



Mechanics & Materials 1

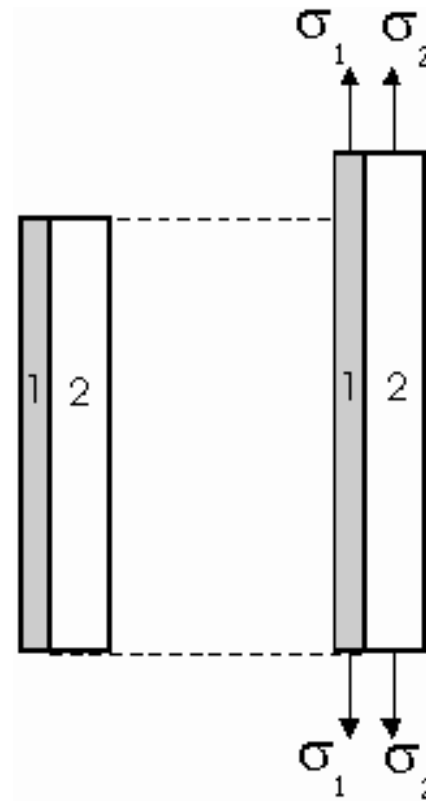
Chapter 10 - continued

Axial Loading

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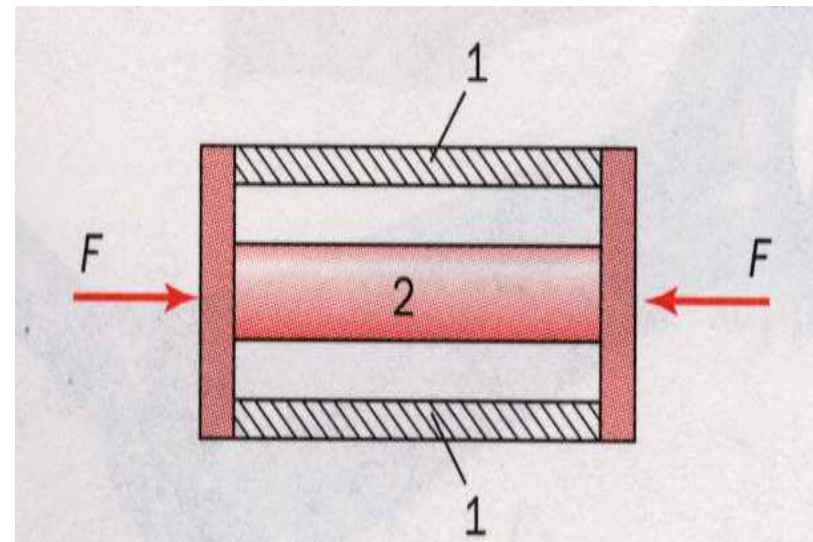
Composite Bars in Tension or Compression

- Any tensile or compressive member which consists of two or more bars or tubes **in parallel**, usually of different materials, is called a compound bar.
- A composite bar is one made of two materials, such as steel rods embedded in concrete.
- The construction of the bar is such that **constituent components extend or contract equally** under load.



Example: Indeterminate Structure

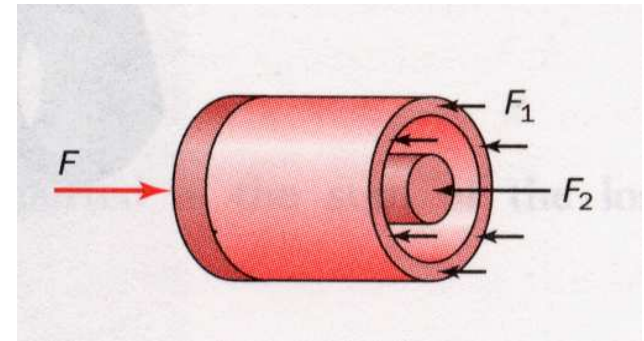
- Two components of different materials are arranged concentrically and loaded through rigid end plates as shown.
- Determine the force carried by each component.



Solution

• Equilibrium: In this case the load is shared in some unknown proportions between two parts so that

$$F_1 + F_2 = F$$



• Geometry of deformation (compatibility): If the unloaded lengths, l , are initially the same, then they will remain the same under load; hence

$$\delta_1 = \delta_2 = \delta$$

➤ Stress-strain relations: For a simple uniaxial situation,

$$\frac{\sigma_1}{\epsilon_1} = E_1$$

and

$$\frac{\sigma_2}{\epsilon_2} = E_2$$

Solution

- From equilibrium and stress equations

$$F_1 = E_1 A_1 \frac{\delta}{l} \quad \text{and} \quad F_2 = E_2 A_2 \frac{\delta}{l}$$

- Substituting the equilibrium equation

$$E_1 A_1 \frac{\delta}{l} + E_2 A_2 \frac{\delta}{l} = F$$

- thus

$$\delta = \frac{Fl}{E_1 A_1 + E_2 A_2}$$

$$F_1 = \frac{FE_1 A_1}{E_1 A_1 + E_2 A_2} \quad \text{and} \quad F_2 = \frac{FE_2 A_2}{E_1 A_1 + E_2 A_2}$$

Composite Bars in Tension or Compression

- To illustrate the behavior of such bars consider a rod made of two materials, 1 and 2, A_1 , A_2 are the cross-sectional area of the bars, and E_1 , E_2 are values of Young's modulus.
- We imagine the bars to be rigidly connected together at the ends; then for the longitudinal strain to be the same when the composite bar is stretched we must have

$$\varepsilon = \frac{\sigma_1}{E_1} = \frac{\sigma_2}{E_2}$$

Composite Bars in Tension or Compression

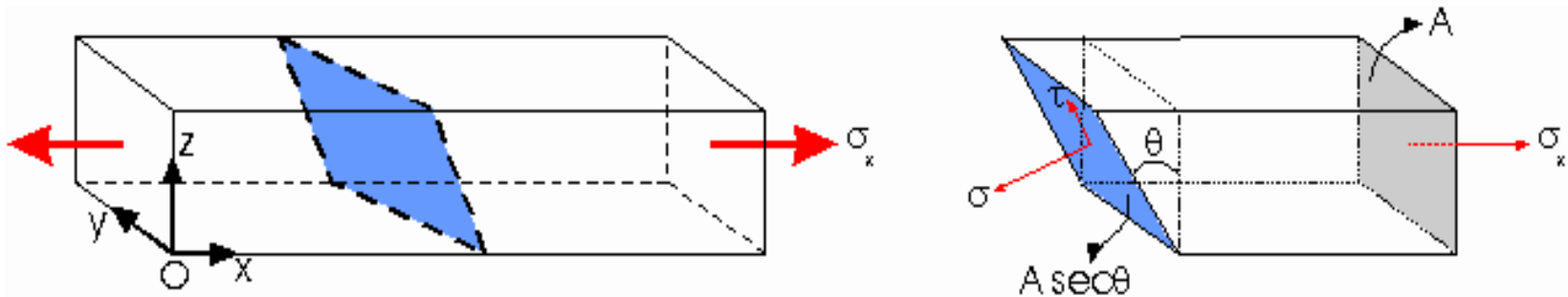
- where σ_1 and σ_2 are the stresses in the two bars.
But the total tensile load is

$$P = \sigma_1 A_1 + \sigma_2 A_2$$

- Together with the strain equation we obtain

$$\sigma_1 = \frac{PE_1}{A_1 E_1 + A_2 E_2}$$
$$\sigma_2 = \frac{PE_2}{A_1 E_1 + A_2 E_2}$$

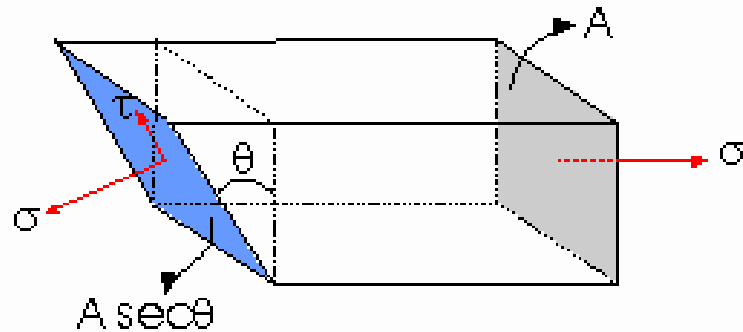
Shearing Stress in Axial Loaded Member



Stresses on an inclined plane through a tensile test piece

- The bar is uniformly stressed in tension in the x direction, the tensile stress on a cross-section of the bar parallel to Ox being σ_x .
- Consider the stresses acting on an inclined cross-section of the bar; an inclined plane is taken at an angle θ to the yz -plane.

Shearing Stress in Axial Loaded Member



$$F_n = A' \sigma_x \cos \theta$$
$$F_t = A' \sigma_x \sin \theta$$

- For equilibrium the resultant force parallel to Ox on an inclined cross section is also $P = A' \sigma_x$
- At the inclined cross-section, resolve the force $A' \sigma_x$ into two components-one perpendicular, and the other tangential, to the inclined cross-section,.

Shearing Stress in Axial Loaded Member

- The normal and tangential forces acting on the inclined surface are given as:
- the area of the inclined surface A' is:
- so that the normal and tangential stresses acting on the inclined cross-section are :

$$F_n = A' \sigma_x \cos \theta$$

$$F_t = A' \sigma_x \sin \theta$$

$$A' = A \sec \theta = \frac{A}{\cos \theta}$$

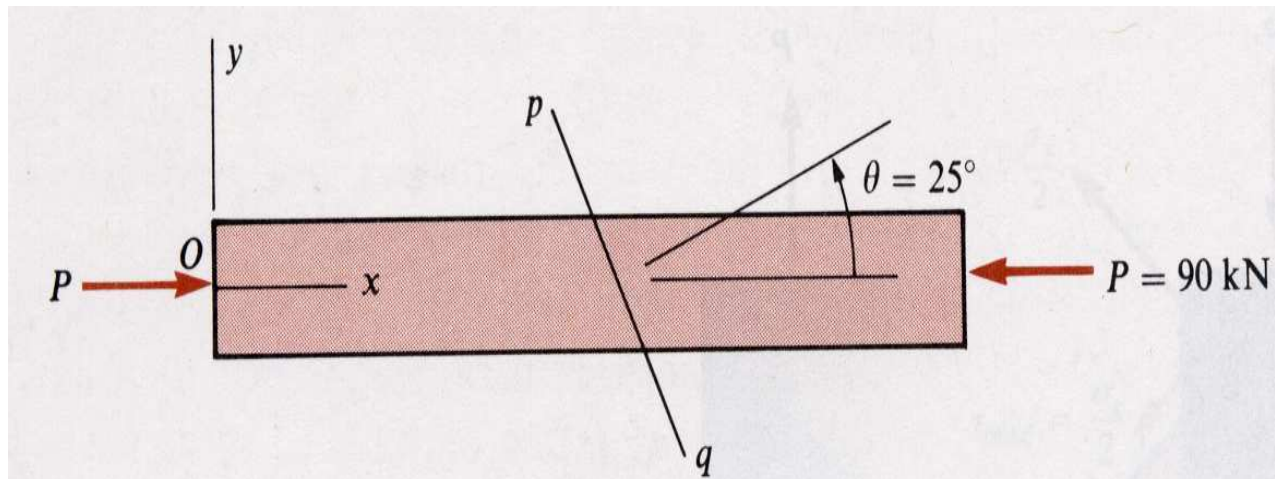
$$A' = A \sec \theta = \frac{A}{\cos \theta}$$

$$\sigma'_x = \frac{F_n}{A'} = \frac{A \sigma_x \cos \theta}{A \sec \theta} = \sigma_x \cos^2 \theta$$

$$\tau' = \frac{F_t}{A'} = \frac{A \sigma_x \sin \theta}{A \sec \theta} = \sigma_x \cos \theta \sin \theta$$

Example: Stress on Inclined Surface

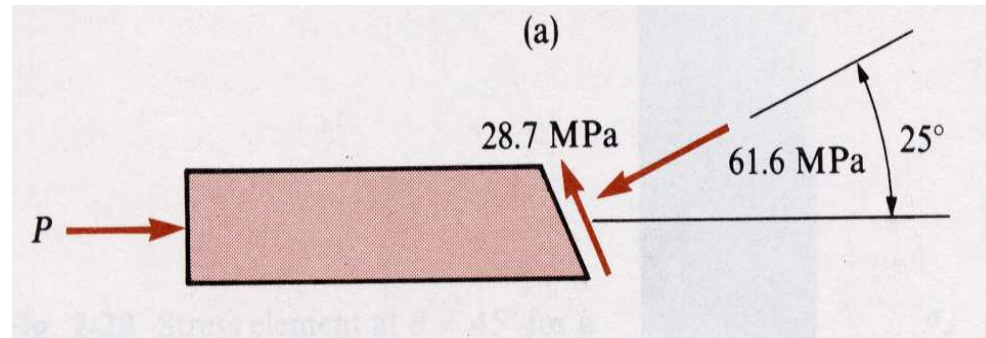
- A prismatic bar has a cross-sectional area $A=1200\text{mm}^2$ and is compressed by a load $P=90\text{kN}$. Determine the stresses acting on the plane cut through the bar at an angle $\theta = 25^\circ$. Then show the complete state of stress for the angle by determining the stresses on all faces of the stress element.



Solution

- The normal stress acting on a cross section is:

$$\sigma_x = -\frac{P}{A} = -\frac{90 \text{ kN}}{1200 \text{ mm}^2} = -75 \text{ MPa (compression)}$$



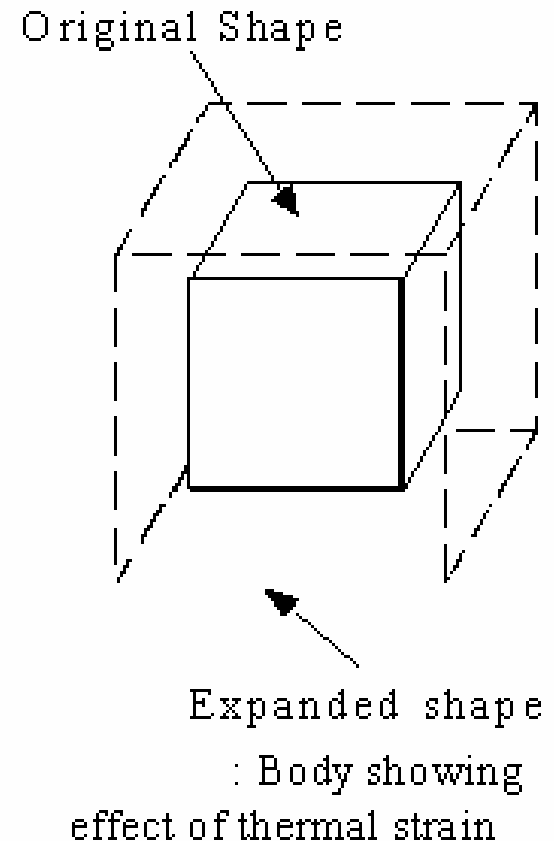
Substituting this value into the equations and using $\theta = 25^\circ$ we obtain

$$\sigma_\theta = \sigma_x \cos^2 \theta = (-75 \text{ MPa})(\cos 25^\circ)^2 = -61.6 \text{ MPa}$$

$$\tau_\theta = -\sigma_x \sin \theta \cos \theta = (75 \text{ MPa})(\sin 25^\circ)(\cos 25^\circ) = 28.7 \text{ MPa}$$

Thermal Stress

- When the temperature of a body is changed its overall size will also change.
- If the temperature is increased all directions will increase, and if the temperature is decreased all dimensions will contract.
- For the majority of engineering materials this relationship is linear.



Thermal Stress

- The thermal strain formula is then:

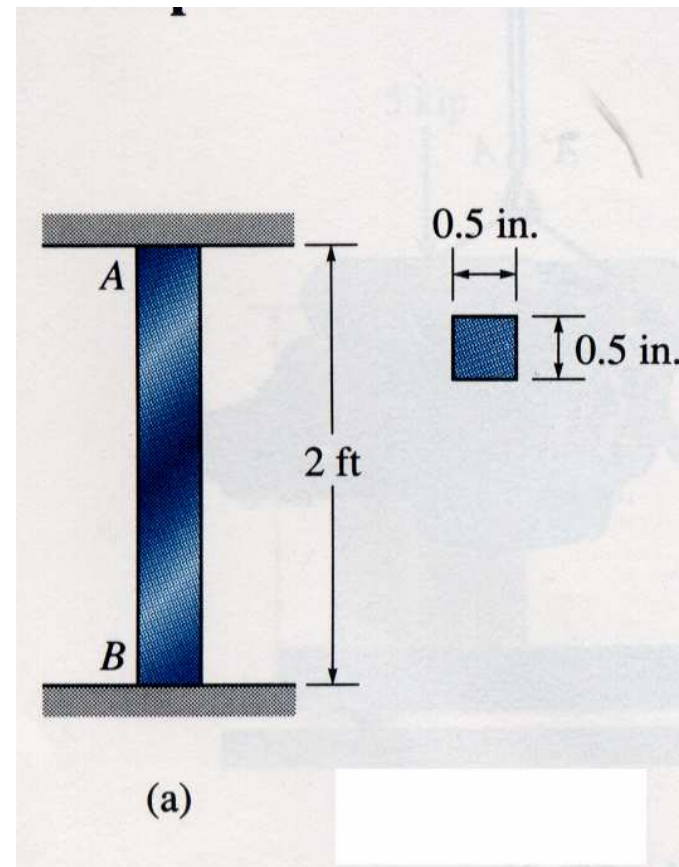
$$\Delta L = \alpha \Delta T L$$

Or, in terms of strain: $\epsilon_{\text{thermal}} = \alpha \Delta T$

- Where
 - α is the coefficient of thermal expansion in units of 1/temperature
 - L is the original length of the sample
 - ΔL is the change in length due to the change in temperature

Example : Thermal Stress

- The steel bar shown is constrained to just fit between two fixed supports when $T_1 = 60^\circ \text{ F}$. If the temperature is raised to $T_2 = 120^\circ \text{ F}$, determine the average normal thermal stress developed in the bar. Take $E_{ST} = 29(10^3) \text{ ksi}$ and $\alpha_{ST} = 6.5(10^{-6})/^\circ \text{ F}$.



Solution

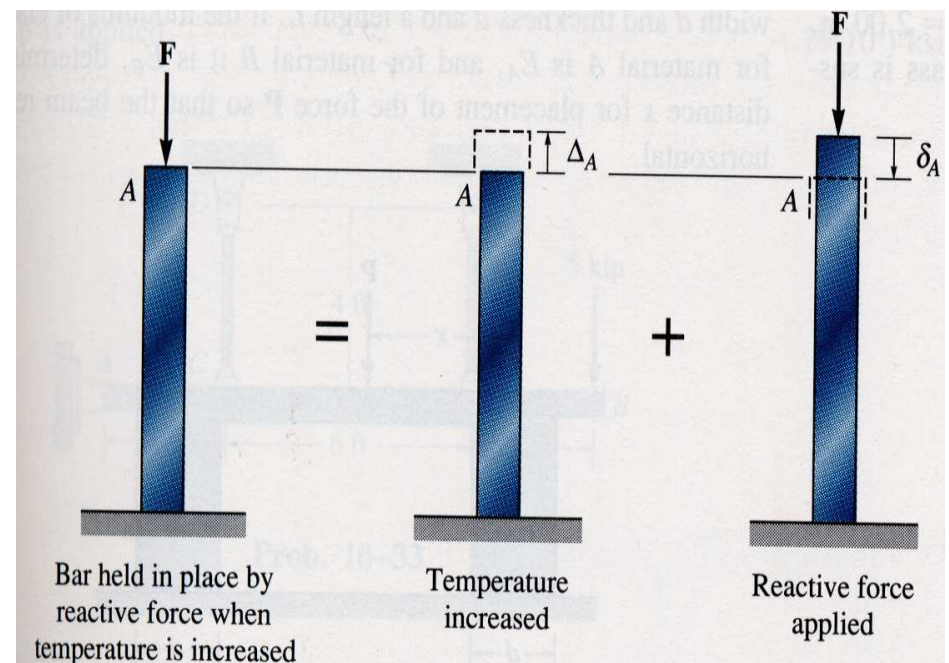
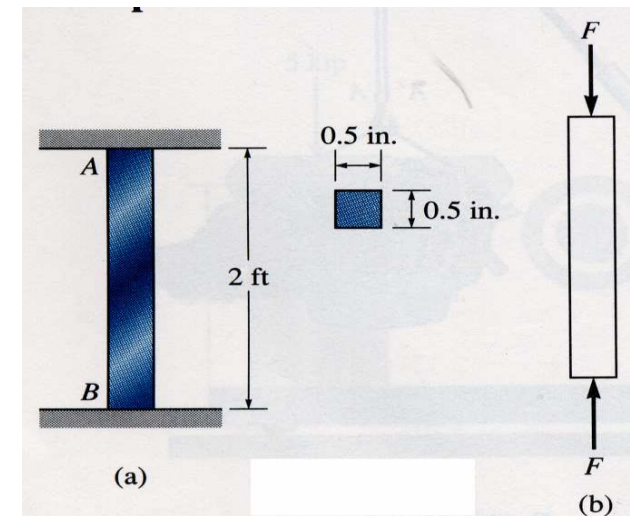
- Since there is no external load, the force at A is equal but opposite to the force acting on B.

$$F_A = F_B = F$$

- The problem is **statistically indeterminate**, since this force cannot be determined from equilibrium

Using the principle of **superposition**, the redundant support at A is removed, and the thermal displacement Δ_A at A occurs. The force F , developed at the redundant, A, then pushes the bar δ_A back to its original position; i.e., the compatibility condition at A becomes

$$0 = \Delta_A - \delta_A$$



Solution

- Applying the thermal and load-displacement relationships:

- Thus
$$0 = \alpha \Delta T L - \frac{FL}{AE}$$

$$F = \alpha \Delta T A E$$

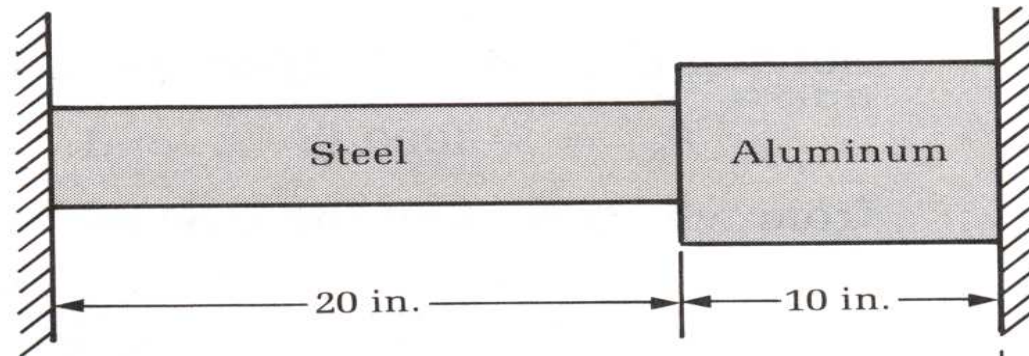
$$\begin{aligned} &= [6.5(10^{-6})/^{\circ}F](120^{\circ}F - 60^{\circ}F)(0.5in.)^2 [29(10^3)]kip/in^2 \\ &= 2.83kip \end{aligned}$$

$$\sigma = \frac{F}{A} = \frac{2.83 \text{ kip}}{(0.5 \text{ in.})^2} = 11.3 \text{ ksi}$$

What would the stress be if the temperature had DECREASED?

Example : Thermal Stress

- Suppose that the temperature is allowed to increase by 10°F . Determine the stress in each material due to the temperature change alone. The thermal expansion coefficients for steel and aluminum are $6.5 \times 10^{-6}\text{in/in/}^{\circ}\text{F}$ and $13 \times 10^{-6}\text{in/in/}^{\circ}\text{F}$, respectively.



- From the Free-body diagram

$$\xrightarrow{+} \sum F_x = 0; \quad P_{ST} - P_{AL} = 0$$

- Under the unconstrained thermal expansion the composite rod will elongate an amount

$$\Delta = \Delta_{ST} + \Delta_{AL}$$

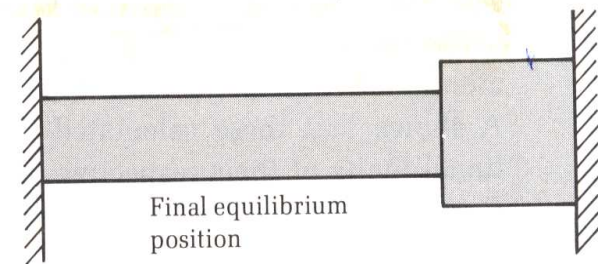
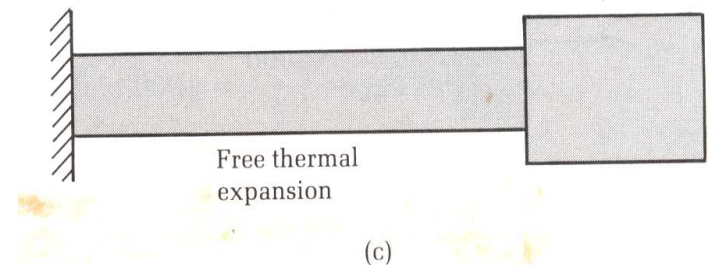
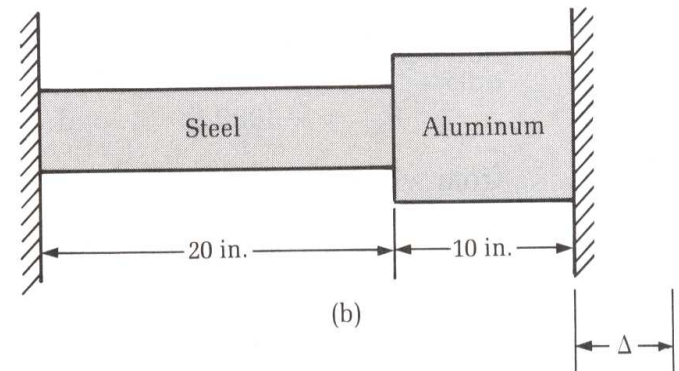
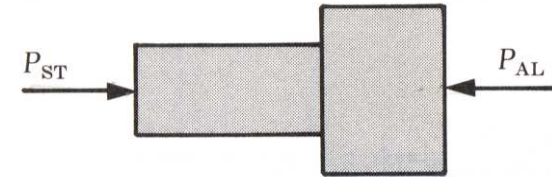
- They are:

$$\Delta_{ST} = \alpha_{ST} l_{ST} \Delta T$$

$$\Delta_{AL} = \alpha_{AL} l_{AL} \Delta T$$

- Thus

$$\begin{aligned} \Delta &= (6.5 * 10^{-6})(20)(10) + (13 * 10^{-6})(10)(10) \\ &= 0.0013 + 0.0013 \\ &= 0.0026 \text{ in.} \end{aligned}$$



Solution

• Now the walls constrain the composite rod from expanding this amount. The net change in length of the composite rod is zero. Therefore, internal strains must be induced in the steel and the aluminum so that the sum of the deformations in the steel and the aluminum caused by thermal expansion and internal forces is zero.

$$\delta_{ST} + \delta_{AL} - 0.0026 = 0$$

$$\delta_{ST} = \frac{P_{ST} l_{ST}}{A_{ST} E_{ST}} \quad \text{and} \quad \delta_{AL} = \frac{P_{AL} l_{AL}}{A_{AL} E_{AL}}$$

$$\text{and} \quad P_{ST} - P_{AL} = 0$$

Solution

- These equations yield:

$$P_{ST} \left\{ 1 + \frac{E_{ST}}{E_{AL}} \frac{A_{ST}}{A_{AL}} \frac{l_{AL}}{l_{ST}} \right\} = \frac{E_{ST} A_{ST}}{l_{ST}} 0.0026$$

$$P_{ST} \left\{ 1 + \frac{30}{10} \frac{2}{4} \frac{10}{20} \right\} = \frac{30 \times 10^6 (2)}{20} 0.0026$$

$$\frac{7}{4} P_{ST} = 7800$$

- Consequently:

$$P_{ST} = P_{AL} = 4457 \text{ lb}$$

The corresponding stresses are:

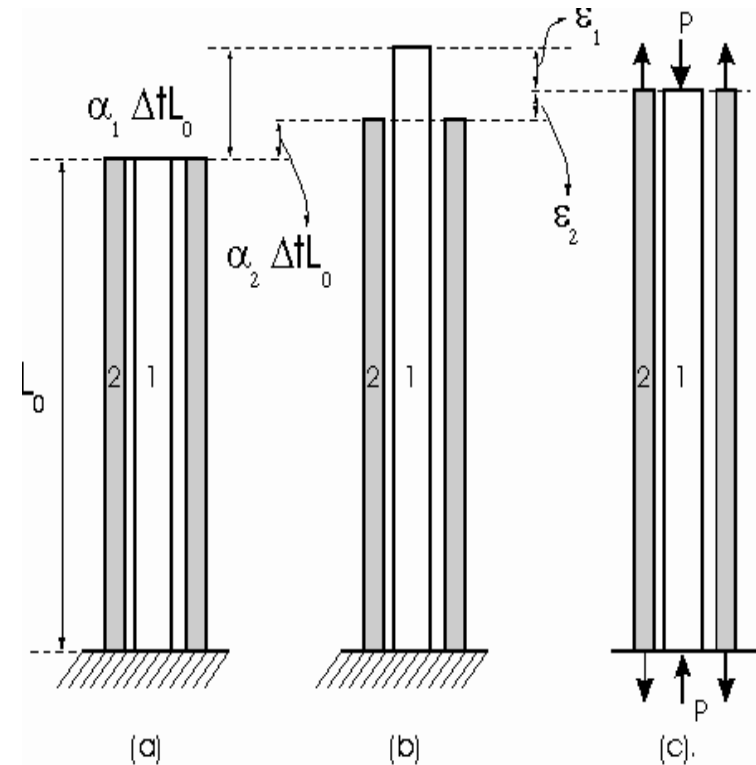
$$\sigma_{ST} = \frac{4457}{2} = 2228.5 \text{ psi (C)}$$

Why is the stress in the Al so much higher?

$$\sigma_{AL} = \frac{4457}{4} = 1114.25 \text{ psi (C)}$$

Example: Thermal Stress

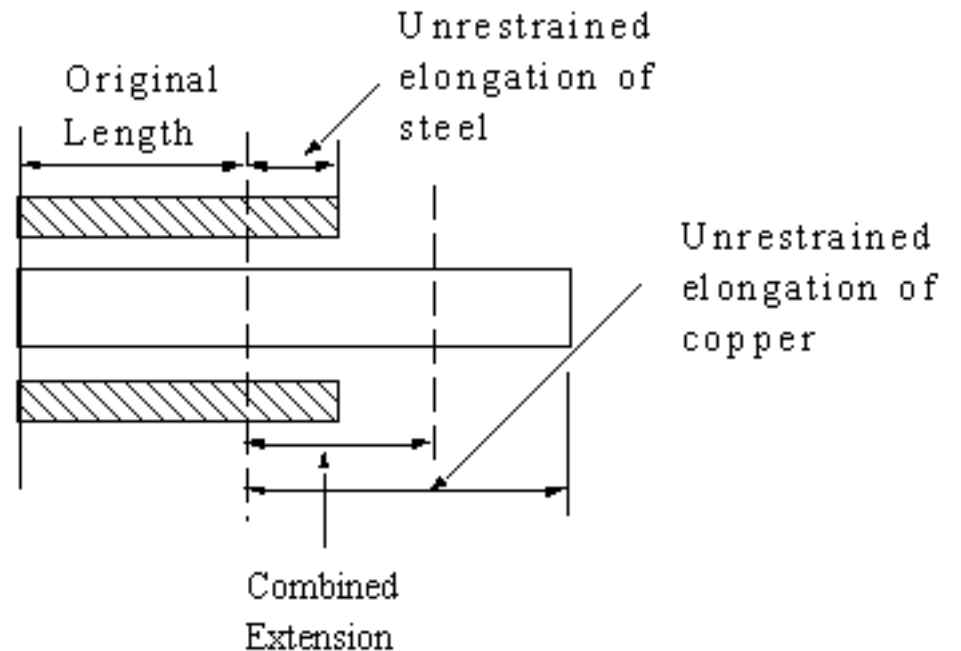
- A steel tube 24 mm OD, 18 mm ID encloses a copper rod 15 mm OD, they are both rigidly attached to a wall at one end and are attached to a rigid plate at the other. At 10°C there is no stress. Calculate the stress in the rod and tube at 200°C. For steel $E_s = 210$ GPa, $\alpha_s = 11 \times 10^{-6} / ^\circ\text{C}$; for copper $E_c = 100$ GPa, $\alpha_c = 18 \times 10^{-6} / ^\circ\text{C}$



Temperature stress in a composite bar

Solution

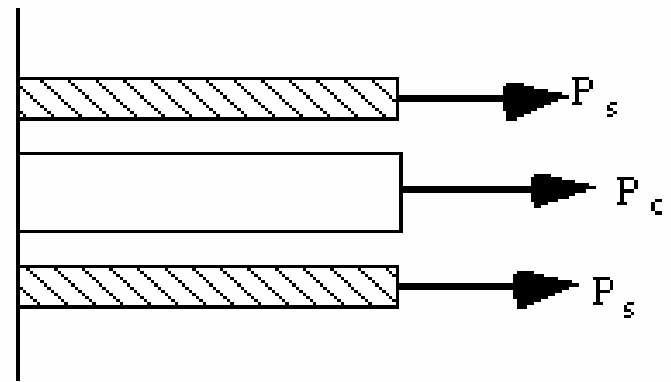
- If both were not attached to the rigid plate then they would just extend freely, however because they are both attached to a rigid plate, the copper is going to be prevented from being fully expanded by the steel. And the steel is going to be expanded further by the copper. Producing a combined elongation as shown.



Steel tube and copper rod shown when freely elongated by change in temperature together with combined elongation

Solution

- Because both the steel and copper are attached, the change in temperature is going to produce an internal load on both. But because we don't know their orientation we assume them to be positive, and draw the following FBD.



FBD of cut combined structure indicating unknown forces carried by each material

Solution

- Using the equations of statics:

$$\overset{+}{\longrightarrow} \sum F_x = 0 = P_c + P_s = 0$$

- Which gives us only one equation with two unknowns. As no more equations of statics can be used, we now need to come up with a compatibility equation relating the displacement of both the tube and the rod.
- Because both the rod and tube are attached to the rigid plate then their displacement must be the same. Which gives that:

$$\delta_c = \delta_s$$

Solution

- Divide δ by the length, since both materials have same length
- The stress strain relation for combined thermal and mechanical loading
- Putting stress in terms of load and area
- From statics we have $P_c = -P_s$
Substitute into the stress equation

Substitute the values given we find the stresses

$$\epsilon_c = \epsilon_s$$

$$\epsilon_c = \epsilon_{stress} + \epsilon_{thermal} = \alpha_c \Delta T + \frac{\sigma_c}{E_c}$$

$$\epsilon_s = \epsilon_{stress} + \epsilon_{thermal} = \alpha_s \Delta T + \frac{\sigma_s}{E_s}$$

$$\alpha_c \Delta T + \frac{\sigma_c}{E_c} = \alpha_s \Delta T + \frac{\sigma_s}{E_s}$$

$$\alpha_c \Delta T + \frac{P_c}{A_c E_c} = \alpha_s \Delta T + \frac{P_s}{A_s E_s}$$

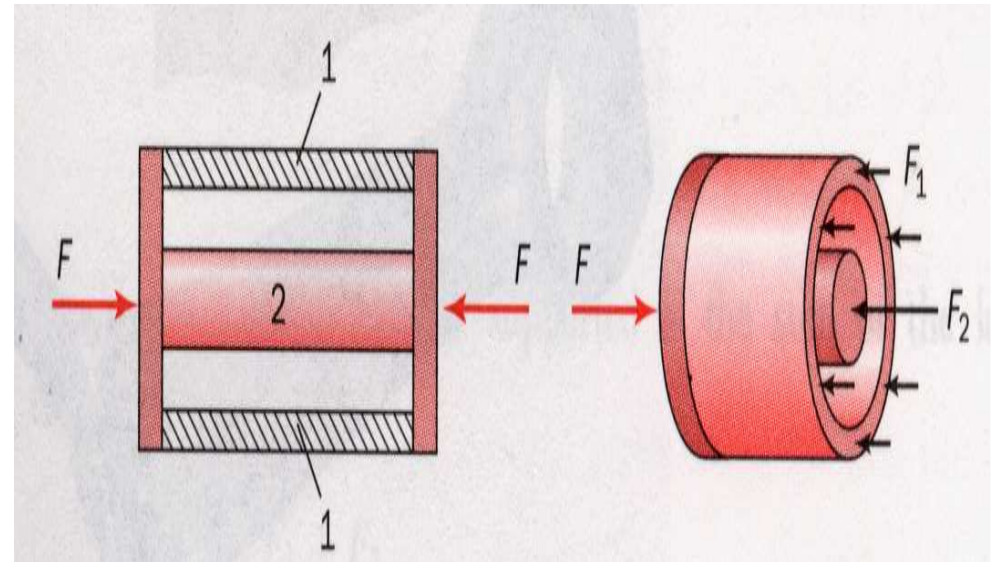
$$P_c = \frac{(\alpha_s - \alpha_c) \Delta T}{\frac{1}{A_c E_c} + \frac{1}{A_s E_s}}$$

$$\sigma_c = \frac{P_c}{A_c} = -93.3 \text{ MPa} \text{ and } \sigma_s = \frac{P_s}{A_s} = 83.9 \text{ MPa}$$

Example: Statically Indeterminate Member with Thermal Stress

- The bimetallic component illustrated in the figure consists of a steel rod of cross sectional area 600mm^2 coaxially surrounded by a copper tube of cross-sectional area 1200mm^2 . It is not subjected to any external load but its temperature is changed from 20°C to 100°C .
- Determine the axial stresses set up in the copper and the steel.

$$E_s = 205\text{GN} / \text{m}^2, E_c = 115\text{GN} / \text{m}^2$$
$$\alpha_s = 11 \times 10^{-6} / ^\circ\text{C}, \alpha_c = 16 \times 10^{-6} / ^\circ\text{C}$$



Solution

- Since there is no applied external force, the sum of the internal forces in the copper and steel must be zero. Therefore

$$F_c + F_s = 0$$

or

$$\sigma_c A_c + \sigma_s A_s = 0$$

- Since the two materials are initially stress-free and their ends are fixed together the total strain must be the same for each.

Therefore

$$\epsilon_c = \epsilon_s \quad \text{or} \quad (\epsilon_\sigma + \epsilon_T)_c = (\epsilon_\sigma + \epsilon_T)_s$$

- Where ϵ_σ = strain due to stress, ϵ_T = strain due to temperature change

Solution

•Stress- strain relation:

$$\varepsilon_c = \frac{\sigma_c}{E_c} + \alpha_c (T - T_0)$$

•Equating ε_c and ε_s from the equations:

$$\varepsilon_c = \frac{\sigma_s}{E_s} + \alpha_s (T - T_0)$$

$$\frac{\sigma_c}{E_c} + \alpha_c (T - T_0) = \frac{\sigma_s}{E_s} + \alpha_s (T - T_0)$$

•Eliminating σ_s gives

$$\frac{\sigma_c}{E_c} + \alpha_c (T - T_0) = -\frac{\sigma_c}{E_s} \frac{A_c}{A_s} + \alpha_s (T - T_0)$$

$$\sigma_c \left(\frac{1}{E_c} + \frac{A_c}{E_s A_s} \right) = (T - T_0)(\alpha_s - \alpha_c)$$

or

$$\sigma_c = \frac{A_s E_s E_c (T - T_0)(\alpha_s - \alpha_c)}{A_s E_s + A_c E_c}$$

Solution

and
$$\sigma_s = -\frac{A_c E_c E_s (T - T_0)(\alpha_s - \alpha_c)}{A_s E_s + A_c E_c}$$

The negative sign for σ_s does not necessarily indicate a component of stress but simply that it is opposite in sign to σ_c . The type of stress in the material is determined by the numerical values of the quantities $(T - T_0)$ and the difference in the coefficient of thermal expansions. Substituting the values into the equations gives :

$$\begin{aligned}\sigma_c &= -21.7 \text{ MPa} && \text{and} \\ \sigma_s &= 43.4 \text{ MPa}\end{aligned}$$

Thus for an increase in temperature σ_c being greater than σ_s , the component is prevented from expanding as much as if it were free and is put in compression.

Example: Statically Indeterminate with Thermal Stress

- A copper ring having an internal diameter of 150mm and an external diameter of 154mm is to be shrunk into a steel ring, of the same width having internal and external diameters of 140mm and 150.5mm respectively.
- What change in temperature is required in the copper ring so that it will just slide on to the steel ring?
- What will the uniform circumferential stress in each ring and the interface pressure when assembled and back at room temperature?
- Assume $E_c = 100\text{GN/m}^2$, $\alpha_c = 18 \times 10^{-6}/^\circ\text{C}$

Solution

- The circumferential length of the copper ring has to be increased by heat till it is fractionally larger than the circumferential length of the steel ring
- Minimum required change in circumference = $\pi d_s - \pi d_c$
 $= \pi \times 0.05$

$$\text{Change in } C \text{ due to heating} = \pi d_c \times \alpha(T - T_0)$$

$$= \pi \times 150 \times 18 \times 10^{-6}(T - T_0)$$

- Therefore

$$T - T_0 = \frac{0.05}{2700 \times 10^{-6}} = 18.5^\circ C$$

Let the width of the ring be w ; then, the thickness of 2 and 5mm respectively

$$(w \times 2)\sigma_c + (w \times 5)\sigma_s = 0$$

$$\sigma_c = -2.5\sigma_s$$

Solution

- The circumferential strains of in the copper and steel must be the same at the mating surface, so

$$\varepsilon_s = \varepsilon_c$$

- Stress-strain relations

$$\varepsilon_s = \frac{\sigma_s}{E_s} \quad \text{and} \quad \varepsilon_c = \frac{\sigma_c}{E_c} + \alpha_c \Delta T$$

Since it is only the copper ring that has the thermal strain component.

$$\frac{\sigma_s}{E_s} = \frac{\sigma_c}{E_c} + \alpha_c \Delta T$$
$$\sigma_s \left(\frac{1}{E_s} + \frac{2.5}{E_c} \right) = -18 \times 10^{-6} \times 18.5$$

Solution

- The negative sign is due to ΔT being a reduction in temperature

- Substituting for E_s and E_c

- $\sigma_s = -11.15\text{MN/m}^2$ and $\sigma_c = 27.9\text{MN/m}^2$

- The radial pressure at the interface between the two rings may be treated as a thin cylinder under internal or external pressure so that

$$\rho = \sigma_c t/r = 27.9 \times 2/75 = 745\text{kN/m}^2$$