Mechanics & Materials 1

Chapter 10

Axial Loading

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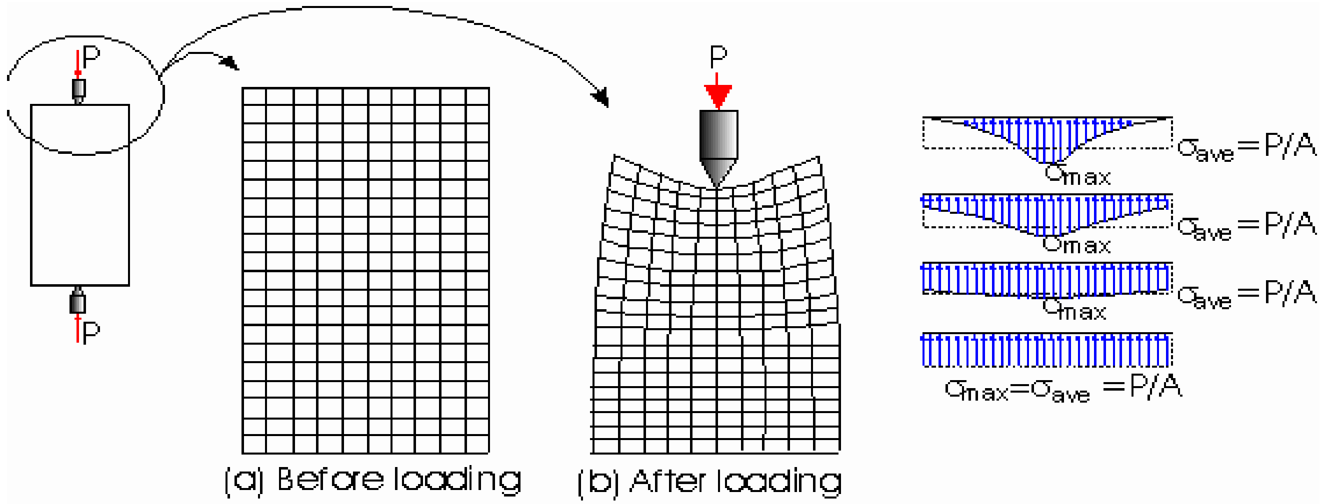
Department of Mechanical Engineering

Saint-Venant's Principle

- This principle states that: "*the actual distribution of the load over the surface of its application will not affect the distribution of stress or strain on sections of the body which are at an appreciable distance (relative to the dimensions) away from the load*".
- Any convenient statically equivalent loading may therefore be substituted for the actual load distribution, provided that the stress analysis in the region of the load is not required.



Deformation of a grid pattern on an axially loaded rubber bar.
(Top part of the bar shown in detail)

- The concentrated load produces a highly nonuniform stress distribution and large local stresses near the load. However, the stress smoothes out to a nearly uniform distribution
- This smoothing out of the stress distribution is an illustration of Saint-Venant's principle. He observed that near loads, high localized stresses may occur, but away from the load at a distance equal to the width or depth of the member, the localized effect disappears and the value of the stress can be determined from an elementary formula such as

$$\sigma = \frac{P}{A}$$

Saint-Venant's Principle

- The St- Venant's principle is important because it applies to almost every other type of member and load as well.
- It allow us to develop simple relationships between loads and stresses and loads and deformations.
- The determination of the local effect of loads is then considered as as separate problem usually by experiment or the theory of elasticity.

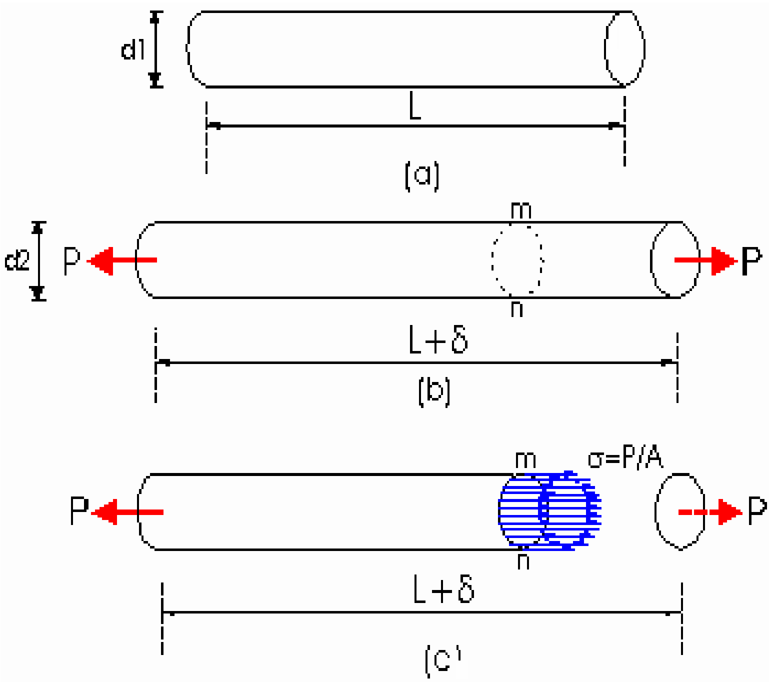
Elastic Deformation of an Axially Loaded Member

- The stress in the element *mn*

$$\sigma = \frac{P}{A}$$

- The strain in the same element is

$$\varepsilon = \frac{\delta}{L}$$



- Using Hooks law, since the material is assumed to deform elastically

$$\sigma = \frac{P}{A} \qquad \varepsilon = \frac{\delta}{L}$$

$$\sigma = E\varepsilon = E\frac{\delta}{L} \Rightarrow \delta = \frac{\sigma L}{E} = \frac{PL}{AE}$$

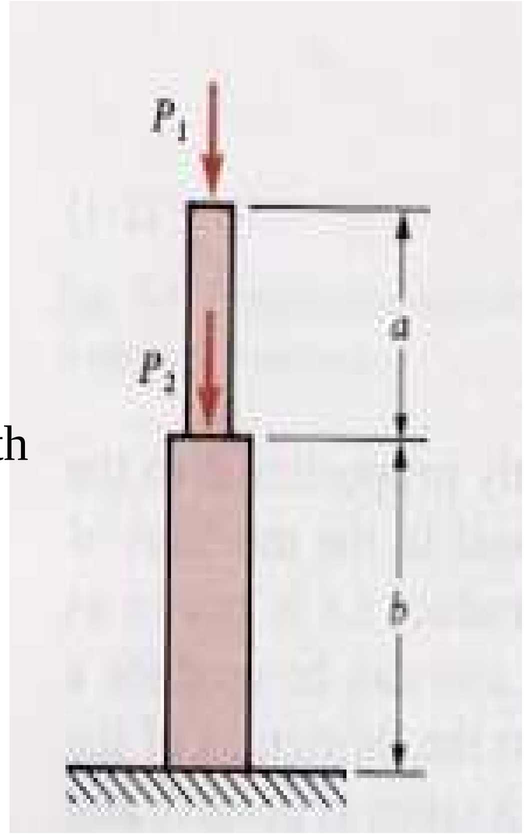
- δ is called the deformation, elongation, or flexibility

$$\delta = \frac{PL}{AE}$$

Deformation of Nonuniform Axially Loaded Member

- For a bar consisting of several prismatic parts having different axial forces, dimensions , or materials, the total elongation is given as

$$\delta = \sum_{i=1}^{n} \frac{P_i L_i}{E_i A_i}$$



i : the numbering index for the various parts of the bar with

n : Total number of parts

P_i : Resultant axial force at the member

L_i : Length of the member

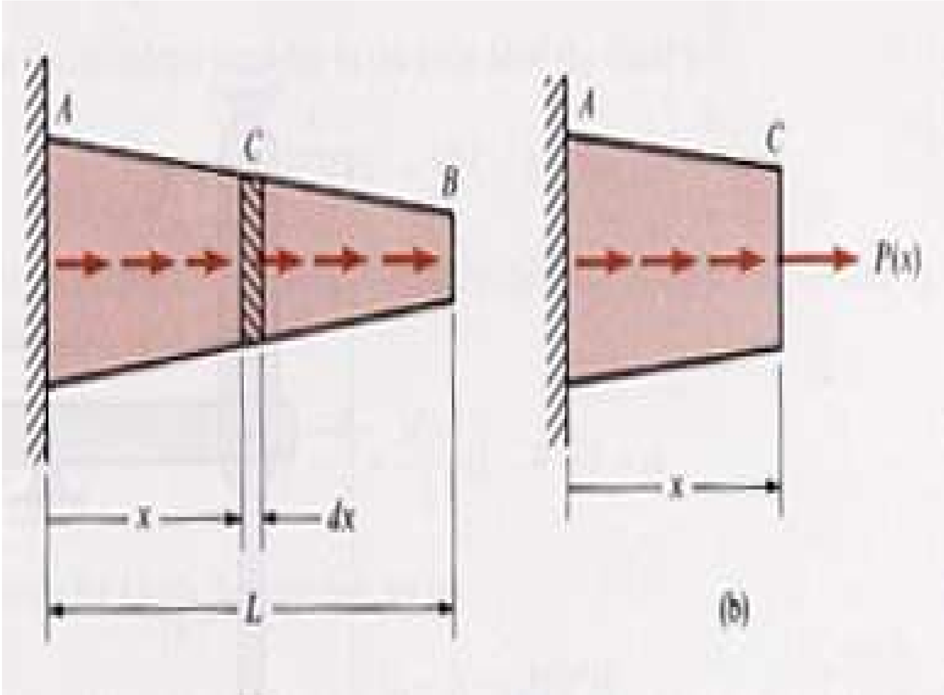
E_i : Young's Modulus of the member

A_i : Cross sectional area of the memebr

- δ can be positive if resultant force are tension, or negative when force is negative

Deformation of Nonuniform Axially Loaded Member

- When the axial force or the cross sectional area varies continuously along the axis of the bar, then we integrate to find the elongation;



$$\delta = \int_0^L \frac{P(x)\,dx}{EA(x)}$$

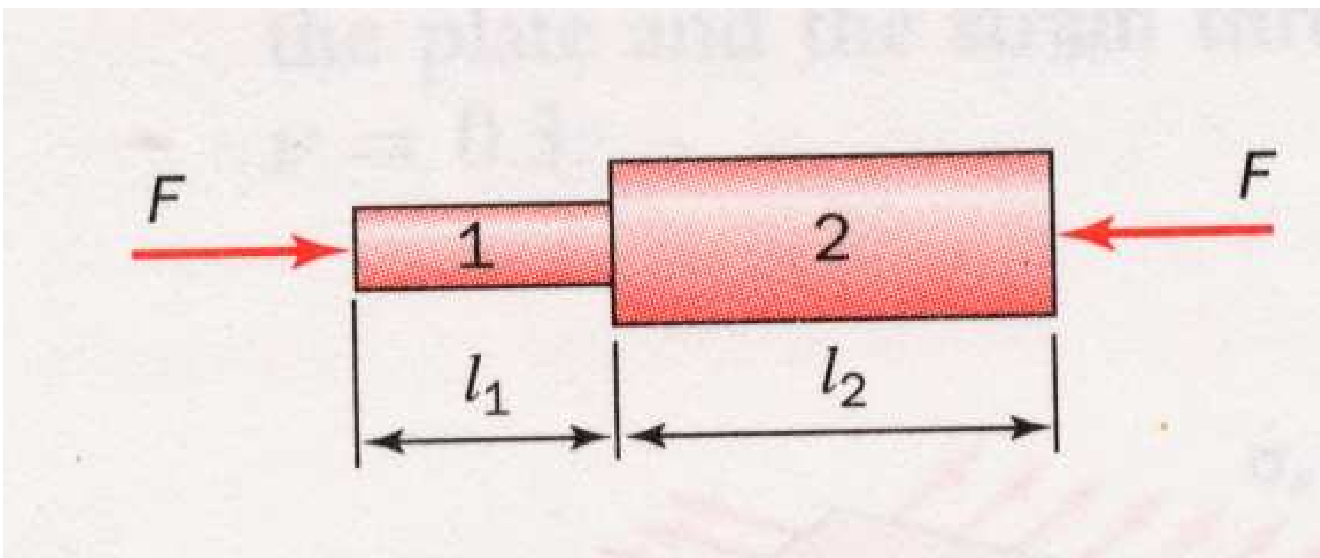
P(x): axial force at a section located a distance x from one end.

A(x) : cross sectional area of the bar as a function of x

E: modulus of Elasticity

Example: Axial Loading

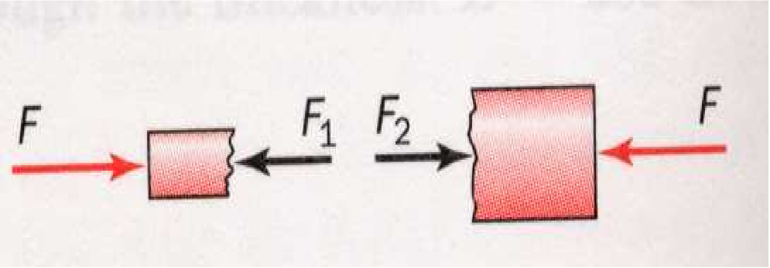
- A bimetallic rod is subjected to a compressive force, F, as shown.
- Determine the overall change in length



Solution

•Equilibrium: It should be clear that the same force is carried through each material so that

$$F_1 = F_2 = F$$



•The overall change in length is the sum of the changes in the two parts of the rod , so that

$$\delta = \delta_1 + \delta_2$$

•Stress and strain relations: Since it is a simple uniaxial stress system

$$\frac{\sigma_1}{\varepsilon_1} = E_1$$

$$\frac{\sigma_2}{\varepsilon_2} = E_2$$

Solution Cont.

- Let the cross-sectional areas be A_1 and A_2

$$\frac{F_1}{A_1} = E_1 \frac{\delta_1}{l_1} \quad \text{and} \quad \frac{F_2}{A_2} = E_2 \frac{\delta_2}{l_2}$$

- Substituting δ_1 and δ_2 into the equation.

$$\delta = \frac{F_1 l_1}{A_1 E_1} + \frac{F_2 l_2}{A_2 E_2} = F\left(\frac{l_1}{A_1 E_1} + \frac{l_2}{A_2 E_2}\right)$$

- In general , for a series of n bars

$$\delta = F \sum_{i=1}^{n} \frac{l_1}{A_1 E_1}$$

Principle of Superposition for Axial Members

- *The relative displacement between two sections along the rod due to several applied forces acting simultaneously is equal to the algebraic sum of the relative displacements between the same two sections due to each external force acting separately*
- To use superposition method :

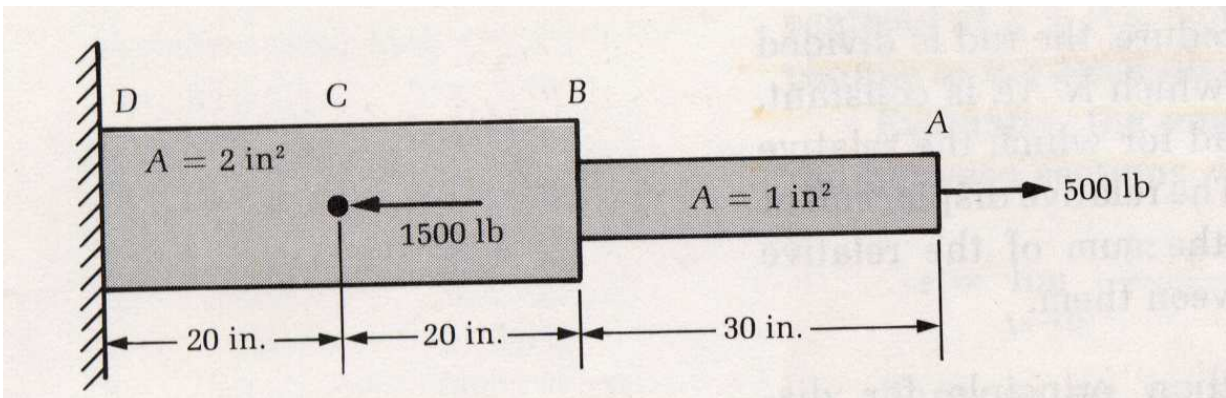
The loading must be linearly-elastic related to the stress or the displacement that is to be determined e.g,

$$\sigma = \frac{P}{A} \text{ and } \delta = \frac{PL}{AE}$$

Involve an elastic linear relation between the load and stress and strain

Example: Axial Stress at Non-uniform Member

- An aluminum step-shaft is loaded by two axial forces as shown.
- Determine the relative displacement of the free end at A with respect to the fixed end at D. The modulus of elasticity for aluminum is 10×10^6 psi
- Solve the problem using the (a)discrete element method, and (b) the superposition method.

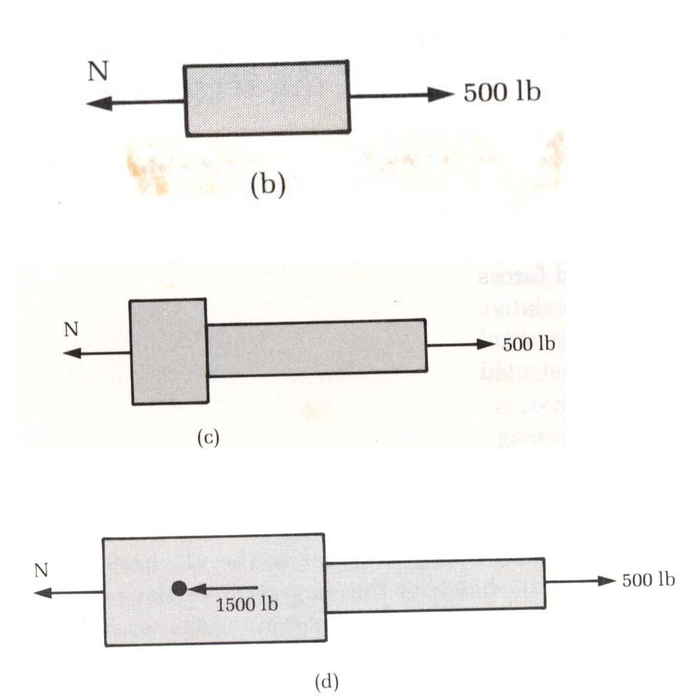


(a)

(a)Solution: Discrete Element Method

- The discrete element method requires the rod to be divided into finite segments for each of which P/AE is constant over its length. Consequently the complete rod is viewed as a series of connected uniform rods. The relative displacement of a series of such segments is then given by the formula:

$$\delta = \sum \delta_i$$



- Three discrete elements are required for the present problem.

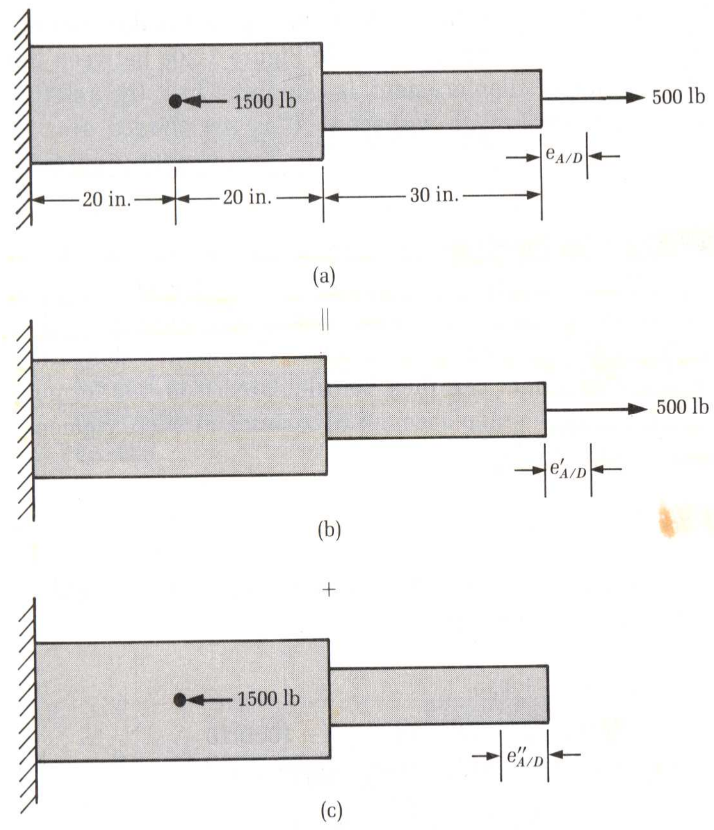
$$\delta_{A/D} = \delta_{A/B} + \delta_{B/C} + \delta_{C/D}$$

$$= \frac{500(30)}{(1)10*10^6} + \frac{500(20)}{(2)10*10^6} + \frac{(-1000)20}{(2)10*10^6}$$

$$= 0.0015 + 0.0005 - 0.001 = 0.001 in.$$

(b)Solution :Superposition Method

- The superposition principle asserts that the relative displacement between two sections of a straight rod due to several external loads acting simultaneously is equal to the algebraic sum of the relative displacement between the same two sections due to each load acting separately.



$$e'_{A/D} = \frac{500(30)}{(1)10*10^6} + \frac{500(40)}{(2)10*10^6} = 0.0025\,in.$$

and

$$e''_{A/D} = \frac{1500(20)}{(2)10*10^6} = -0.0015\,in.$$

- When forces act simultaneously:

$$e_{A/D} = 0.0025 - 0.0015 = 0.001 in.$$

Statically Indeterminate Axial Members

- **Statically determinate Structure:**

The equations of equilibrium determines all member forces and support reactions

# of Equilibrium equations $\geq$ # of Unknown reactions

- **Statically Indeterminate Structure**

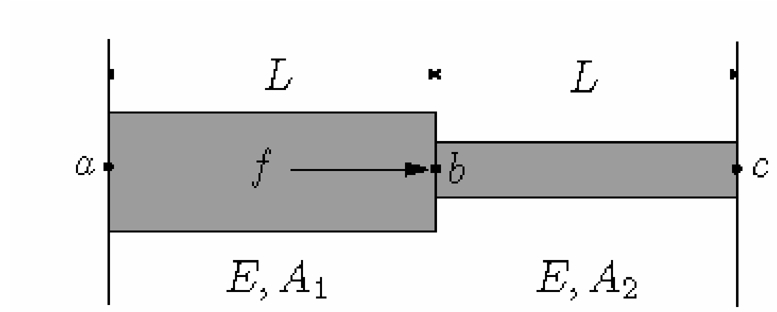
The equations of equilibrium by itself is not sufficient to determine all member forces and support reactions,

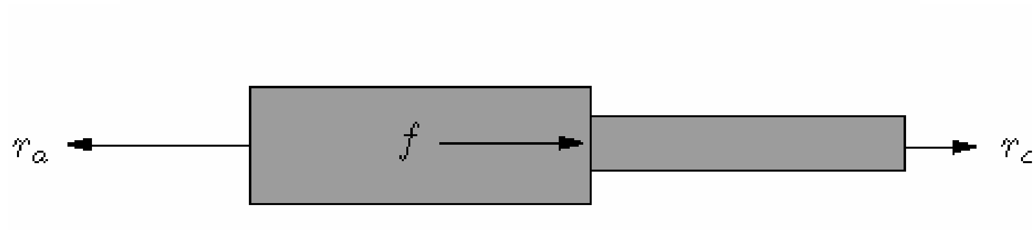
- # of Equilibrium equations < # of Unknown reactions
- additional equations involve the displacements of the structure are required. These are called Compatibility Equations

Method-1:Flexibility Method

- Consider the structure shown on the side.
- 1-Free Body Diagram
- Force balance

$$\sum F_x = 0 \Rightarrow f + r_c - r_a = 0$$

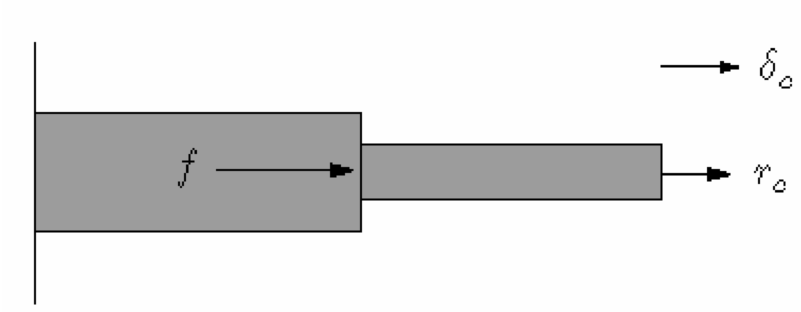


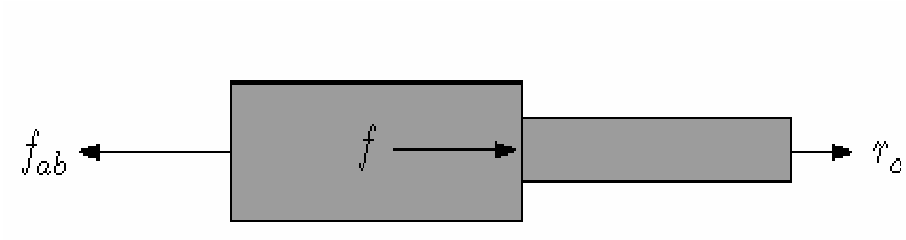


$$\left.\begin{array}{l}\text{Number of Unknowns } 2\ (r_a, r_c) \\ \text{Number of Equations } 1\end{array}\right\} \Rightarrow \text{Statically Indeterminate}$$

Flexibility Method

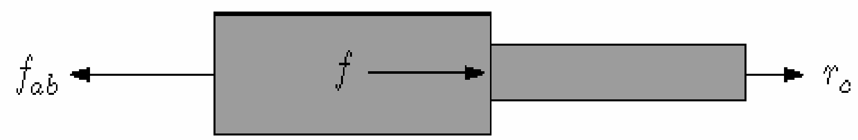
- 2-Select one force or reaction as redundant
- 3-Release the redundant force by cutting the structure to render it statically determinate
- 4-Use the equations of equilibrium to determine the forces in the cut structure





Flexibility Method

$$\sum F_x = 0 \Rightarrow f_{ab} = f + r_c$$



$$\sum F_x = 0 \Rightarrow f_{bc} = r_c$$

- 5-Use flexibility (elongation) to determine the displacements of each section due to the release (compatibility)

$$\delta_{ab} = \frac{f_{ab} L_{ab}}{EA_1} = \frac{(f + r_c) L_{ab}}{EA_1}$$

$$\delta_{bc} = \frac{f_{bc} L_{bc}}{EA_2} = \frac{r_c L_{bc}}{EA_2}$$

Flexibility Method

- 6-Use compatibility of the displacements at the release cut to solve for redundant force

$$\delta_c = \delta_{ab} + \delta_{bc} = 0 \;\Rightarrow r_c = -\frac{fA_2}{A_1 + A_2}$$

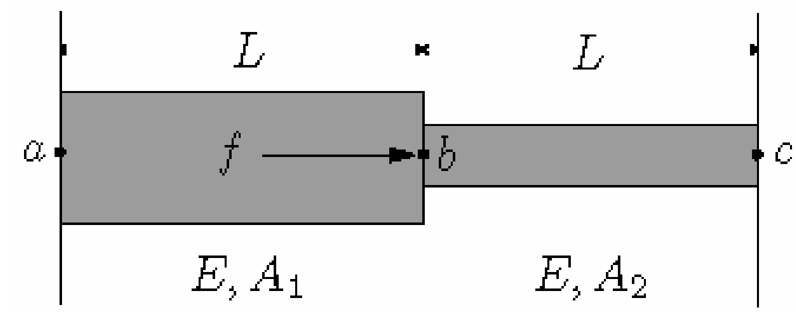
- 7-Substitute back in the equilibrium equations

$$f_{ab} = \frac{fA_1}{A_1 + A_2}$$

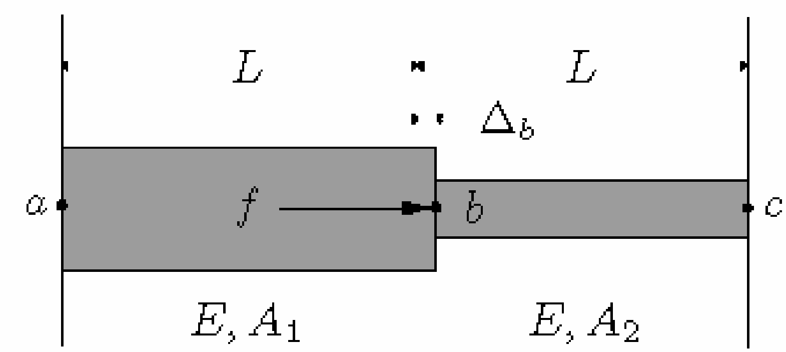
$$f_{bc} = -\frac{fA_2}{A_1 + A_2}$$

Method 2:Stiffness Method

- Displacement are treated as the primary unknowns and determined directly from the equations of Equilibrium.

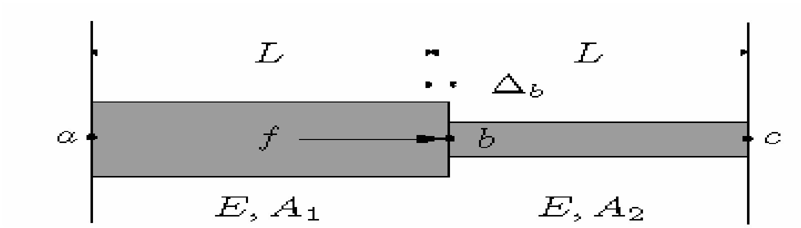


- 1. Select a suitable displacement as the unknown(displacement degree of freedom)



Stiffness Method

- 2. Use compatibility to release the displacement degree of freedom to the displacement of the individual structural member
- 3. Express the forces in the members in terms of the displacement,using the expression for elongation



$$\delta_{ab} = \Delta_b, \quad \delta_{bc} = -\Delta_b$$

$$f_{ab} = \frac{EA_1}{L}\delta_{ab} = \frac{EA_1\Delta_b}{L}$$

$$f_{bc} = \frac{EA_2}{L}\delta_{bc} = \frac{EA_2(-\Delta_b)}{L}$$

Stiffness Method

- 4. Use equilibrium at the joints to solve for the unknown displacements

$$\sum F_x = 0 = -f_{ab} + f + f_{bc}$$

$$\Rightarrow f = \frac{\overbrace{EA_1\Delta_b}^{f_{ab}}}{L} - \frac{\overbrace{EA_2(-\Delta_b -}^{f_{bc}}}{L}$$

$$\Rightarrow \Delta_b = \frac{fL}{E(A_1 + A_2)}$$

5. Determine the forces from the calculated displacement

$$f_{ab} = \frac{fA_1}{A_1 + A_1}$$

$$f_{bc} = \frac{fA_2}{A_1 + A_1}$$

- Notice that these results are identical to those obtained by flexibility method.

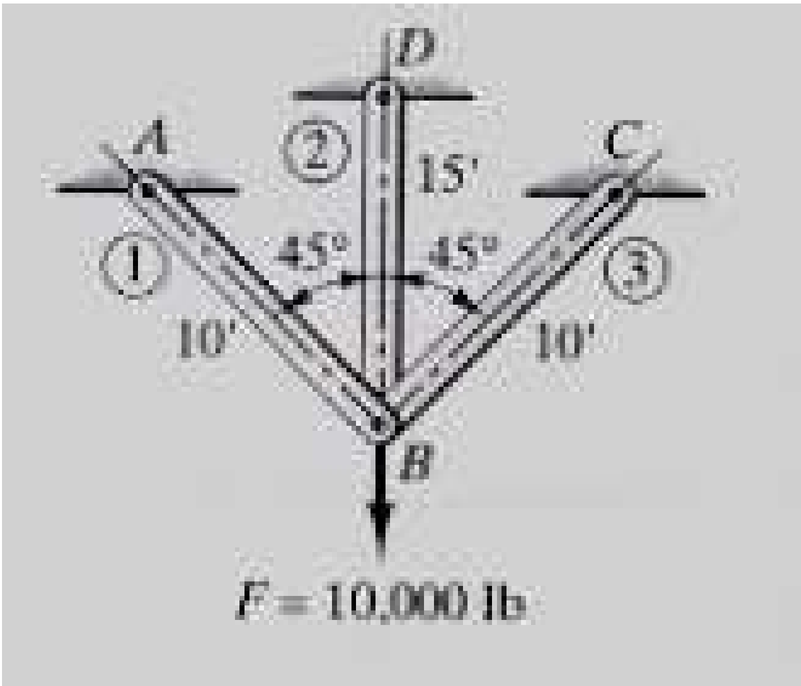
Comparison of the Flexibility and Stiffness Methods

- The flexibility method generates one equation per redundant force.
- Since the # of redundant forces is typically small in comparison to displacement degrees of freedom, the flexibility method is preferred for hand calculations

- The stiffness method generates one equation per displacement degree of freedom.
- the stiffness method is generally easier to automate than flexibility method and therefore is preferred for computer calculation

Example

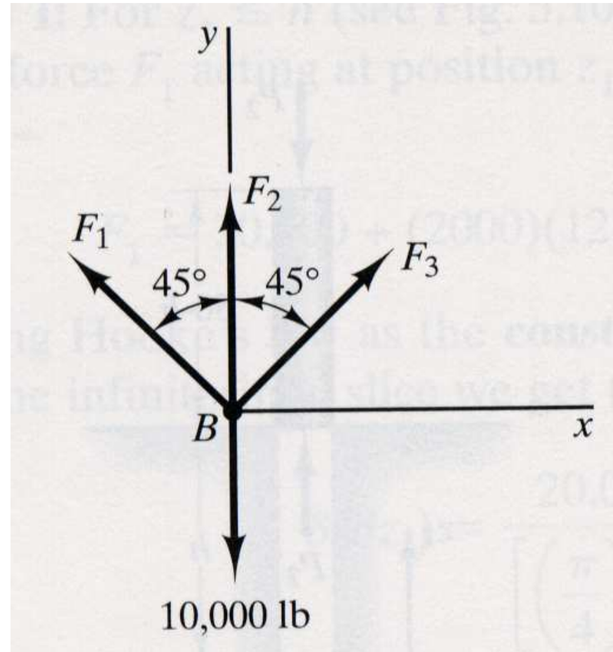
- A simple structure is shown wherein three members support load F. We wish to compute the forces in the members and the deflection of Joint B. Take the cross-section of each member to be in A in^2 and the modulus of elasticity to be in E lb/in^2 for all members.



From equilibrium it is clear that
$F_1 = F_3$ and that:

$$F_2 + 2F_1 \cos 45^o = 10{,}000$$
$$\therefore F_2 + 1.141 F_1 = 10{,}000$$

$$\delta_1 = \delta_3 = \frac{F_1 (10)}{AE}$$
$$\delta_2 = \frac{F_2 (15)}{AE}$$

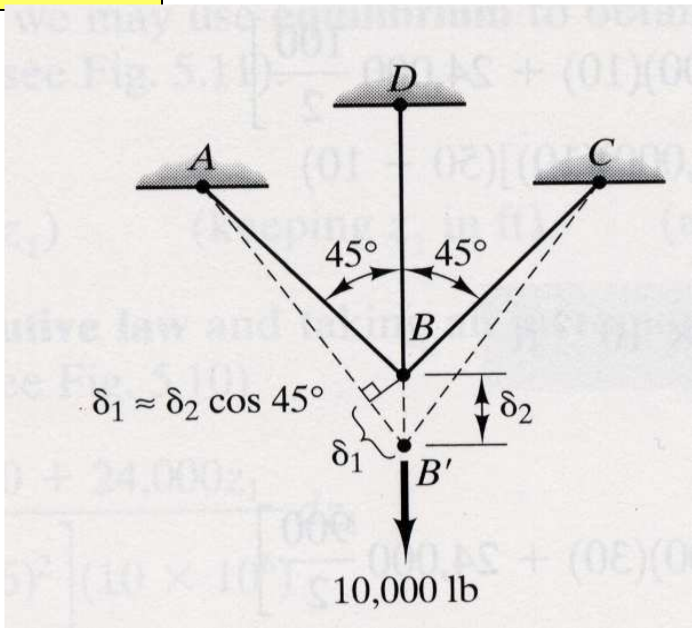


In order for the displacement of pin B to be single-valued, it is necessary that the displacements δ_1 and δ_2 be so related as to give the same vertical reflection of pin B. We find the following deformation:

$$\delta_1 = \delta_2 \cos 45^o = \delta_2 (.707)$$

Solution

In order for the displacement of pin B to be single-valued, it is necessary that the displacements δ_1 and δ_2 be so related as to give the same vertical reflection of pin B. We find the following deformation:



•Now solving the forces:

$$\frac{AE}{15}\delta_2 + 1.414\frac{AE}{10}\delta_1 = 10{,}000$$

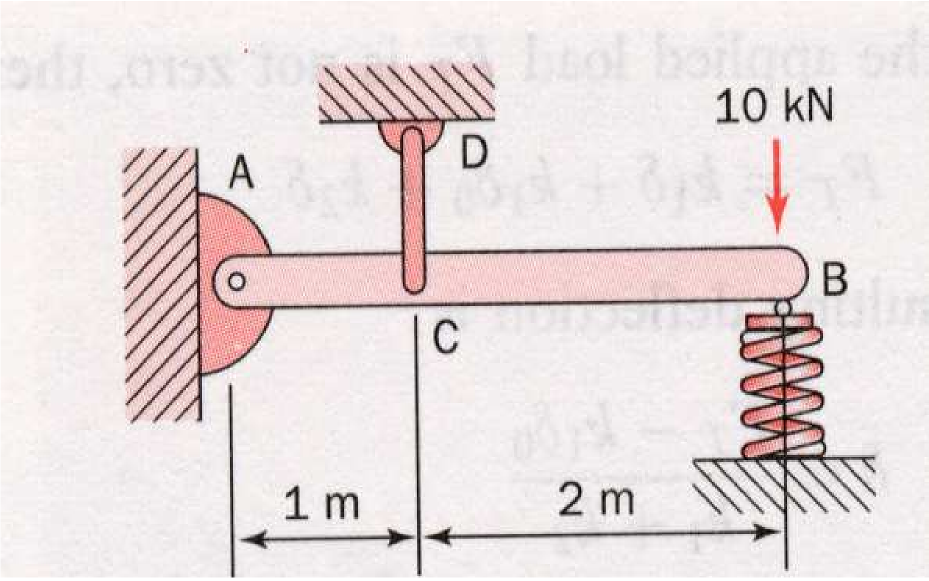
$$\therefore \delta_2 + 2.12\delta_1 = \frac{15{,}000}{AE}$$

•Finally we have:

$$\delta_2 = \frac{60{,}000}{AE}$$

Example: Statically Indeterminate Structure

- A rigid member AB, the weight of which can be neglected, is supported horizontally at the pin joints A and C and by the spring at B as shown. The stiffness of member CD is 2kN/m and of the spring is 5kN/m.
- Calculate the force on CD, which is initially unstressed, and the reaction at A when the vertical load of 10 kN is applied at B.

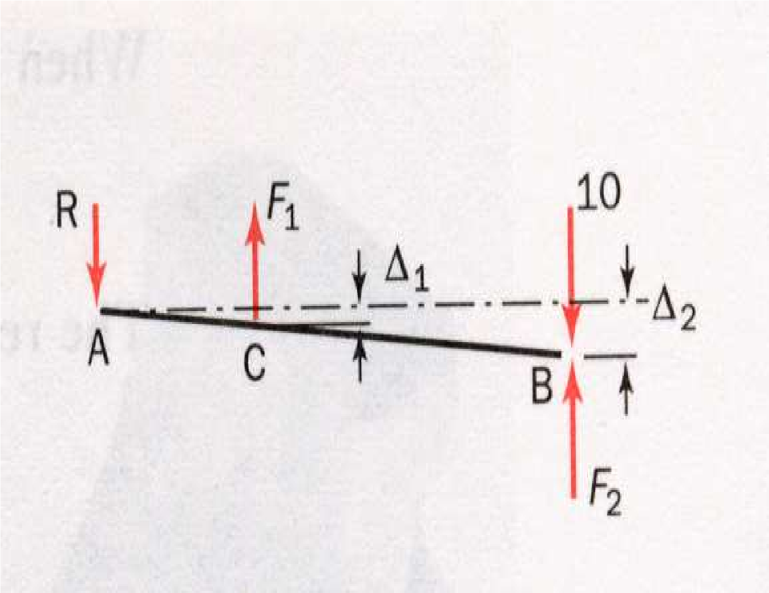


Solution

- Let the reaction at A be R and the forces in CD and in the spring be F_1 and F_2 respectively

Vertical: $10 + R - F_1 - F_2 = 0$
Moments about B: $3R - 2F_1 = 0$

- The deformation is as shown in the figure from similar triangles.



$$\frac{\Delta_1}{\Delta_2} = \frac{1}{3}$$

- Load deformation relations

$$\frac{F_1}{\Delta_1} = 2kN/m \quad \text{and} \quad \frac{F_2}{\Delta_2} = 5kN/m$$

From equations:

$$F_1 = \frac{2}{15} F_2$$

Solution

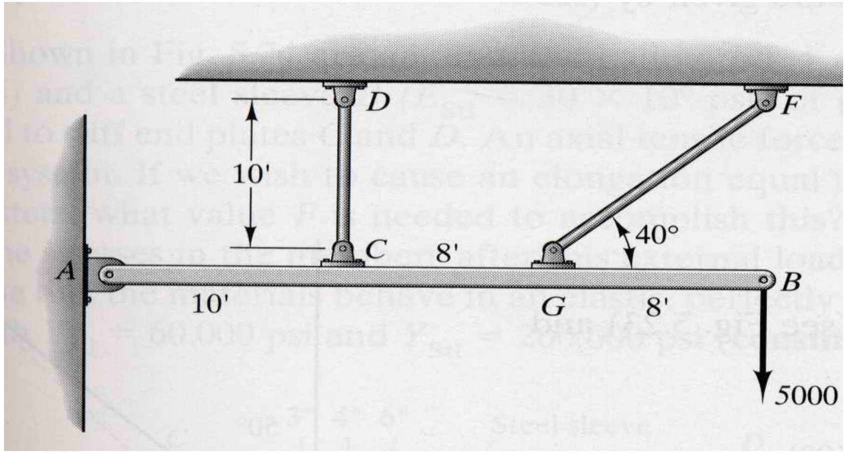
- Eliminating R

$$F_1 + 3F_2 = 30$$

- Hence the force in CD is $F_1 = 1.276\text{kN}$ and the reaction at A is $R = .67F_1 = 0.851\text{kN}$

Example

- A light rigid member AB is pinned at A and is further supported by light elastic rods CD and GF. Member AB supports a 5000lb load at B. the elastic modulus for the two elastic rods CD and GF is given as $E = 30*10^6$psi and the cross-sectional area for each of these rods is $1in^2$.
- What are the forces in the two supporting elastic rods as a result of the loading?



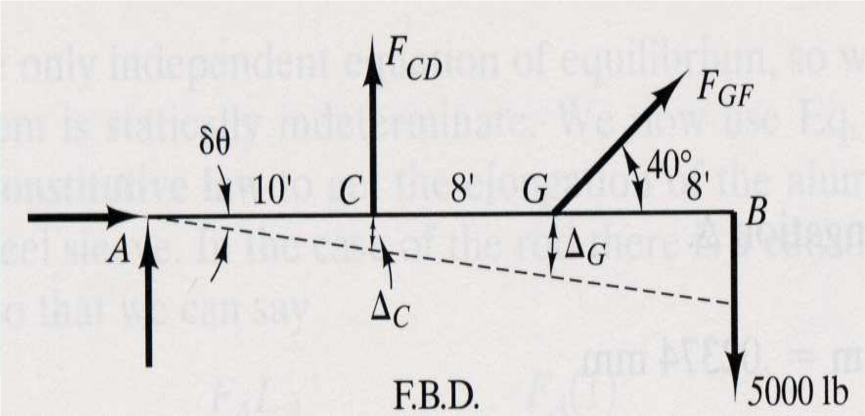
Solution

- With equilibrium we have;

$$\sum M_A = 0$$

$$(F_{CD})(10) + (F_{GF})(\sin 40^o) - (5000)26) = 0$$

$$\therefore F_{CD} + 1.157F_{GF} = 13{,}000$$



- Now we use the deformation law for linear elastic behavior of the rods

$$\Delta_{CD} = \frac{(F_{CD})(10)}{(1)(30 * 10^6)}$$

$$\Delta_{GF} = \frac{(F_{GF})(10)/\sin 40^o}{(1)(30 * 10^6)}$$

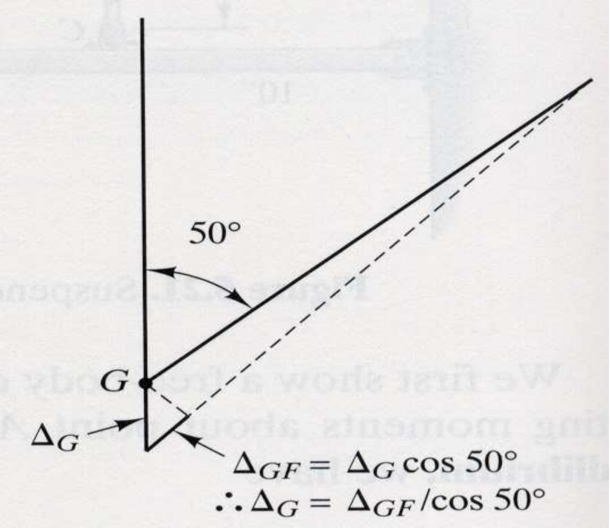
We find that compatibility in this case is given by:

$$\Delta_C = 10(\delta\theta) \quad \text{and} \quad \Delta_G = 18(\delta\theta)$$

$$so$$

$$\Delta_C = \frac{10}{18}(\Delta_G)$$

Solution



- Using compatibility considerations at pin G and realizing $\Delta_C = \Delta_{CD}$ we can say

$$\Delta_C \approx \Delta_{CD} = \frac{10}{18}\Delta_G = \frac{10}{18}(\Delta_G / \cos 50^o)$$

- Now going back to previous equations and inserting displacements into the above equation we get:

$$\frac{(F_{CD})(10)}{(1)(30*10^6)} = \frac{10}{18}\frac{(F_{GF})(10)/\sin 40^o}{(30*10^6)\cos 50^o}$$

$$\therefore F_{CD} = 1.345 F_{GF}$$

$$1.345 F_{GF} + 1.157 F_{GF} = 13{,}000$$

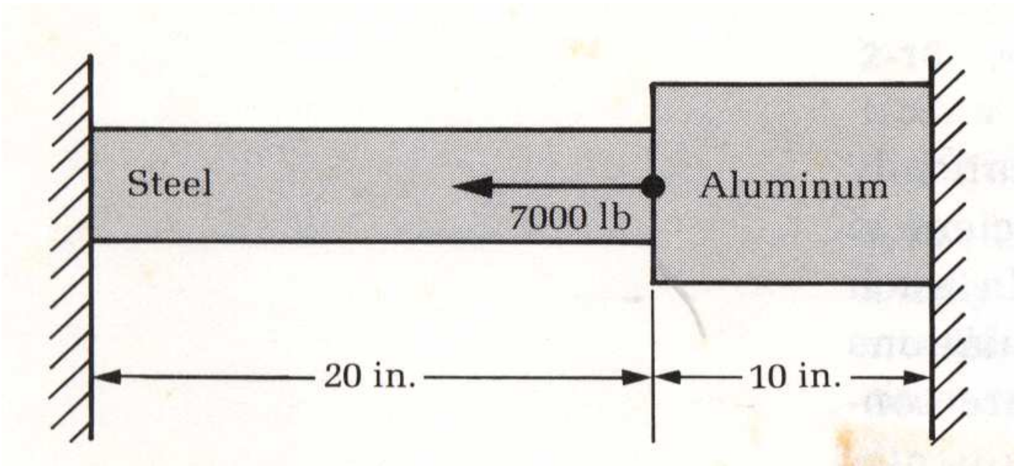
$$F_{GF} = 5197 lb$$

$$F_{CD} = 6990 lb$$

- Now replacing F_{CD} we get

Example:Statically Indeterminate Axial Loading

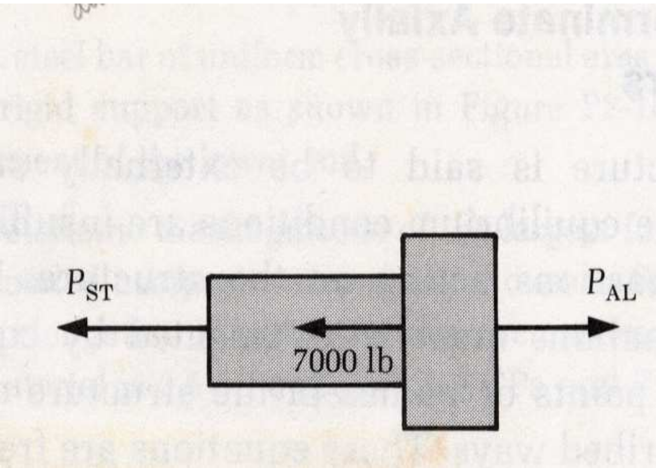
- Determine the stresses induced in the aluminum and steel portions of the composite rod shown due to a force of 7000lb. The cross-sectional areas of the steel and aluminum portions are $2in^2$ and $4in^2$, respectively, and their moduli of elasticity are $30*10^6$ psi and $10*10^6$psi, respectively.



Solution

- From the Free-body diagram

$$\xrightarrow{+} \sum F_x = 0; \qquad -P_{ST} + P_{AL} - 7000 = 0$$



- We must augment this equation with an equation that expresses that the total deformation of the composite rod from end to end is zero.

$$\delta = \delta_{ST} + \delta_{AL} = 0$$

where

$$\delta_{ST} = \frac{P_{ST} l_{ST}}{A_{ST} E_{ST}} \quad \text{and} \quad \delta_{AL} = \frac{P_{AL} l_{AL}}{A_{AL} E_{AL}}$$

Solution

- From this we obtain

$$P_{ST} = -\frac{l_{AL}}{l_{ST}}\frac{E_{ST}}{E_{AL}}\frac{A_{ST}}{A_{AL}}P_{AL} = -\frac{10}{20}\frac{30}{10}\frac{2}{4}P_{AL}$$

$$P_{ST} = -\frac{3}{4}P_{AL}$$

- Solving simultaneously yields

$$P_{ST} = -3000lb \quad \text{and} \quad P_{AL} = 4000lb$$

➢ For which we find stresses

$$\sigma_{ST} = -\frac{3000}{4} = -1500\ psi = 1500\ psi\,(C)$$

$$\sigma_{AL} = \frac{4000}{4} = 1000\ psi\,(T)$$