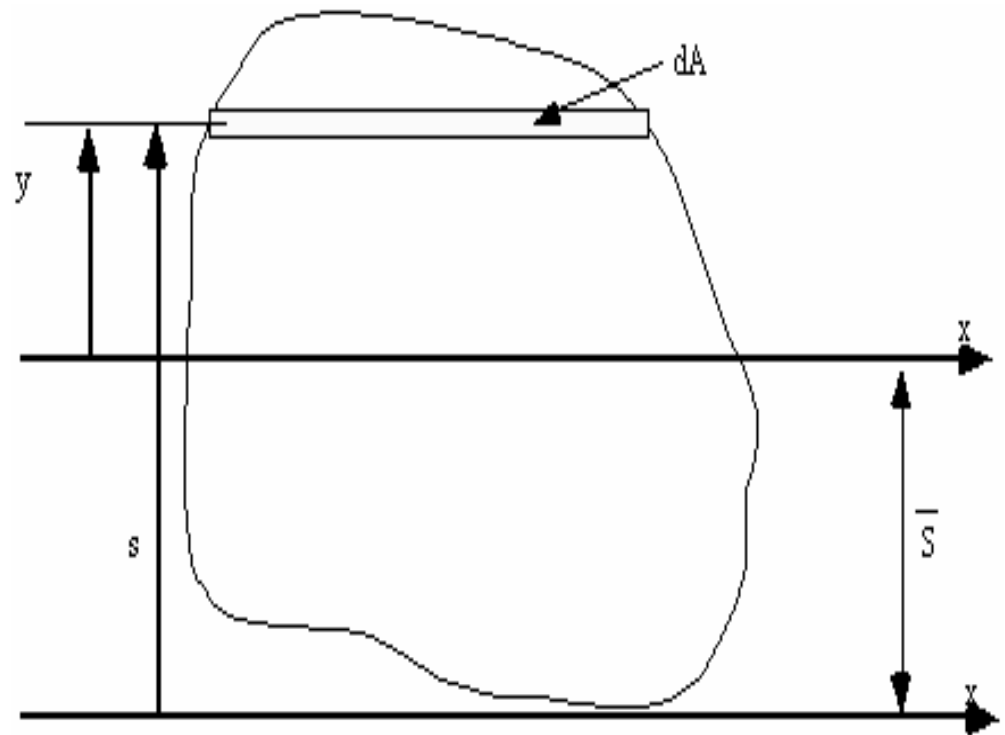
- Substituting the moment of inertia into the equation for stress gives

$$\sigma_x = \frac{My}{I} = \frac{Ey}{R}$$

- σ : Normal stress in the member due to bending moment
- M : the resultant internal moment, computed about the neutral axis of the cross section
- I : the moment of inertia of the cross sectional area computed about the neutral axis
- y : the perpendicular distance from the neutral axis to a point where the stress is to be determined,
- The resulting stress will be **tensile** if positive or **compressive** if negative

Determination Of Beam's Neutral Axis

- In order to find the centroid it is often best to find it in reference to the bottom of the beam cross section.
- If we do this, and because the centroid equation is integrated about the neutral plane we firstly need to change the axis from y to s .
- Change beam axis from y (distance away from Neutral Plane) to s (distance away from bottom of beam) have that:



Arbitrary cross section showing an infinitesimal section of area dA a distance y from the reference axis and the Neutral Plane.

Determination Of Beam's Neutral Axis

$$y = s - \bar{S}$$
$$\int_A (s - \bar{S}) dA = \int_A s dA - \int_A \bar{S} dA = 0$$
$$\bar{S} \int_A dA = \int_A s dA$$

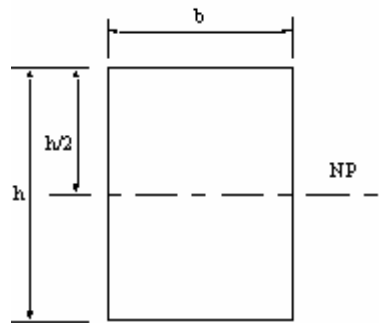
but since \bar{S} is the distance to the centroid, it is a constant and can be taken out of the integral equation, giving :

$$\bar{S} = \frac{\int_A s dA}{\int_A dA}$$

As most engineering beams are made of regular shapes, for which you know the areas and the centroids of these areas, then this equation can be used

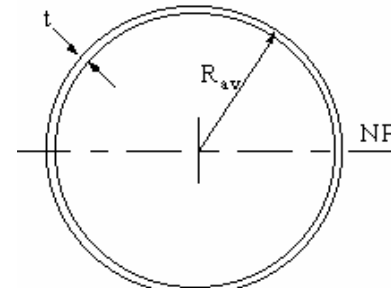
$$\bar{S} = \frac{\sum s_i A_i}{A_i}$$

Determination of Neutral Axis



Rectangular section

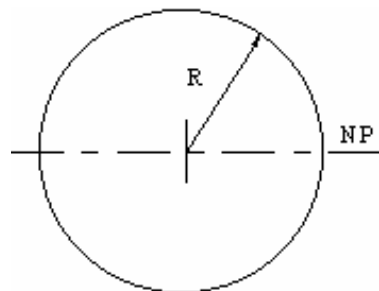
$$I = \int_{-\frac{h}{2}}^{\frac{h}{2}} y^2 b dy = \frac{bh^3}{12}$$



Thin tubular section

$$A = 2\pi R_{av} t$$

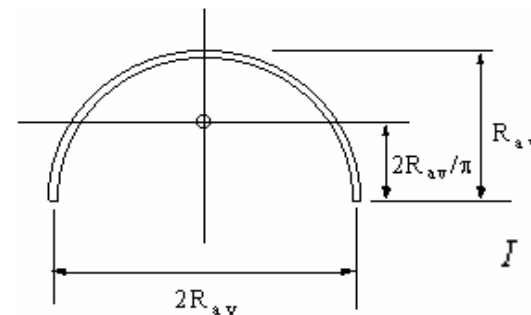
$$I = \pi R_{av}^3 t$$



Circular section

$$I = \frac{\pi R^4}{4}$$

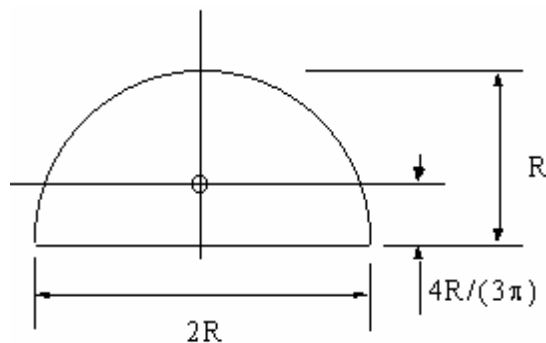
$$A = \frac{\pi R^2}{2}$$



Half-Thin tubular section

$$A = \pi R_{av} t$$

$$I = 0.095 \pi R_{av}^3 t$$

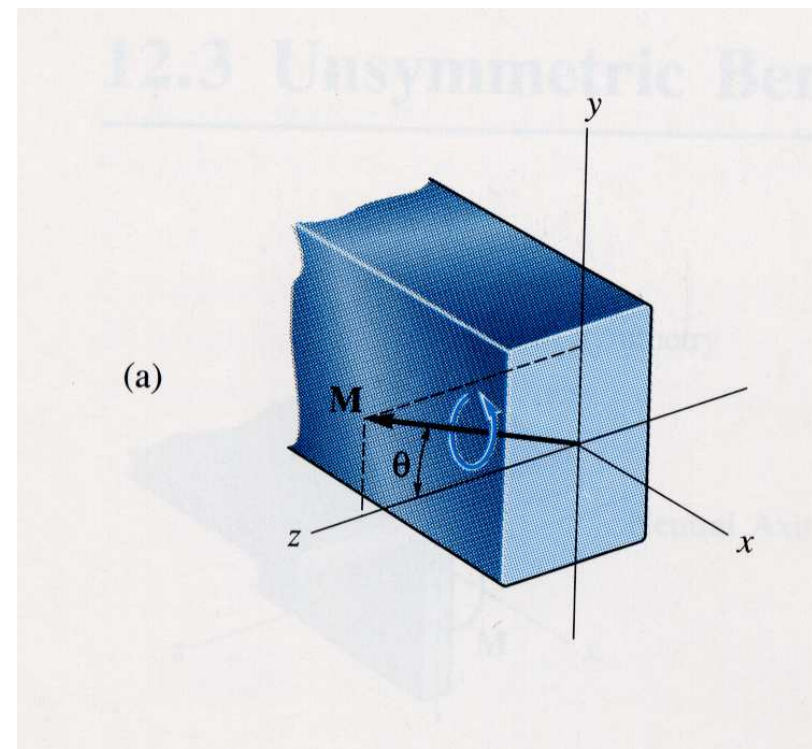


Semicircular section

$$I = 0.110 R^4$$

Unsymmetric Bending

- When the resultant moment does not act about one of the principal axis of the cross section, then the moment should be resolved into components directed along the principal axis. The flexure formula can then be used to determine the normal stress caused by each component of the moment



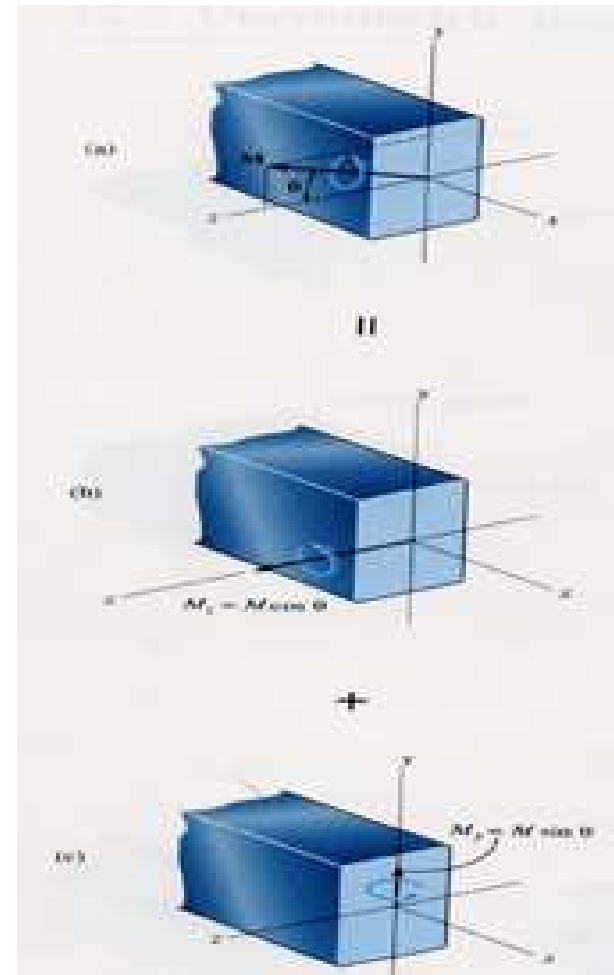
Unsymmetric Bending

- The moment in the previous figure is resolved along the y and z axis

$$M_z = M \cos \theta$$

$$M_y = M \sin \theta$$

- Applying the Flexure formula twice, and taking into account the direction of the moments

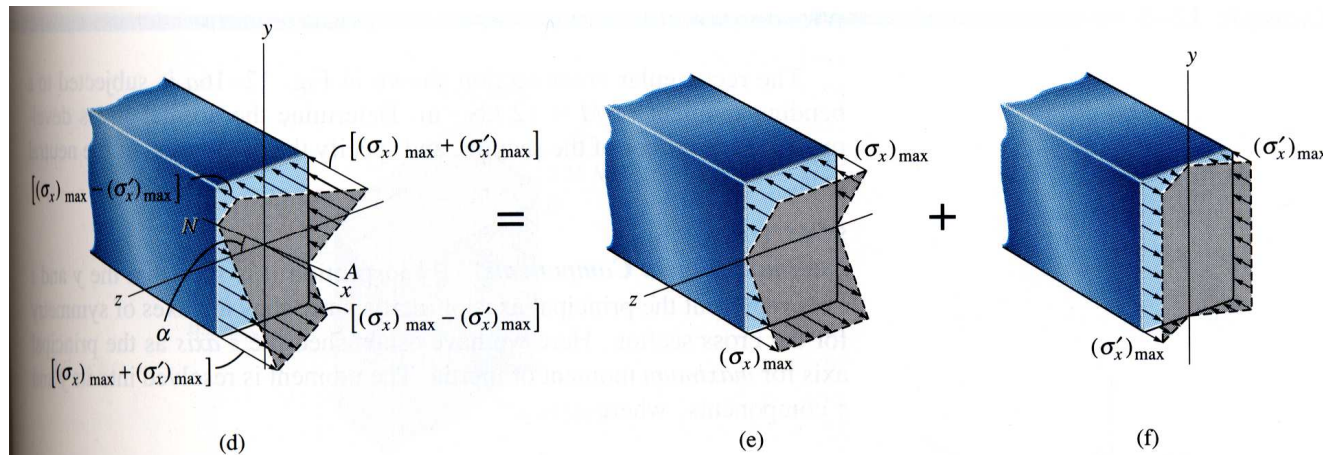


Unsymmetric Bending

$$\sigma = -\frac{M_z y}{I_z} + \frac{M_y z}{I_y}$$

- σ : the normal stress at the point
- y, z : The coordinates of the point measured from the x, y, z axis located at the centroid of the cross sectional area
- M_y, M_z : The resultant internal moment components along the principal axis, they are + when directed along the +z and +y axes.
- I_y, I_z : the principal moments of inertia computed about the z and y axes respectively

Orientation of Neutral Axes



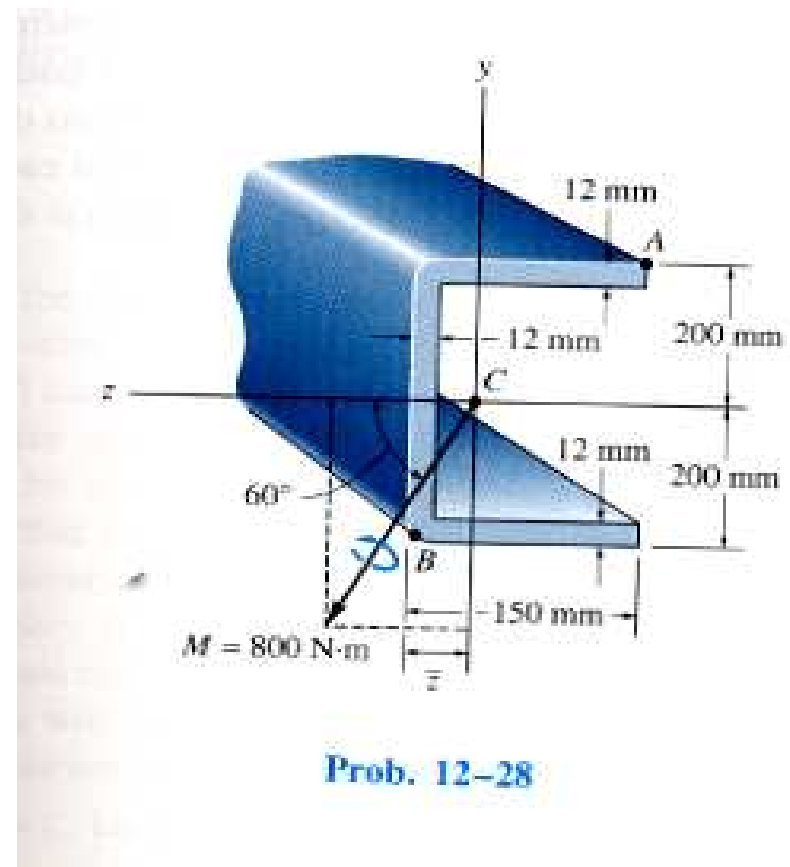
The angle α of the neutral axes can be determined from the fact that there is no normal stresses along the neutral plane $\sigma = 0$

$$y = \frac{M_y I_z}{M_z I_y} z \xrightarrow{u \sin \theta} \begin{cases} M_z = M \cos \theta \\ M_y = M \sin \theta \end{cases}$$

$$y = \left(\frac{I_z}{I_y} \tan \theta \right) z \Rightarrow \Rightarrow \tan \alpha = \frac{I_z}{I_y} \tan \theta$$

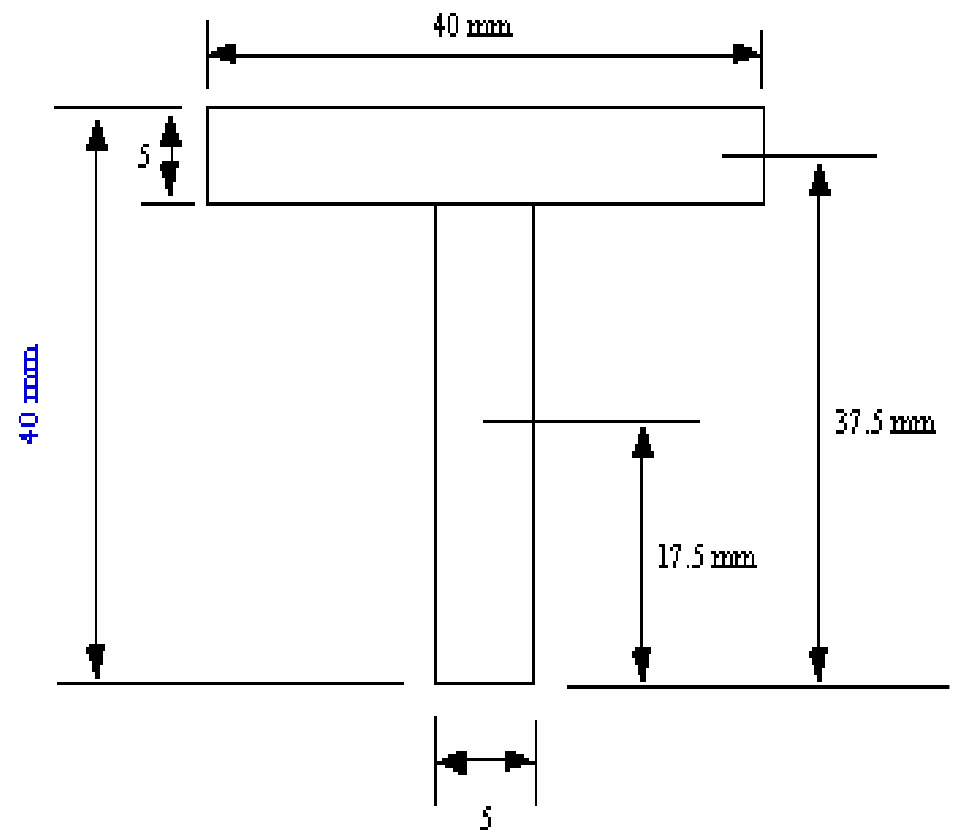
Problem. 12.28

- If the internal moment acting on the cross section of the strut has a magnitude of $M = 800\text{N}\cdot\text{m}$ and is directed as shown.
- Determine the bending stress at points A and B.
- The location of the centroid C of the strut's cross sectional area must be determined . Also, specify the neutral axis.



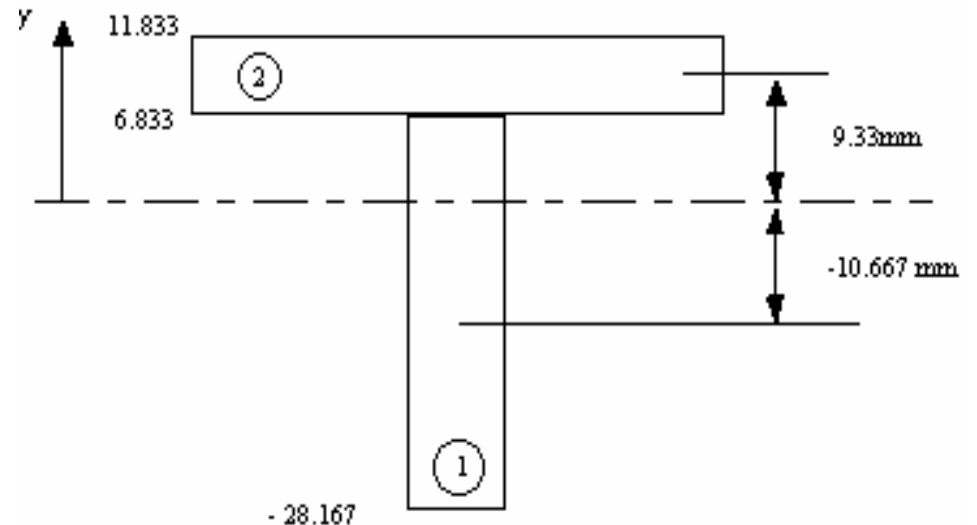
Example

- Determine the stresses in the given section due to an applied bending moment M_x of 1000 Nm.



Answer

- For all these type of problems it is best to follow the following methodology when solving them:
- a) Determine the location of the Neutral Plane. Do this by firstly subdividing the beam's cross section into regular geometric shapes.
- For this example the beam can be divided into two rectangular sections 1 and 2.



Beam cross section with vertical axis shifted to the position of the centroid (Neutral Axis)

$$\bar{S} = \frac{35 * 35 * 17.5 + 40 * 5 * 37.5}{35 * 5 + 40 * 5} = 28.167 \text{ mm}$$

Solution

If we use the formulas straight forward for finding the moment of inertia

$$\begin{aligned} I &= \frac{1}{12} bh^3 |_1 + \frac{1}{12} bh^3 |_2 \\ &= \frac{1}{12} 5 (35)^3 + \frac{1}{12} 40 (5)^3 \\ &= 18281.25 * 10^{-12} m^4 \end{aligned}$$

Did we get the moment of inertia right???????

Solution

- b) Determine the sections Second Moment of Area,
- In our example we need to use the parallel axis theorem

$$I_{N.A} = I_{local} + \bar{y}^2 A$$

$$I = I_1 + \bar{y}_1^2 A_1 + I_2 + \bar{y}_2^2 A_2$$

$$= \frac{5 * 35^3}{12} + (-10.667)^2 * 35 * 5 + \frac{40 * 5^3}{12} + (9.33)^2 * 40 * 5$$

$$= 55614.6 \text{ mm}^4$$

Or we can find I by the integration method as follows

$$I = \int_A y^2 dA = \int_{A_1} y^2 dA + \int_{A_2} y^2 dA$$

$$I = \int_{-28.167}^{6.833} y^2 5 dy + \int_{6.833}^{11.833} y^2 40 dy$$

$$= 17837 + 37777 = 55613 \text{ mm}^4 = 55613 * 10^{-12}$$

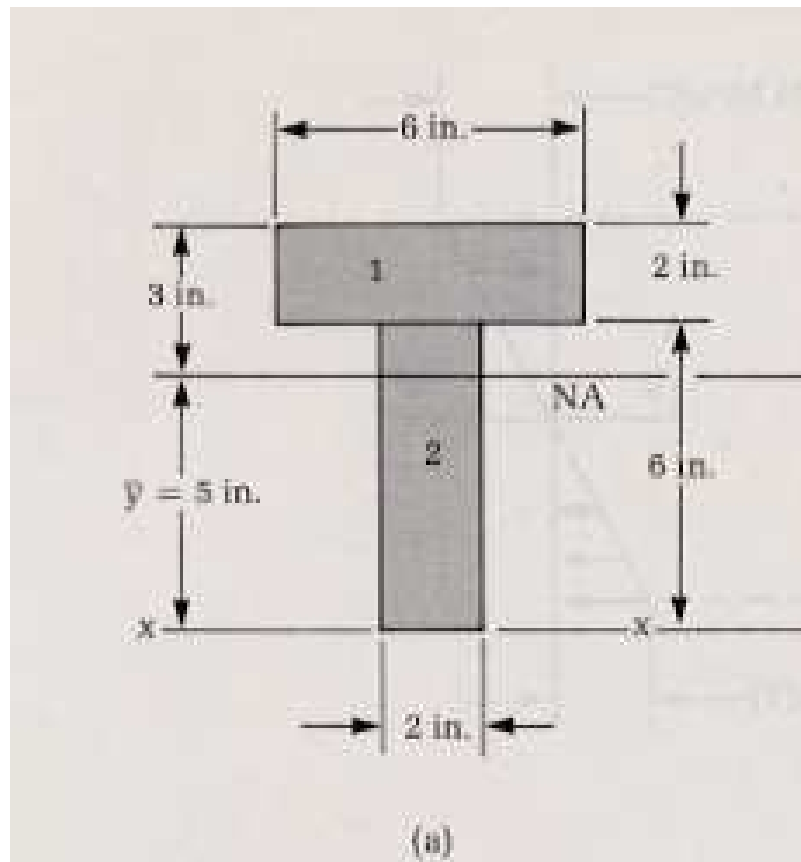
Solution

- c. Substituting this value for I and the bending moment into the flexure formula

$$\sigma_x = - \frac{1000}{55614} \frac{y}{* 10^{-12}} Pa$$

- and at the furthest points away from the neutral plane the stresses are:
- at $y = 11.833$ mm , $y = 0.011833$, $\sigma = -212.8$ MPa,
and
- at $y = -28.167$ mm, $y = -0.028167$, $\sigma = 506.48$ MPa

Example



- Determine the normal stress distribution on the T section shown if the bending moment $M=136,000\text{in.-lb.}$

Solution

1. Find the neutral axis location

$$A\bar{y} = A_1\bar{y}_1 + A_2\bar{y}_2 = 12(7) + 12(3) = 120 \text{ in}^3$$

so that

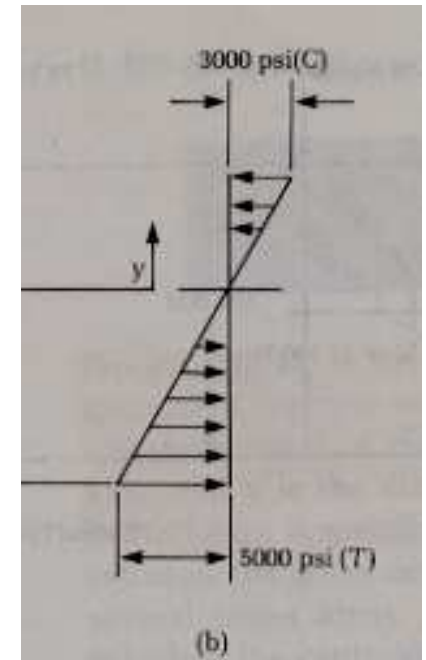
$$\bar{y} = \frac{120}{24} = 5 \text{ in.}$$

2. Moment of inertia:

$$I = \sum_{i=1}^2 (\bar{I} + A\bar{y}^2)_i = \left[\frac{1}{12} (6)2^3 + 12(2)^2 \right] + \left[\frac{1}{12} (2)6^3 + 12(2)^2 \right]$$
$$= 136 \text{ in}^4$$

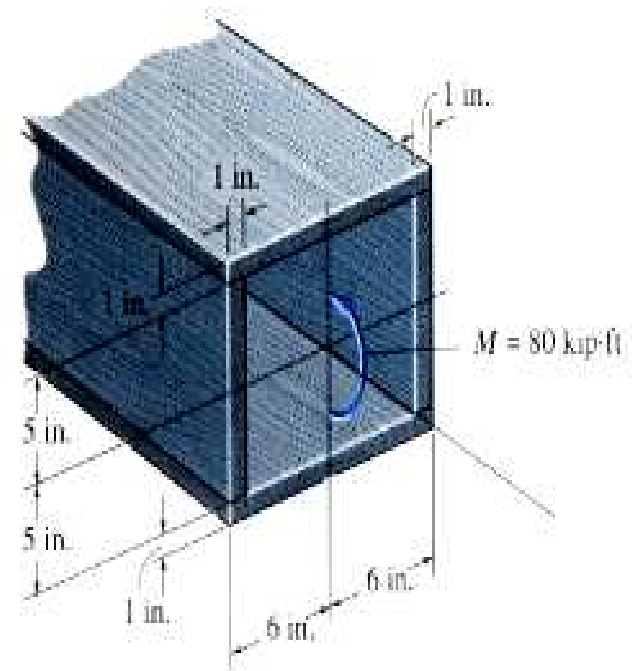
• Stress distribution:

$$\sigma_x = -\frac{136000}{136} y = -1000y \text{ psi}$$



Problem 12.2

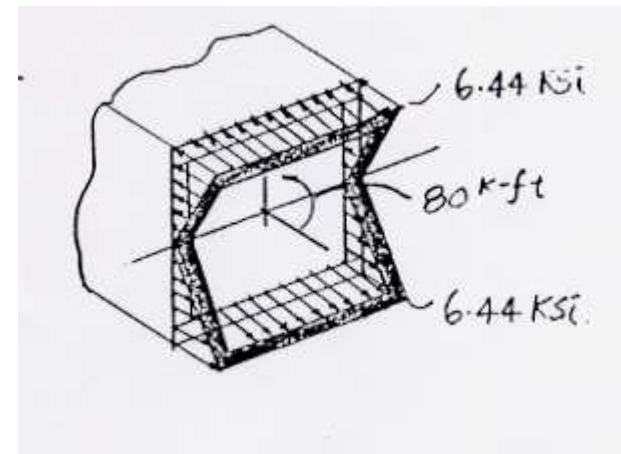
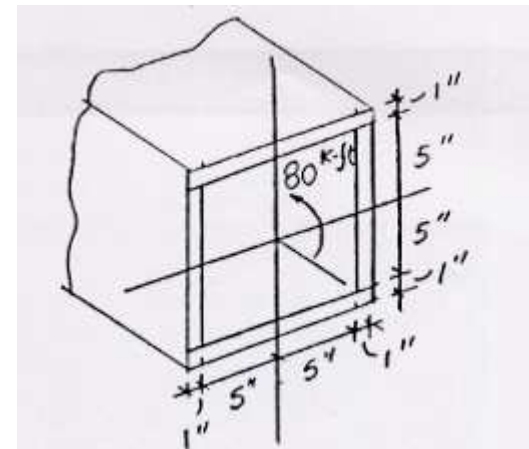
- A beam is constructed from four pieces of wood, glued together as shown.
- If the internal bending moment is $M = 800 \text{ kip}\cdot\text{ft}$, determine the maximum bending stress in the beam.
- Sketch a three-dimensional view of the stress distribution acting over the cross section



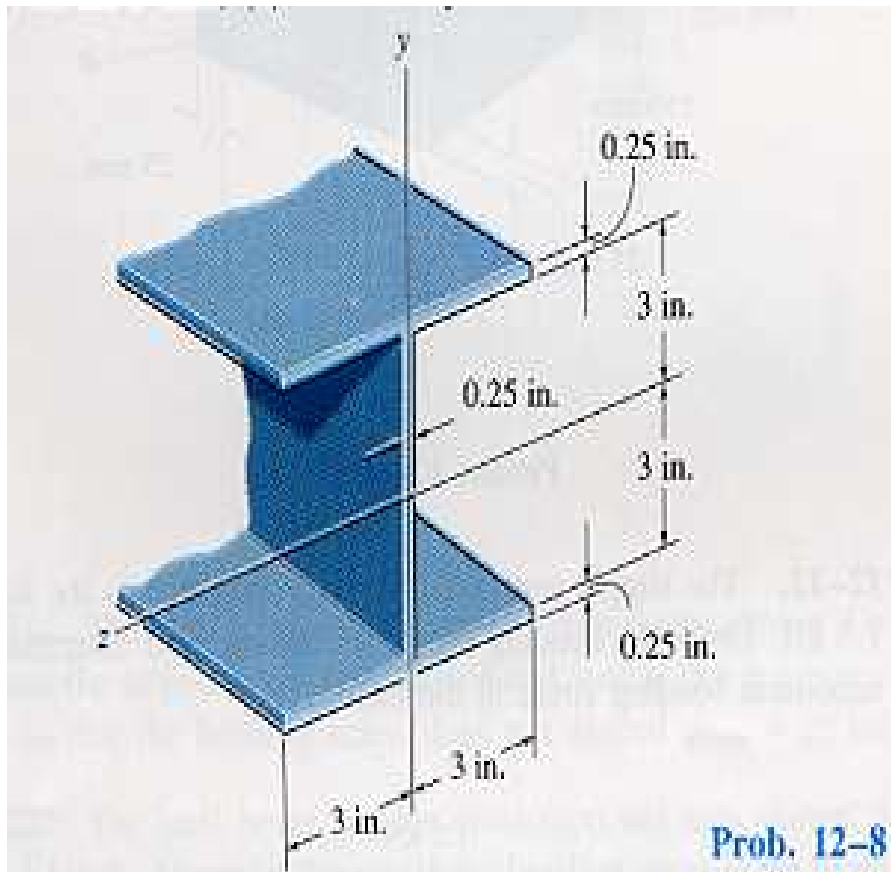
Solution

$$I = \frac{1}{12}(12)(12^3) - \frac{1}{12}(10)(10^3)$$
$$= 894.67 \text{ in}^4$$

$$\sigma_{\max} = \frac{Mc}{I}$$
$$= \frac{80(12)(6)}{894.67}$$
$$= 6.44 \text{ ksi}$$



Problem.12.8



- A beam has the cross section shown. If it is made of steel that has an allowable stress of $\sigma_{\text{allow}} = 24\text{ksi}$.
- Determine the largest internal moment the beam can resist if the moment is applied (a) about the z axis, (b) about the y axis.

Solution

$$I_z = \frac{1}{12} (6)(6.5^3) - \frac{1}{12} (5.75)(6^3)$$
$$= 33.8125 \text{ in.}^4$$

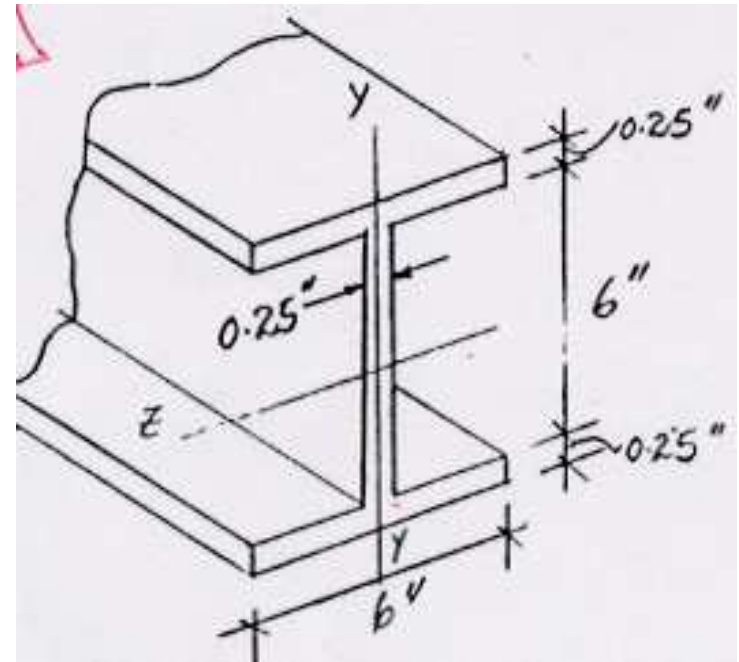
$$I_y = \frac{1}{12} (6)(0.25^3) + 2 \left(\frac{1}{12} \right) (0.25)(6^3)$$
$$= 9.0078125 \text{ in.}^4$$

a) $\sigma_{\max} = \frac{M_z c}{I_z}$

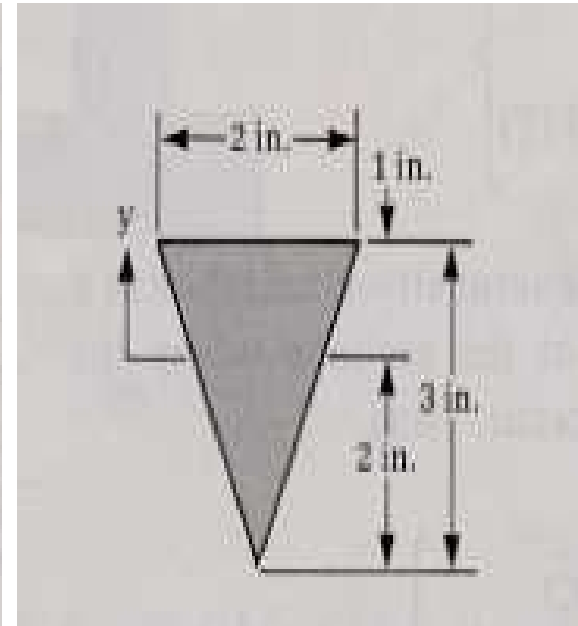
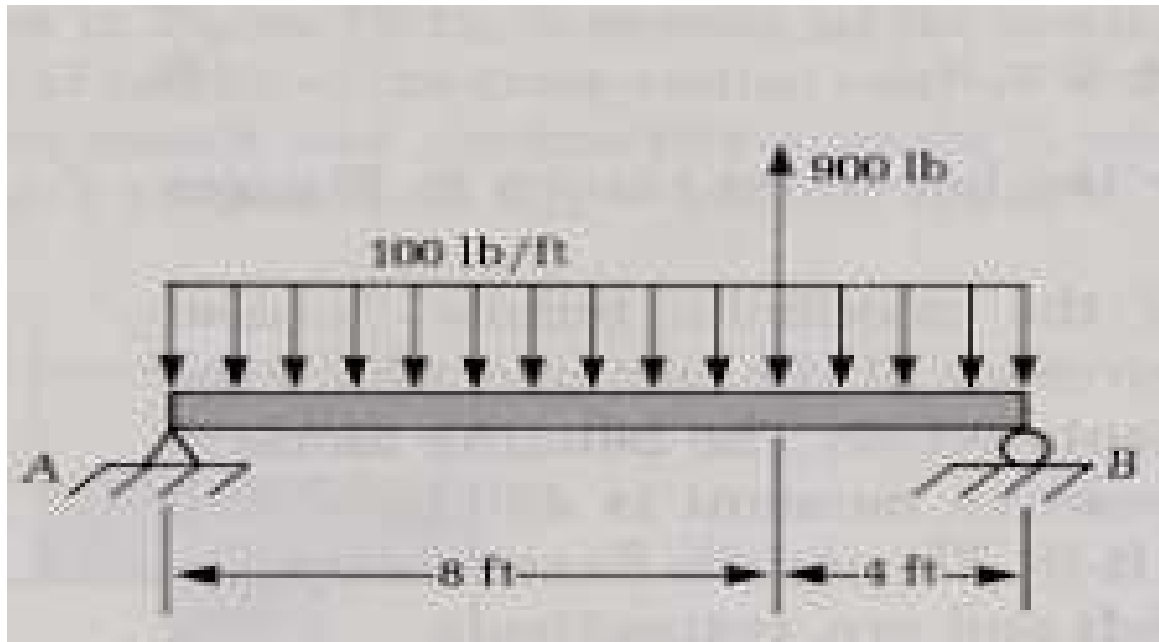
$$24(10^3) = \frac{M_z (3.25)}{33.8125}$$
$$M_z = 249692 \text{ lb} \cdot \text{in} = 20.8 \text{ kip} \cdot \text{ft}$$

b) $\sigma_{\max} = \frac{M_y c}{I_y}$

$$24(10^3) = \frac{M_z (3)}{9.0078125}$$
$$M_z = 72063 \text{ lb} \cdot \text{in} = 6.00 \text{ kip} \cdot \text{ft}$$



Example



- Determine the maximum tensile flexural stress and the maximum compressive flexural stress in the beam shown. The cross-section of the beam is also shown.

Solution

Find I

$$I = \frac{1}{36}bh^3 = \frac{1}{36}(2)(3)^3 = 1.5 \text{ in}^4$$

Draw shear and moment diagrams to figure out the maximum M

$$\sum M_A = 0: 12B + 900(8) - 1200(6) = 0 \quad \text{or} \quad B = 0$$

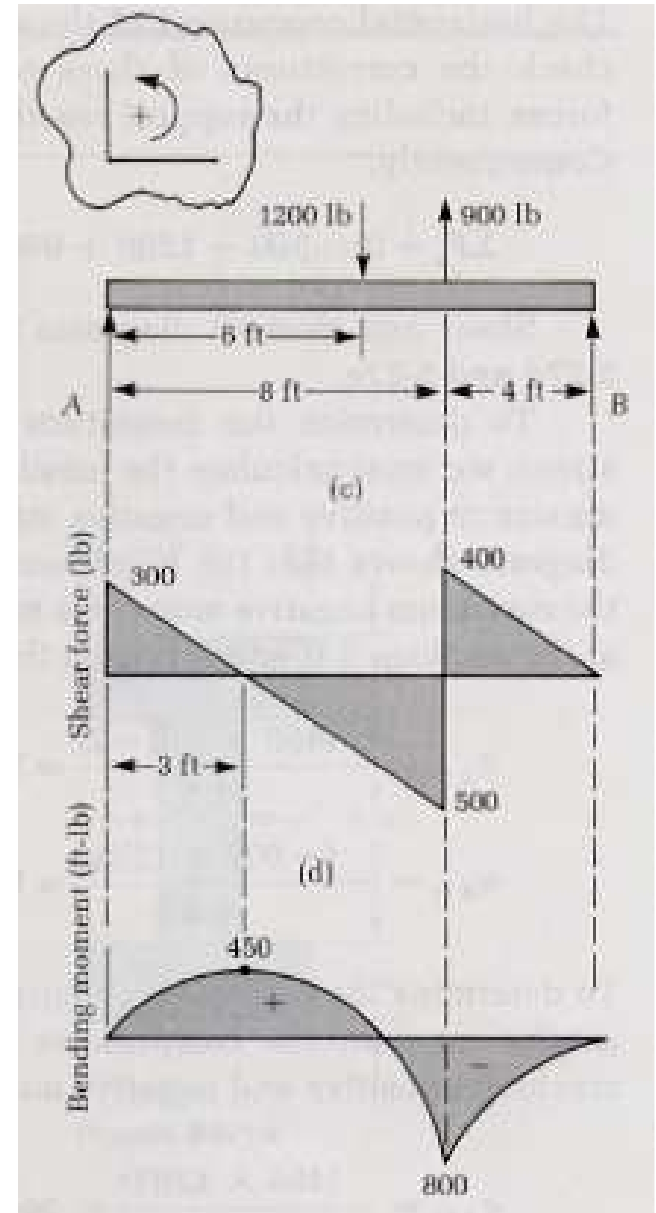
and

$$\sum M_B = 0: -12A - 900(4) + 1200(6) = 0 \quad \text{or} \quad A = 300 \text{ lb}$$

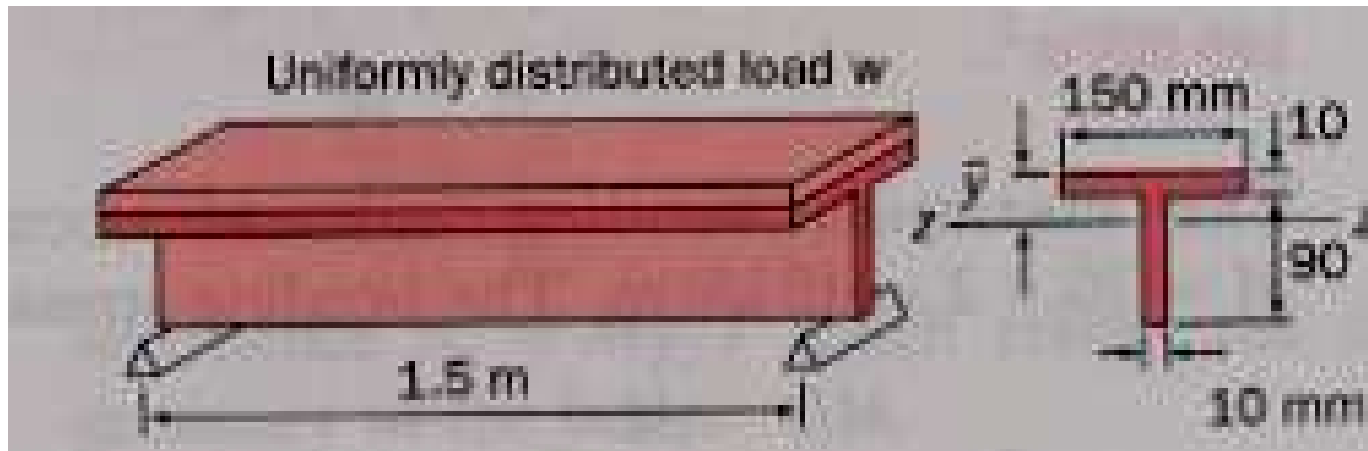
Investigate the points with high positive and negative moments

$$\left. \begin{aligned} \sigma_{3 \text{ ft}} &= -\frac{(450 * 12)(-2)}{1.5} = 7200 \text{ psi (T)} \\ \sigma_{8 \text{ ft}} &= -\frac{(-800 * 12)(1)}{1.5} = 6400 \text{ psi (T)} \end{aligned} \right\}$$

$$\left. \begin{aligned} \sigma_{3 \text{ ft}} &= -\frac{(450 * 12)(1)}{1.5} = 3600 \text{ psi (C)} \\ \sigma_{8 \text{ ft}} &= -\frac{(-800 * 12)(-2)}{1.5} = 12800 \text{ psi (C)} \end{aligned} \right\}$$



Example



- A T section bar has dimensions as shown. The bar is used as a simply supported beam of span 1.5m, the flange being horizontal as shown.
- Calculate the uniformly distributed load which can be applied if the maximum tensile stress in the material is not to exceed 100MN/m.
- What is the greatest bending stress in the flange?

Solution

Find the neutral axis location

$$(240 * 10) \bar{y} = (1500 * 5) + (900 * 55)$$

$$\bar{y} = 23.8 \text{ mm}$$

Find the moment of inertia(using parallel axes theorem)

$$I_z = \frac{150 * 10^3}{12} + (150 * 10 * 18.8^2) + \frac{10 * 90^3}{12} + (10 * 90 * 31.2^2) = 2.28 * 10^3 \text{ mm}^4$$

Use the distributed load to find the moment

$$M_{\max} = \frac{w * 1.5^2}{8} = 0.281 w \text{ Nm}$$

• The max. tensile stress will occur at the bottom so $y = y_{\max} = 76.2 \text{ mm}$

$$\frac{100 * 10^6}{0.0762} = \frac{0.281 w}{2028 * 10^{-9}}$$

$$w = \frac{202.8}{0.0762 * 0.281} = 9.5 \text{ kN / m}$$

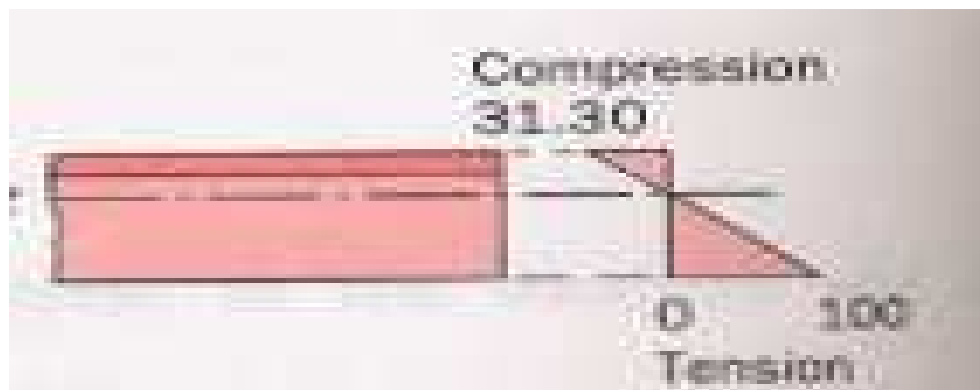


Solution

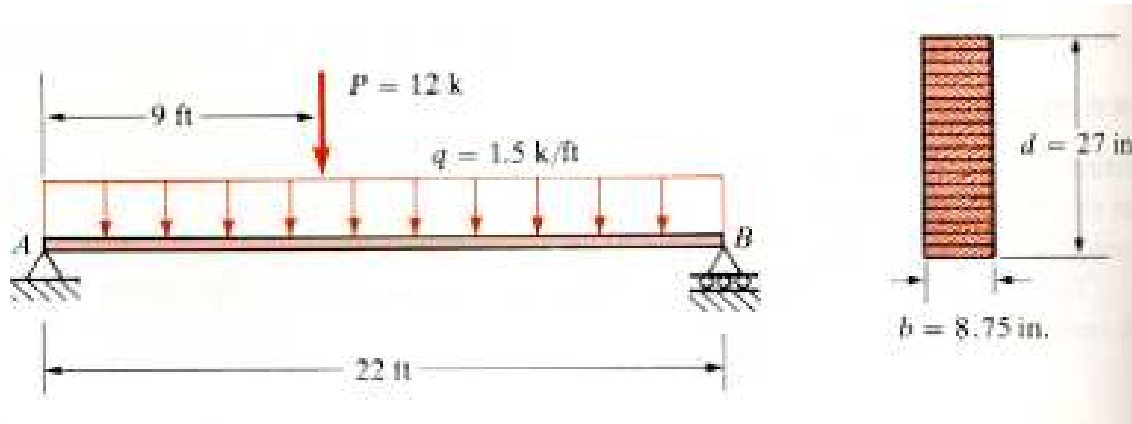
- Since stress is proportional to distance from the neutral axis:

$$\frac{\sigma_c}{y_c} = \frac{\sigma_t}{y_t}$$

$$\sigma_c = 100 * 10^6 * \frac{23.8}{76.2} = 31.3 \text{ MN} / \text{m}^2$$



Example



- A simple beam AB of span length $L=22\text{ft}$ supports a uniform load of intensity $q = 1.5\text{kip/ft}$ and a concentrated load $P = 12\text{kip}$. The beam is constructed of glued laminated wood with width $b = 8.75\text{in.}$ and depth $d = 27\text{in.}$
- Determine the maximum tensile and compressive stresses in the beam due to bending.

Solution

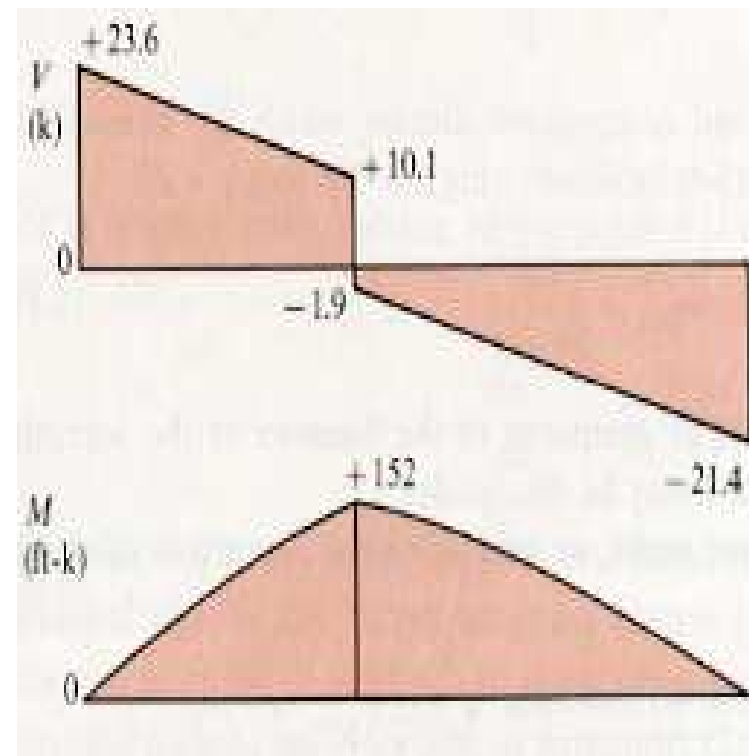
Draw the shear and bending Moment diagrams

$$M_{\max} = 152 \text{ ft} - \text{kip}$$

- Section modulus of the cross sectional area:

$$\text{Section modulus} \Rightarrow S = \frac{I}{y}$$

- Max. stresses, using flexure formula



$$S = \frac{bh^2}{6} = \frac{1}{6}(8.75\text{in.})(27\text{in})^2 = 1063\text{in}^3$$

$$\sigma_t = \sigma_1 = \frac{M}{S} = \frac{152 \text{ ft} - \text{k}}{1063\text{in}^3} = 1720 \text{ psi}$$

$$\sigma_t = \sigma_1 = -\frac{M}{S} = -1720 \text{ psi}$$

Solution

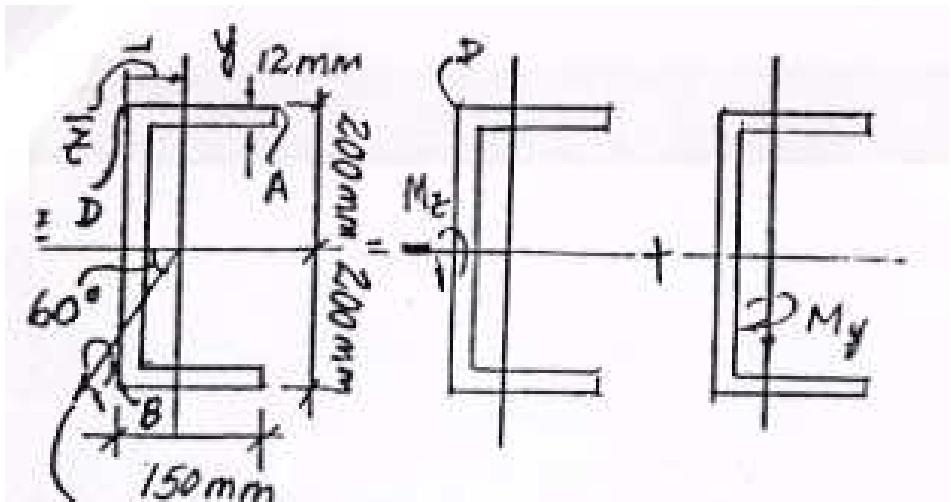
$$M = 800N * m$$

$$M_z = 800 \cos 60 = 400N * m$$

$$M_y = 800 \sin 60 = 692.82N * m$$

$$\bar{z} = \frac{400(12)(6) + 2(138)(12)(81)}{400(12) + 2(138)(12)}$$

$$= 36.6\text{mm}$$



$$I_z = \frac{1}{12} (0.15)(0.4^3) - \frac{1}{12} (0.138)(0.376^3)$$

$$= 1.8869(10^{-4})m^4$$

$$I_z = \frac{1}{12} (0.4)(0.012^3) - (0.4)(0.012)(0.03062^2)$$

$$+ 2 \frac{1}{12} (0.012)(0.138^3) + (0.138)(0.012)(0.04438^2)$$

$$= 1.63374(10^{-5})m^4$$

Solution

$$\sigma = \frac{-M_z y}{I_z} + \frac{M_y z}{I_y}$$

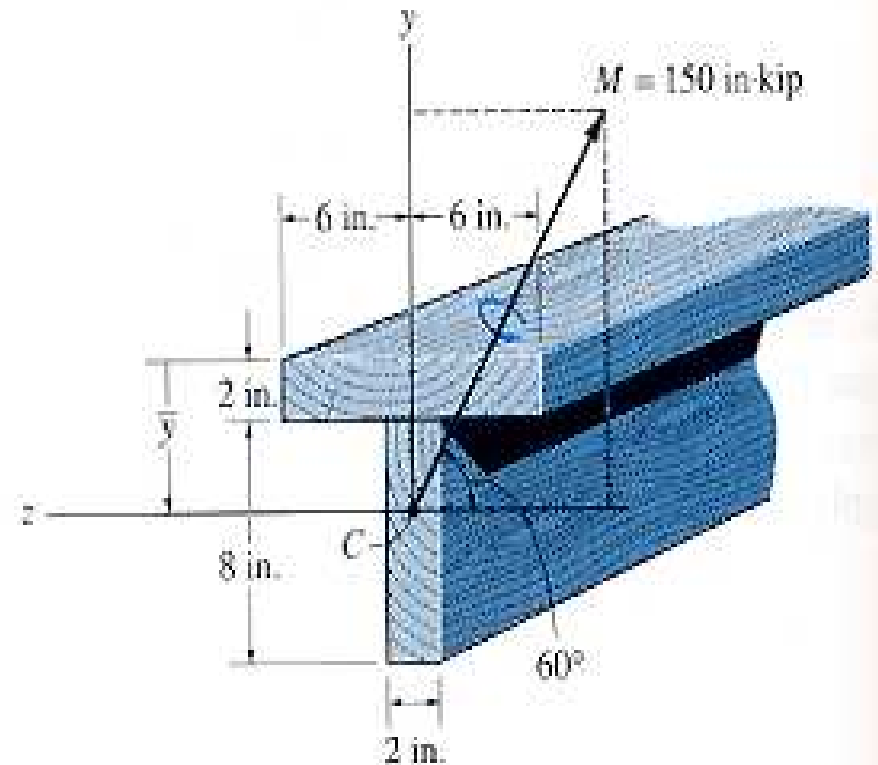
$$\begin{aligned}\sigma_A &= \frac{-400(0.2)}{1.8869(10^{-4})} + \frac{-692.82(-0.11338)}{1.63374(10^{-5})} \\ &= 4.38\text{MPa}\end{aligned}$$

$$\begin{aligned}\sigma_B &= \frac{-400(-0.2)}{1.8869(10^{-4})} + \frac{-692.82(0.036621)}{1.63374(10^{-5})} \\ &= -1.13\text{MPa}\end{aligned}$$

$$\begin{aligned}\sigma_D &= \frac{-400(0.2)}{1.8869(10^{-4})} + \frac{-692.82(0.036621)}{1.63374(10^{-5})} \\ &= -1.977\text{MPa}\end{aligned}$$

Problem 12.26

- The T beam is subjected to a bend moment of $M = 150$ kip-in directed as shown.
- Determine the maximum bending stress in the beam and the orientation of the neutral axis.
- The location y of the centroid, c , must be located.



Prob. 12-26

Solution

$$M_y = 150 \sin 60 = 129.9 \text{ kip} \cdot \text{in}$$

$$M_z = -150 \cos 60 = -75 \text{ kip} \cdot \text{in}$$

$$\bar{y} = \frac{12(2)(1) + 8(2)(6)}{12(2) + (8)(2)} = 3 \text{ in}$$

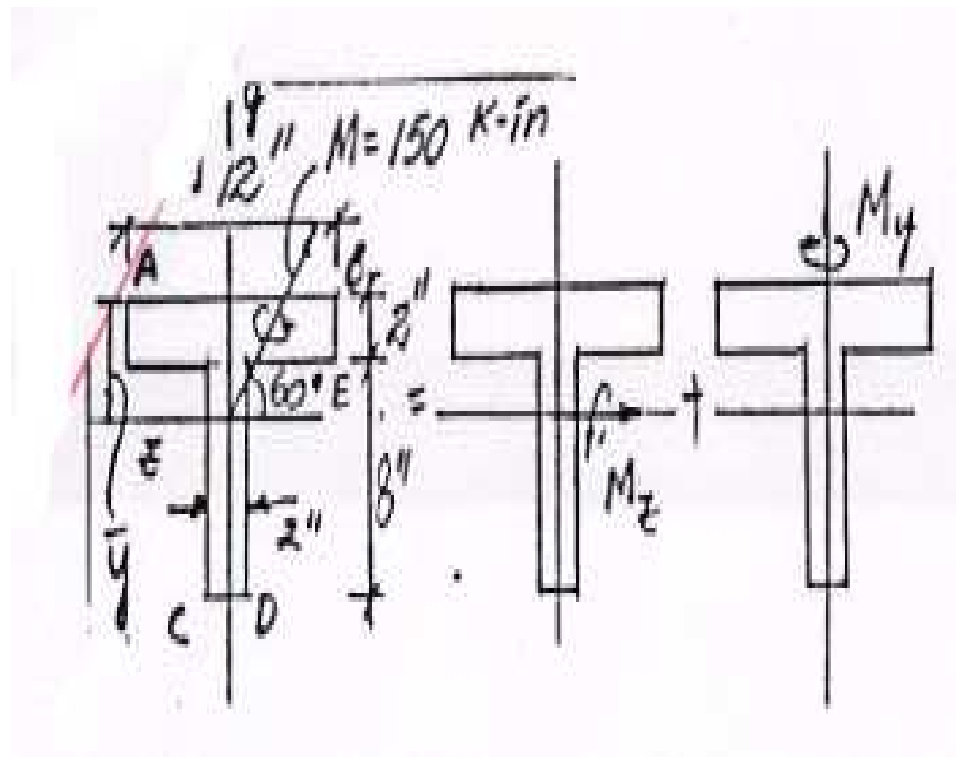
$$I_y = \frac{1}{12} (2)(12^3) + \frac{1}{12} (8)(2^3)$$

$$= 293.33 \text{ in}^4$$

$$I_z = \frac{1}{12} (12)(2^3) + (12)(2)(2^2)$$

$$+ \frac{1}{12} (2)(8^3) + (2)(8)(3^2)$$

$$= 333.33 \text{ in}^4$$



Solution

$$\sigma = -\frac{M_z y}{I_z} + \frac{M_y z}{I_y}$$

$$\sigma_A = \frac{-(-75)(3)}{333.33} + \frac{129.9(6)}{293.33} = 3.33 \text{ksi}$$

$$\sigma_D = \frac{-(-75)(-7)}{333.33} + \frac{129.9(-1)}{293.33} = -2.02 \text{ksi}$$

$$\sigma_B = \frac{-(-75)(3)}{333.33} + \frac{129.9(-6)}{293.33} = -1.982 \text{ksi}$$

$$\frac{z'}{1.982} = \frac{12 - z'}{3.333}$$

$$z' = 4.47 \text{in}$$

$$\tan \alpha = \frac{I_z}{I_y} = \tan \theta$$

$$\tan \alpha = \frac{333.33}{293.33} \tan(-60^\circ)$$

$$\alpha = 63.1^\circ$$

