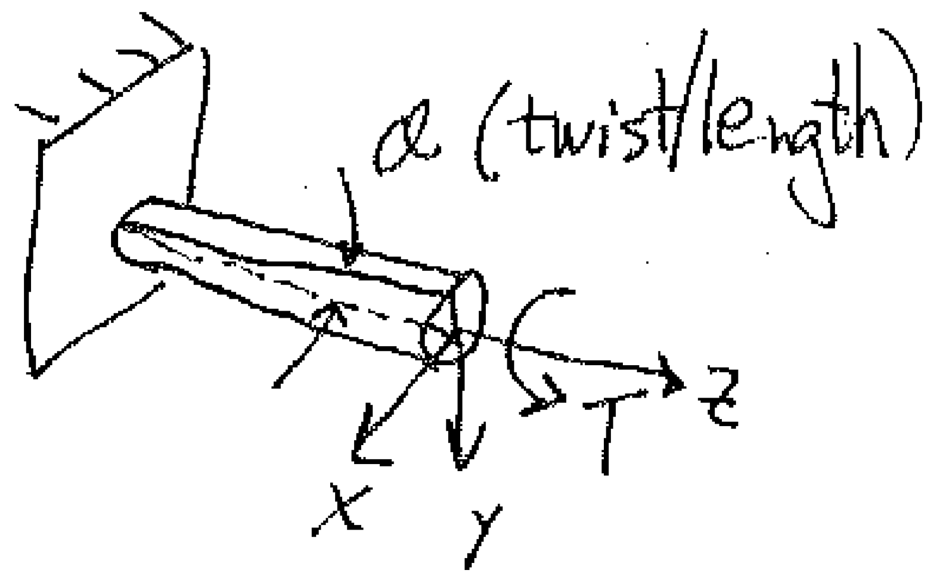
Torsions St. Venant's semi-inverse method

circular bar



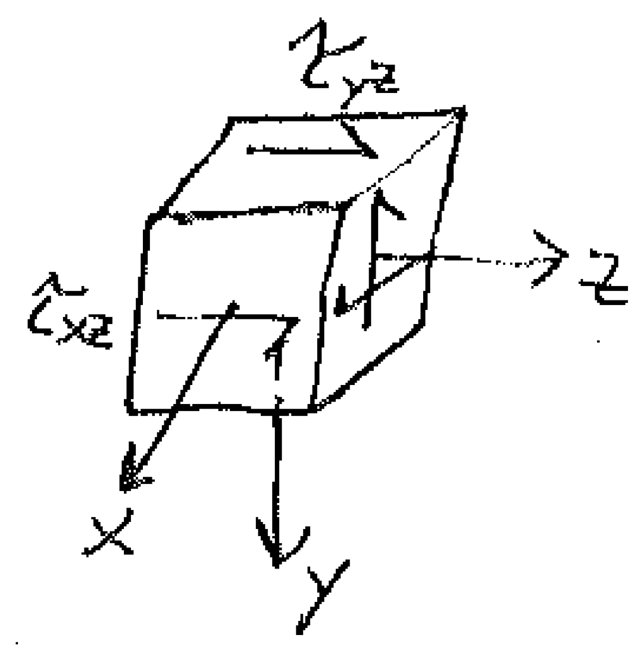
torsion rigidity $C = \frac{T}{\alpha}$

assume: plane sections remain plane

$u = u_x = -y\alpha z$

$v = u_y = \alpha z x$

$w = 0$ (plane sections remain plane)



$\epsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i})$

$\epsilon_{xx} = \epsilon_{yy} = \epsilon_{zz} = 0$

$\epsilon_{xz} = \frac{1}{2}(u_{x,z} + u_{z,x})$

$= \frac{1}{2}(-y\alpha)$

$\epsilon_{yz} = \frac{1}{2}(u_{y,z} + u_{z,y}) = \frac{1}{2}\alpha x$

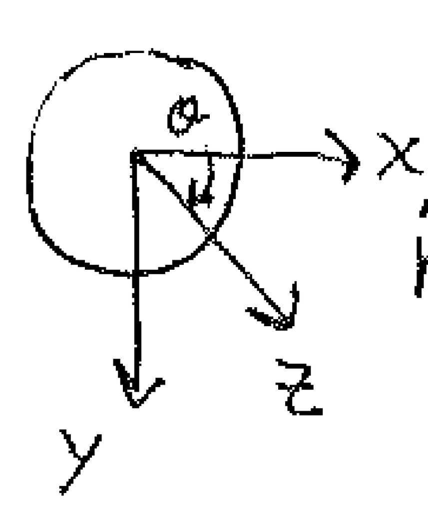
$\tau = G\gamma$

$\tau_{xz} = -G\alpha y$

$\tau_{yz} = G\alpha x$

on surface away from ends

$\underline{t} = 0 = t_x = t_y = t_z$



$\hat{n} = n_x \hat{i} + n_y \hat{j} = \frac{x}{r}\hat{i} + \frac{y}{r}\hat{j}$ can use cosθ, sinθ also

$t_j = \sigma_{ji} n_j$

$$\frac{\partial^2\phi}{\partial y^2}+\frac{\partial^2\phi}{\partial x^2}=G\theta\left(\frac{\partial^2\psi}{\partial x\partial y}-1\right)-G\theta\left(\frac{\partial^2\psi}{\partial y\partial x}+1\right)$$

$$=\underbrace{-2G\theta}_{F}$$

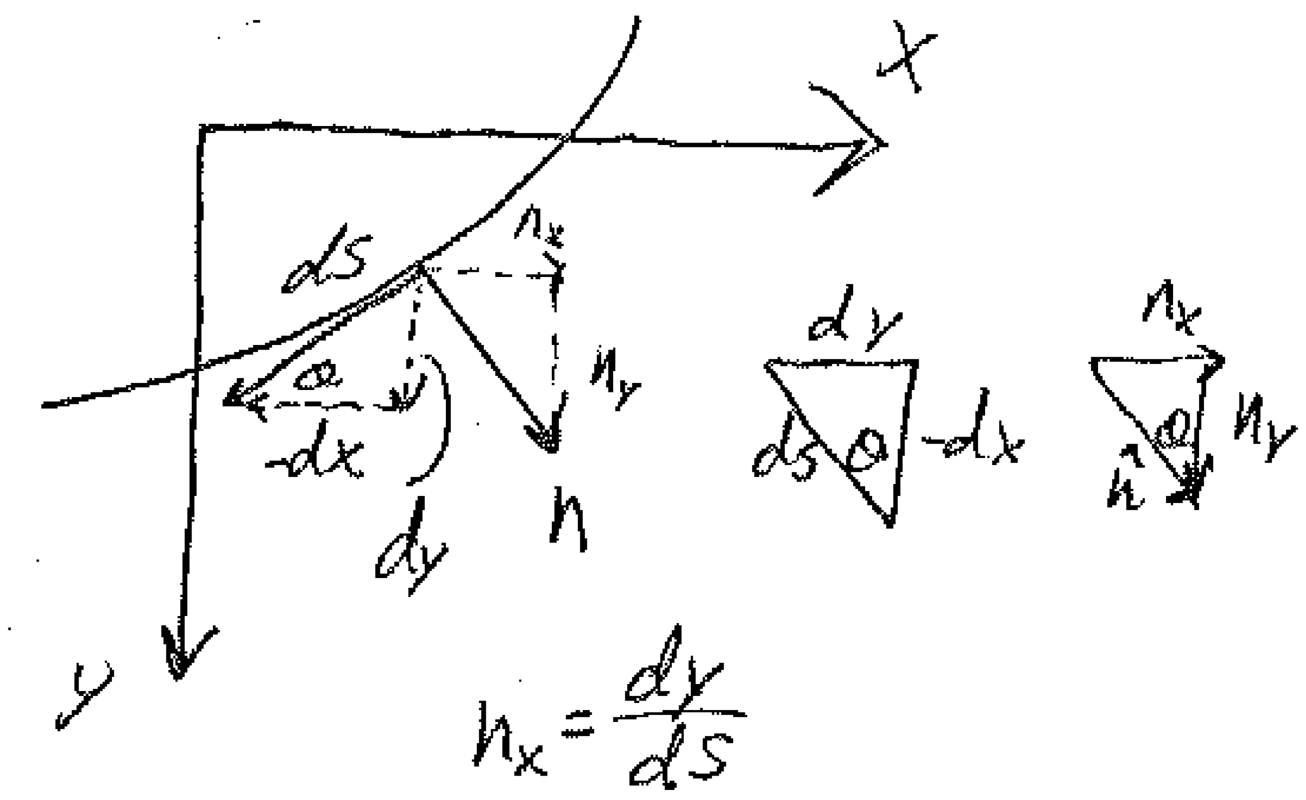
Boundary conditions

on the surface, $\underline{t}=0$

$$\tau_{xz}=\tau_{yz}=0$$

$$\tau_{xz}n_x+\tau_{yz}n_y=0$$

$$\frac{\partial\phi}{\partial y}n_x-\frac{\partial\phi}{\partial x}n_y=0$$



$$n_x=\frac{dy}{ds}$$

$$n_y=-\frac{dx}{ds}$$

$$\frac{\partial\phi}{\partial y}\frac{\partial y}{\partial s}+\frac{\partial\phi}{\partial x}\frac{\partial x}{\partial s}=0=\frac{d\phi}{ds}$$

ϕ is constant on S

$\phi=0$ picked arbitrarily, see 7.6 section in book

$\tau_{xz}=\frac{\partial\phi}{\partial y}$ $\therefore$ it doesn't matter if ϕ=const. or zero

B.C. of bar on end

$\hat{n}=1\cdot\hat{k}$

$n_z=1$

$t_x=\tau_{xz}$

$t_y=\tau_{yz}$

$$\Sigma F_x=0=\int_A \tau_{xz}\,dA=\int_A\frac{\partial\phi}{\partial y}dx\,dy=\int\phi\Big|_{y_1}^{y_2}dx$$

$$\Sigma M=M_t\,,\quad \Sigma F_y=\int_A-\frac{\partial\phi}{\partial x}dx\,dy=-\int\phi\Big|_{x_1}^{x_2}dy=0$$

$$M_t=\int_A(t_y x-t_x y)\,dx\,dy$$

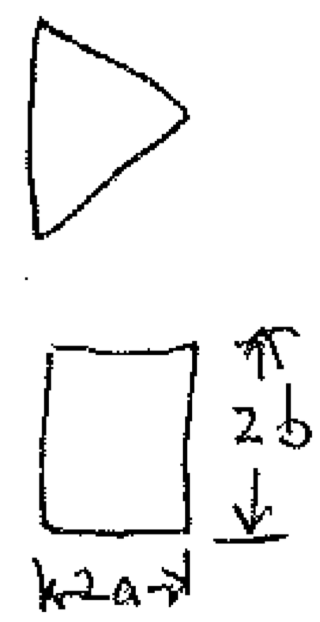
$$=\int_A-\frac{\partial\phi}{\partial x}x\,dx\,dy+\int_A-\frac{\partial\phi}{\partial y}y\,dx\,dy$$

integrate by parts with $\phi=0$ on boundary

giving $M_t=2\int_A\phi\,dx\,dy$

Elliptical cross-sections fairly easy

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$



these are harder

try $\phi = m\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}-1\right)=0$

recall $\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}=-2G\theta=F$

$$F=m\left(\frac{2}{a^2}+\frac{2}{b^2}\right)$$

$$m=\frac{F}{2}\left(\frac{a^2b^2}{a^2+b^2}\right)$$

$$\phi=\frac{F}{2}\frac{a^2b^2}{a^2+b^2}\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}-1\right)=0$$

now use,

$$M_t=2\int_A\phi\,dx\,dy$$

$$=\frac{a^2b^2F}{a^2+b^2}\left[\underbrace{\int_A\frac{x^2}{a^2}dx\,dy}_{\frac{1}{a^2}I_y}+\underbrace{\int_A\frac{y^2}{b^2}dx\,dy}_{\frac{1}{b^2}I_x}-\underbrace{\int dx\,dy}_{A}\right]$$

$$\frac{\pi}{4}\frac{a^3b}{a^2}+\frac{\pi}{4}\frac{b^3a}{b^2}-\pi ab=-\frac{\pi}{2}ab$$

$$M_t=-\frac{\pi a^3b^3F}{2(a^2+b^2)}$$

$$\phi=-\frac{M_t}{\pi ab}\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}-1\right)$$

$$\tau_{xz}=\frac{-2M_ty}{\pi ab^3}$$

$$\tau_{yz}=\frac{2M_tx}{\pi a^3b}$$

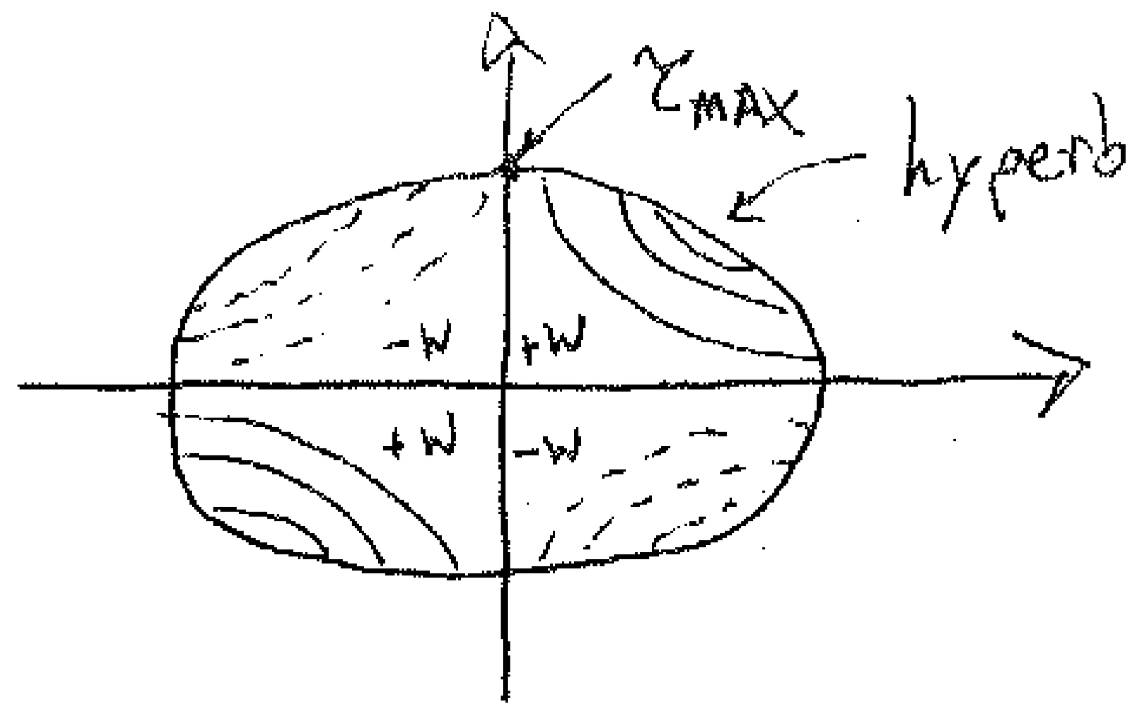
must work back using $w=\theta\psi(x,y)$

$$w=\frac{M_t(b^2-a^2)}{\pi a^3b^3G}xy$$

$$\theta=\frac{M_t(a^2+b^2)}{\pi a^3b^3G}$$

Back substitution using equilibrium and shear stress

$$\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}=-2G\theta=F$$



Torsional rigidity

$$C = \frac{T}{\theta} = \frac{G}{4\pi^2}\frac{A^4}{I_p}$$

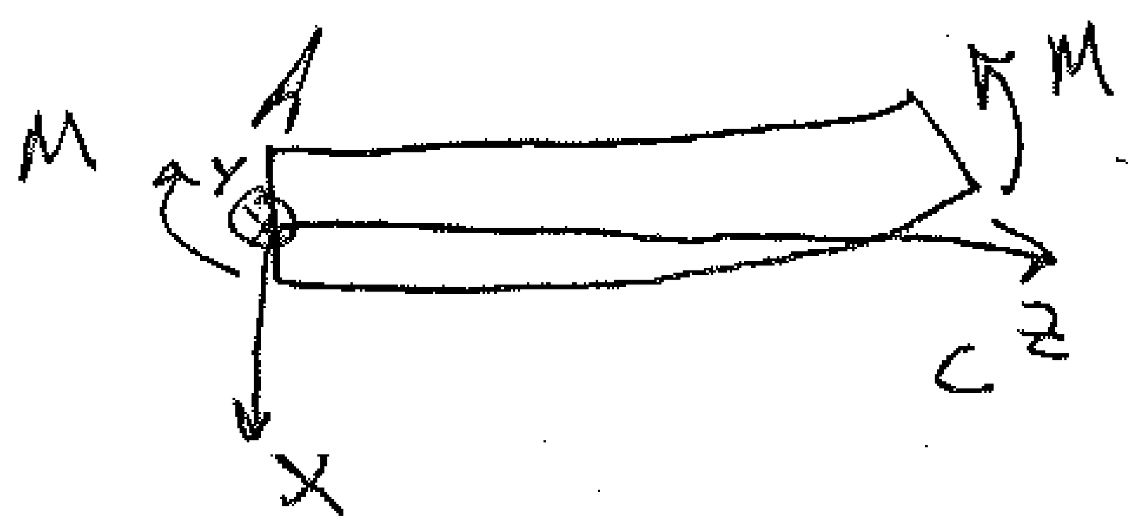
A - area $= \pi ab$

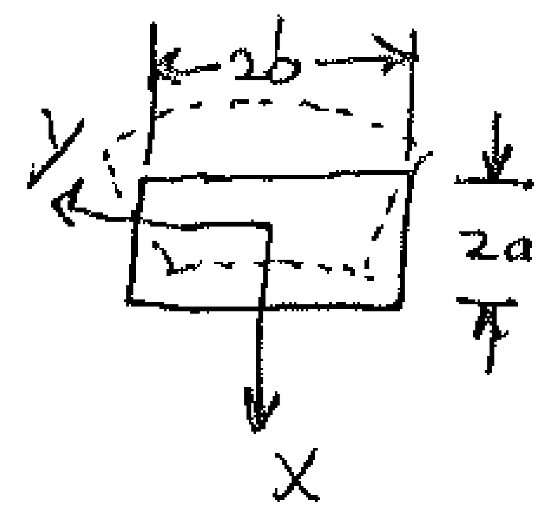
$I_p = I_x + I_y$

τ_{MAX} = on the boundary at $y = b$ (minor axis)

$$= \frac{2M_t}{\pi a b^2}$$

Bending of beams





$$\sigma_{zz} = E\epsilon_{zz} = \frac{Ex}{R}$$

$$\epsilon_{zz} = \frac{x}{R}$$

all other $\sigma = 0$

$$M = \int \sigma_{zz} x \, dA = \int \frac{Ex^2}{R} dA = \frac{EI}{R}$$

$$\epsilon_{zz} = \frac{\sigma_{zz}}{E} = \frac{x}{R} = \frac{\partial u_z}{\partial z}$$

$$\epsilon_{xx} = \epsilon_{yy} = -\frac{\nu x}{R} = \frac{\partial u_x}{\partial x} = \frac{\partial u_y}{\partial y}$$

$$\frac{\partial u_z}{\partial x} + \frac{\partial u_x}{\partial z} = 0 = \gamma_{zx}$$

$$u_z = \frac{xz}{R} + w_0(x, y)$$

$$\gamma_{zx} = 0 = \frac{z}{R} + \frac{\partial w_0}{\partial x} + \frac{\partial u_x}{\partial z}$$

$$\frac{\partial u_x}{\partial z} = -\frac{z}{R} - \frac{\partial w_0}{\partial x}$$

$$-\frac{\partial u_z}{\partial y} = \frac{\partial u_y}{\partial z} = -\frac{\partial w_0}{\partial y}$$

(52)

$$u_x = -\frac{z^2}{2R} - z\frac{\partial w_0}{\partial x} + u_0(x,y)$$

$$u_y = -z\frac{\partial w_0}{\partial y} + v_0(x,y)$$

$$\left.\begin{aligned} \frac{\partial u_x}{\partial x} &= -z\frac{\partial^2 w_0}{\partial x^2} + \frac{\partial u_0}{\partial x} = -\frac{\nu x}{R} \\ \frac{\partial u_y}{\partial y} &= -z\frac{\partial^2 w_0}{\partial y^2} + \frac{\partial v_0}{\partial y} = -\frac{\nu x}{R} \end{aligned}\right\} \text{ must be valid for any } z$$

$$\frac{\partial^2 w_0}{\partial x^2} = \frac{\partial^2 w_0}{\partial y^2} = 0$$

$$\frac{\partial u_0}{\partial x} = -\frac{\nu x}{R}$$

$$u_0 = -\frac{\nu x^2}{2R} + f_1(y)$$

$$v_0 = -\frac{\nu xy}{R} + f_2(x)$$

$$u_x = -\frac{z^2}{2R} - z\frac{\partial w_0}{\partial x} - \frac{\nu x^2}{2R} + f_1(y)$$

$$u_y = -z\frac{\partial w_0}{\partial y} - \frac{\nu xy}{R} + f_2(x)$$

recall $\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x} = 0$

$$-z\frac{\partial^2 w_0}{\partial x \partial y} + \frac{\partial f_1}{\partial y} - \frac{\nu y}{R} + \frac{\partial f_2}{\partial x} - z\frac{\partial^2 w_0}{\partial y \partial x} = 0$$

independent of z

$$\frac{\partial^2 w_0}{\partial x \partial y} = 0$$

collecting like terms
$$\left\{\begin{aligned} \frac{\partial f_2}{\partial x} &= -\alpha \\ \frac{\partial f_1}{\partial y} - \frac{\nu y}{R} &= +\alpha \end{aligned}\right.$$

$f_2 = -\alpha x + \beta$

$f_1 = \alpha y + \frac{\nu y^2}{2R} + \gamma$

$\frac{\partial^2 w_0}{\partial x^2} = \frac{\partial^2 w_0}{\partial y^2} = 0$ (from earlier)

$w_0 = mx + ny + p$

now,

$u_x = -\frac{z^2}{2R} - zm - \frac{\nu x^2}{2R} + \alpha y + \frac{\nu y^2}{2R} + \gamma$

$u_y = -zn - \frac{\nu xy}{R} - \alpha x + \beta$

find constants $m, n, \alpha, \beta, \gamma$

at A $u_x = u_y = 0$

$\frac{\partial u_x}{\partial z} = \frac{\partial u_y}{\partial z} = \frac{\partial u_y}{\partial x} = 0$

this sets all constants to zero

$u_x = -\frac{z^2}{2R} - \frac{\nu x^2}{2R} + \frac{\nu y^2}{2R}$

Timoshenko
$\left(u_x = -\frac{1}{2R}\left(z^2 + \nu(x^2 + y^2)\right)\right)$

$u_y = -\frac{\nu xy}{R}$

deflection at $y = x = 0$

$u_x|_{x=y=0} = -\frac{z^2}{2R}$ deflection curve

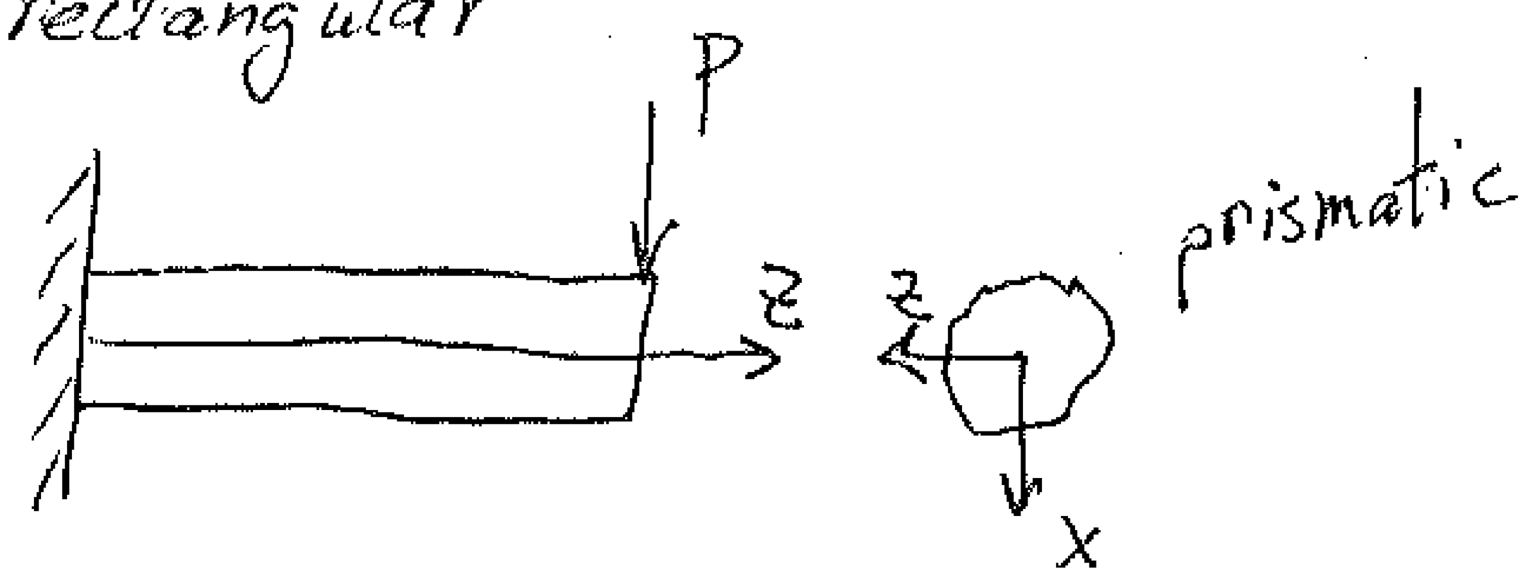
Do plane sections remain plane?

u_z @ $z = c \Rightarrow \frac{cx}{R}$ plane has rotated but consistent along z

$u_z = \frac{xz}{R} + mx + ny + p = \left(\frac{z}{R} + m\right)x + ny + p = \frac{zx}{R}$

$(m = n = p = 0)$

Bending of beams with other cross sections than rectangular



use St. Venant's semi-inverse method

$$\sigma_{zz} = -P(l-z)\frac{x}{I}$$

$$\tau_{xz} \neq 0$$

$$\tau_{yz} \neq 0$$

$$\sigma_{xx}, \sigma_{yy}, \tau_{xy} = 0$$

Equilibrium

$$\frac{\partial \tau_{xz}}{\partial z} = 0 \qquad \frac{\partial \tau_{yz}}{\partial z} = 0$$

$$\frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} = -\frac{Px}{I} = -\frac{\partial \sigma_{zz}}{\partial z}$$

B.C.'s

$$\tau_{xz} n_x + \tau_{yz} n_y = 0$$

$$\tau_{xz} \frac{dy}{ds} - \tau_{yz}\frac{dx}{ds} = 0$$

Compatibility

$$(1+\nu)\nabla^2\sigma_{xx} + \sigma_{kk,xx} = 0$$

$$(1+\nu)\nabla^2\sigma_{yy} + \sigma_{kk,yy} = 0$$

$$(1+\nu)\nabla^2\sigma_{zz} + \sigma_{kk,zz} = 0$$

$$(1+\nu)\nabla^2\sigma_{yz} + \sigma_{kk,yz} = 0$$

also for xz and xy

several components are zero

$$\left.\begin{aligned} &\sigma_{zz,xx} = 0 \\ &\sigma_{zz,yy} = 0 \\ &(1+\nu)\nabla^2\sigma_{zz} + \sigma_{zz,zz} = 0 \end{aligned}\right\} \text{ already satisfied by } \sigma_{zz} \text{ function}$$

$$\nabla^2 \sigma_{yz} = 0$$

$$\sigma_{zz,zx} + (1+\nu)\nabla^2 \sigma_{xz} = 0$$

$$\left.\begin{aligned} &-\frac{P}{I}\frac{1}{(1+\nu)} = \nabla^2 \sigma_{xz} \\ &\nabla^2 \sigma_{yz} = 0 \end{aligned}\right\} \text{ must satisfy compatibility}$$

Introduce a stress function $\phi(x,y)$

$$\tau_{xz} = \frac{\partial \phi}{\partial y} - \frac{Px^2}{2I} + f(y)$$

$$\tau_{yz} = -\frac{\partial \phi}{\partial x}$$

compatibility becomes

$$\frac{\partial}{\partial x}\left(\phi_{,xx} + \phi_{,yy}\right) = 0$$

$$\frac{\partial}{\partial y}\left(\phi_{,xx} + \phi_{,yy}\right) = \frac{\nu}{1+\nu}\frac{P}{I} - \frac{d^2 f}{dy^2}$$

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = \frac{\nu}{1+\nu}\frac{Py}{I} - \frac{df}{dy} + C$$

(can be shown that $C=0$, assuming no twisting) - see Timoshenko

Substitute τ_{xz}, τ_{yz} into B.C.'s

$$\frac{\partial \phi}{\partial y}\frac{dy}{ds} + \frac{\partial \phi}{\partial x}\frac{dx}{ds} = \frac{d\phi}{ds} = \left[\frac{Px^2}{2I} - f(y)\right]\frac{dy}{ds}$$

if $f(y) = \frac{Px^2}{2I}$ then ϕ = constant on boundary

$\phi = 0$ can be used