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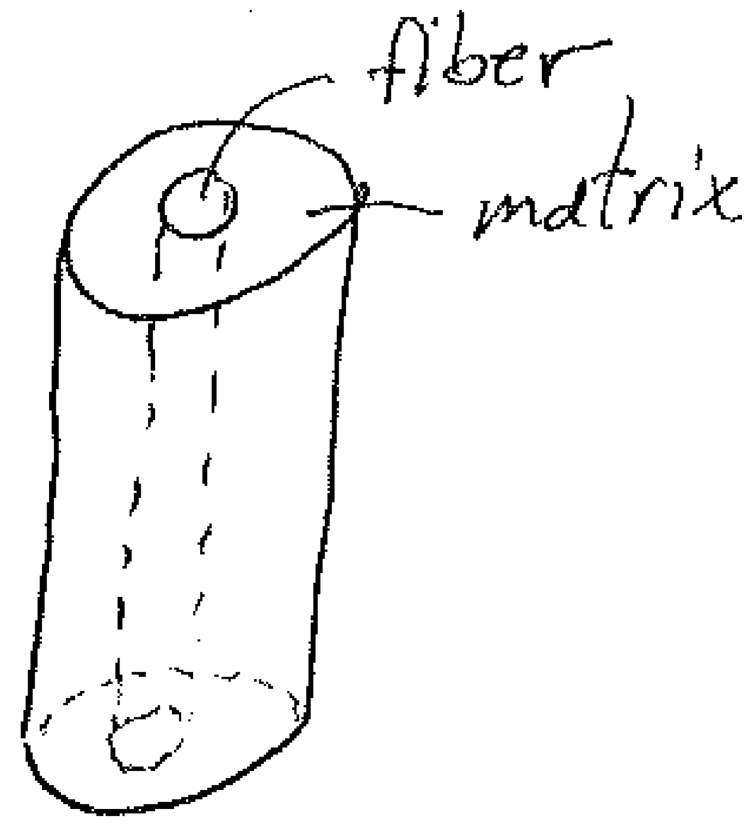
Consider a fiber composite

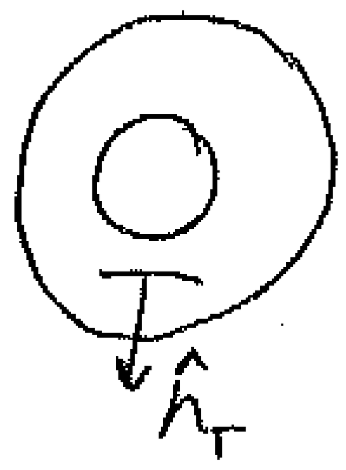


$u_r^f = u_r^m$

$t_i = \sigma_{ij} n_i$

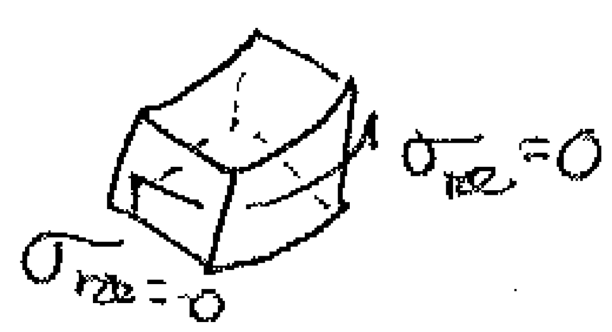
$t_r = \sigma_{rr} n_r + \cancel{\sigma_{\theta r}} \cancel{n_\theta} + \cancel{\sigma_{zr}} \cancel{n_z}$

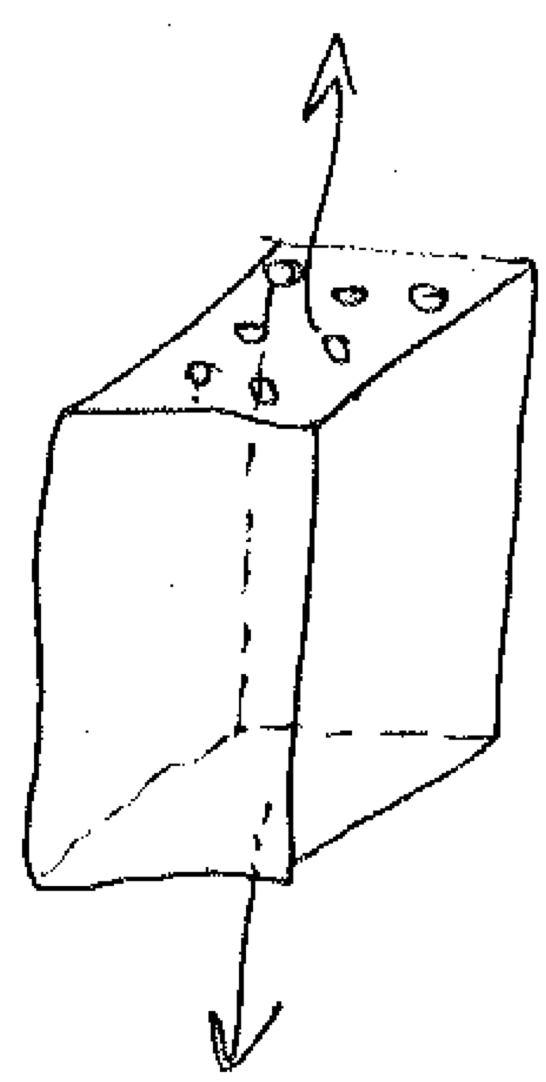
$\sigma_{rr}^f = \sigma_{rr}^m$



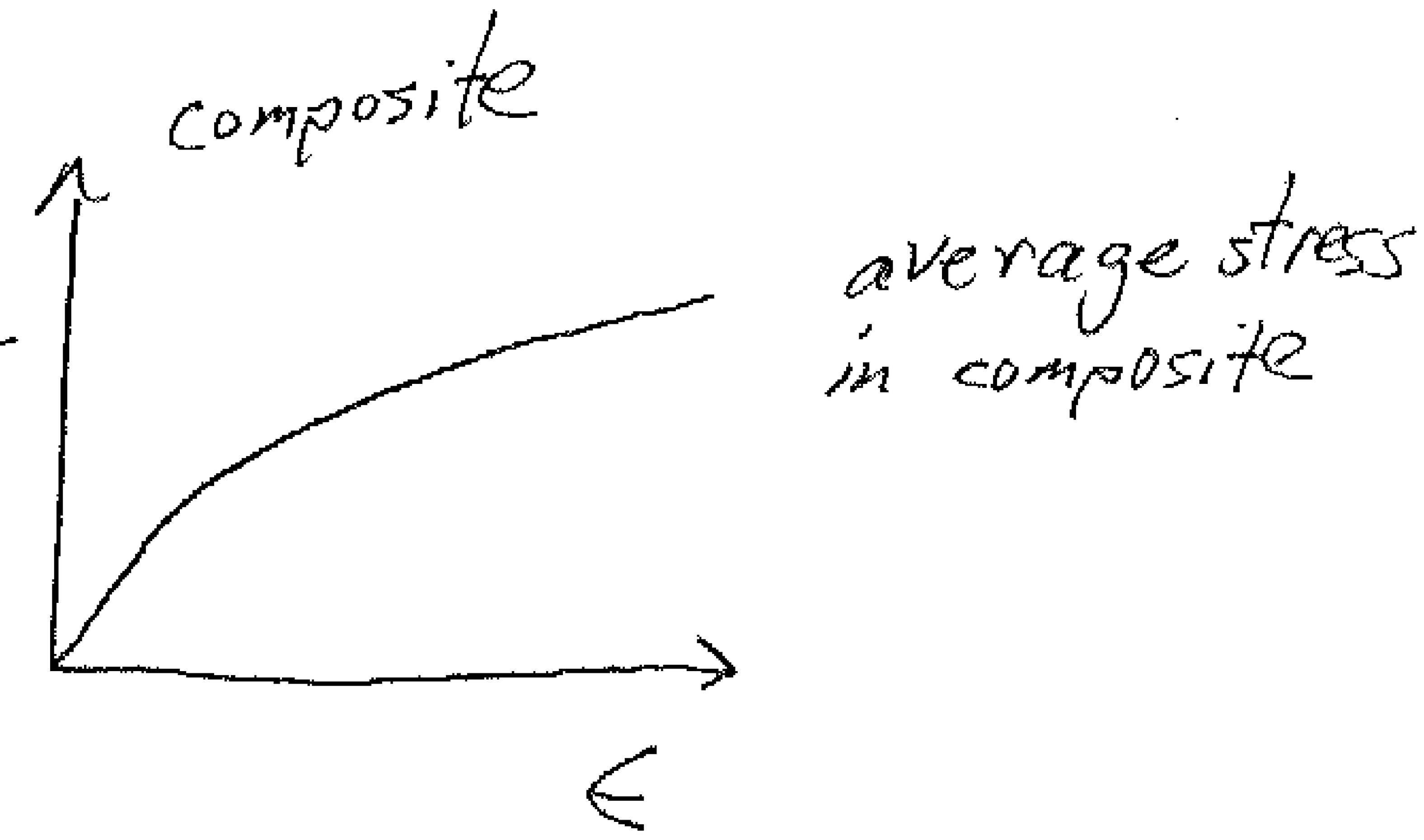
$\hat{n}_\theta = \hat{n}_z = 0$ in radial direction

σ_z uniform in matrix and fiber





$\sigma_{zz}^{AVG} = \frac{F}{A}$



$\alpha^f > \alpha^m$

Find expression for

$\sigma_{rr}^f \quad \sigma_{rr}^m$

$\sigma_{\theta\theta}^f \quad \sigma_{\theta\theta}^m$

in terms of $\alpha^f, \alpha^m, E^m, E^f, \nu^f, \nu^m$ and $\sigma_{zz}^{AVG}, \Delta T$

Find σ_{zz}^{AVG} at which $\sigma_{zz}^m = 0$

Imagine applying ΔT and $\sigma_{zz} = 0$

if $\alpha_1 \neq \alpha_2$ residual stress is generated

Integrating strain to get displacement

$$\sigma_{rr} = \frac{A}{r^2} + B(1+2\ln r) + 2C$$

$$\sigma_{\theta\theta} = -\frac{A}{r^2} + B(3+2\ln r) + 2C$$

$$\epsilon_{rr} = \frac{\sigma_{rr}}{E} - \frac{\nu}{E}(\sigma_{\theta\theta}+\sigma_{zz}) + \alpha\Delta T$$

$$\epsilon_{\theta\theta} = \frac{\sigma_{\theta\theta}}{E} - \frac{\nu}{E}(\sigma_{rr}+\sigma_{zz}) + \alpha\Delta T$$

$$\epsilon_{zz} = \frac{\sigma_{zz}}{E} - \frac{\nu}{E}(\sigma_{rr}+\sigma_{\theta\theta}) + \alpha\Delta T$$

Pl. $\sigma \rightarrow \sigma_{zz} = 0$

pl. $\epsilon \rightarrow \epsilon_{zz} = 0$ (extra $\alpha\Delta T$ from ϵ_{zz} eqn. added to ϵ_{rr} and $\epsilon_{\theta\theta}$ if $\epsilon_{zz} = D$ constant)

start with plane stress

eliminate σ_{zz} in expressions for σ_{rr} & $\sigma_{\theta\theta}$

(1) $\sigma_{zz} = 0$

(2) $\epsilon_{zz} = 0$

(3) $\epsilon_{zz} = 0$

<u>Case 1</u>

$$\epsilon_{rr} = \frac{\sigma_{rr}}{E} - \frac{\nu}{E}\sigma_{\theta\theta} + \alpha\Delta T$$

$$\epsilon_{\theta\theta} = \frac{\sigma_{\theta\theta}}{E} - \frac{\nu}{E}\sigma_{rr} + \alpha\Delta T$$

write $\sigma_{rr}, \sigma_{\theta\theta}$ in terms of stress functions

$$\epsilon_{rr} = \frac{1}{E}\left(\frac{A}{r^2} + B(1+2\ln r) + 2C\right) - \frac{\nu}{E}\left[-\frac{A}{r^2} + B(3+2\ln r) + 2C\right] + \alpha\Delta T$$

$$\frac{\partial u_r}{\partial r} = \epsilon_{rr} = \frac{A}{Er^2}(1+\nu) + \frac{B}{E}(1-3\nu) + \frac{B}{E}(1-\nu)2\ln r + 2(1-\nu)C + \alpha\Delta T$$

Integrate:

$$\frac{d}{dr}\left((1-\nu)2Br\ln r + KBr\right)$$ guess and check with ϵ_{rr}

$$2(1-\nu)B\ln r + 2B(1-\nu) + KB = B(1-3\nu) + B(1-\nu)2\ln r$$

$$K = 1-3\nu-2+2\nu$$
$$K = -(1+\nu)$$

Note:
$$\int \ln ax\,dx = x\ln ax - x + C$$

integrated B term is
$$2(1-\nu)r\ln r - (1+\nu)Br$$

$$u_r = \frac{1}{E}\left[(1+\nu)\left(-\frac{A}{r}\right) + 2(1-\nu)Br\ln r - (1+\nu)Br + 2(1-\nu)Cr\right] + \alpha r\Delta T + f(\theta)$$

assume no variations in z

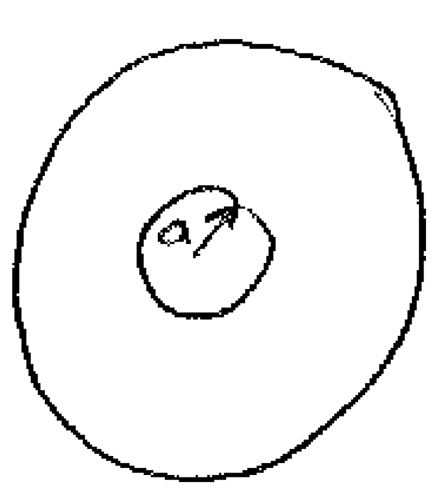
$$\epsilon_{\theta\theta} = \frac{u_r}{r} + \frac{1}{r}\frac{\partial u_\theta}{\partial \theta}$$

sub from constitutive law

$$\epsilon_{r\theta} = \frac{1}{2}\left(\frac{1}{r}\frac{\partial u_r}{\partial \theta} + \frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r}\right) = 0$$

$$u_r\big|_{r=0} = 0$$
$$u_\theta\big|_\theta = 0$$

other integration constants go to zero including B



at $r=a$ $\quad u_r^f = u_r^m$

$$\sigma_{rr}^f = \sigma_{rr}^m$$

go through process for fiber and matrix

2D Problems in Polar Coordinates

$\nabla^4 \phi = 0$ biharmonic eqn.

$\nabla^2(\sigma_{xx} + \sigma_{yy}) = 0$ LaPlace eqn.

stress function $\phi \rightarrow$ compatibility + equilibrium

$$\left.\begin{aligned} \sigma_{rr} &= \frac{1}{r}\frac{\partial \phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \phi}{\partial \theta^2} \\ \sigma_{\theta\theta} &= \frac{\partial^2 \phi}{\partial r^2} \\ \sigma_{r\theta} &= -\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial \phi}{\partial r}\right) \end{aligned}\right\} \text{apply B.C.'s}$$

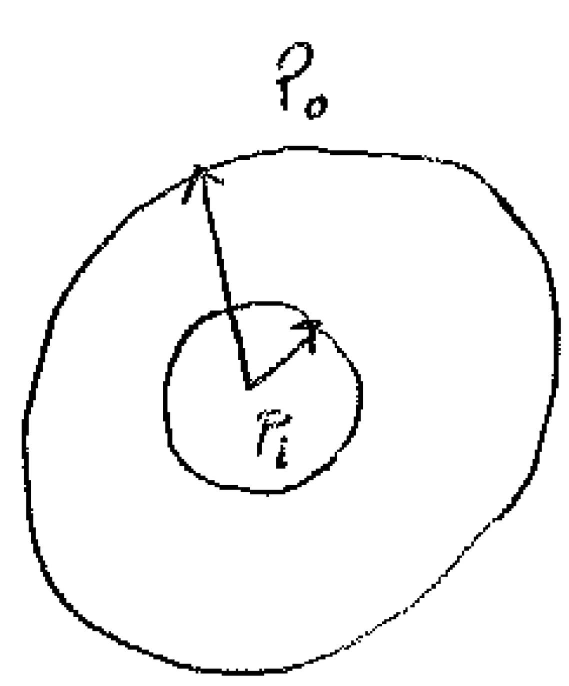
$\underset{=}{\sigma}(\underline{x})$ independent of θ

$\phi = \phi(r)$ integrate $\nabla^4 \phi = 0$

$\phi = A \ln r + B r^2 \ln r + C r^2 + D$

result,

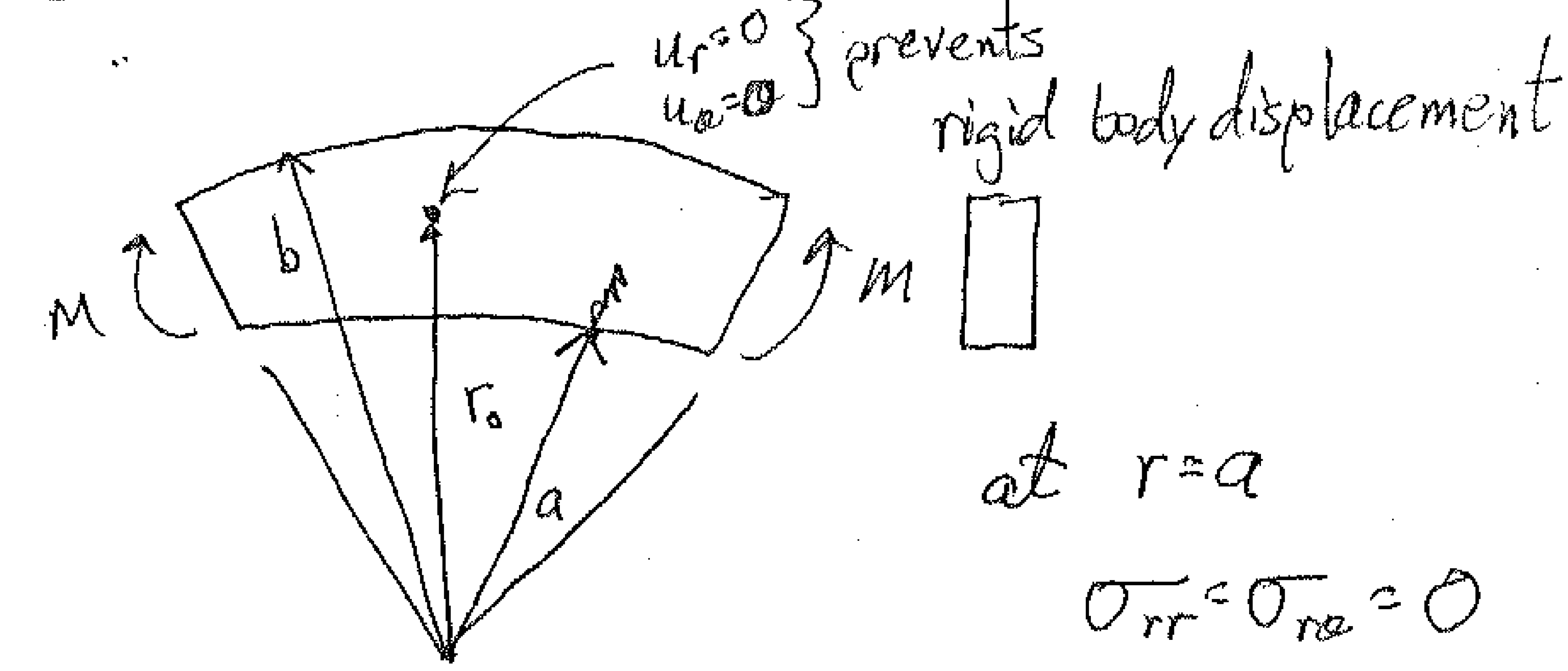
$$\left.\begin{aligned} \sigma_{rr} &= \frac{A}{r^2} + B(2\ln r + 1) + 2C \\ \sigma_{\theta\theta} &= -\frac{A}{r^2} + B(3 + 2\ln r) + 2C \\ \sigma_{r\theta} &= 0 \end{aligned}\right\} \text{apply B.C.'s to get } A, B, C$$



A, C are solved

when subbing σ_{ij} into Hooke's Law

$\epsilon_{\theta\theta}$ will have a θ term — multi-valued displacement



at $r = a$

$\sigma_{rr} = \sigma_{r\theta} = 0 \quad t = 0$

$\frac{\partial u_r}{\partial \theta} = 0$ prevents rigid body rotation

$$\int_a^b \sigma_{\theta\theta} dr = 0$$

$$\int_a^b \sigma_{\theta\theta} r dr = -M$$

see Timoshenko for details

$$\sigma_{rr} = -\frac{4M}{N} f(r, a, b)$$ ← normally neglected in def. bods.

$$N = (b^2 - a^2)^2 - 4a^2b^2\left(\ln \frac{b}{a}\right)^2$$

$$\sigma_{\theta\theta} = -\frac{4M}{N} g(r, a, b)$$

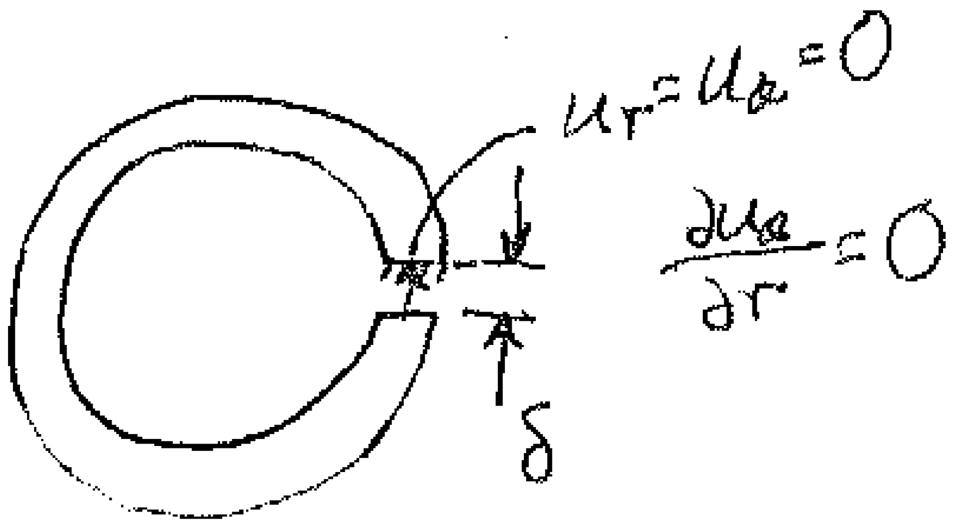
$$\sigma_{r\theta} = 0$$

$$u_\theta = \frac{4Br\theta}{E} + Fr + \underbrace{H\cos\theta + K\sin\theta}_{\text{rigid body}}$$

$\uparrow$

$\theta = 0 \to 2\pi$

u_θ jumps

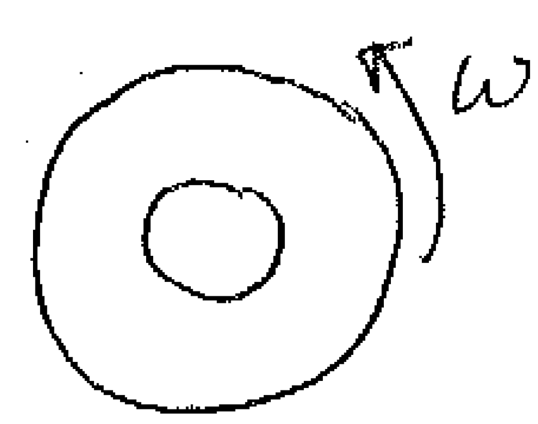


displacement changes as $\theta \to 0 \to 2\pi$ by δ

stress distribution similar to applying pure moment not dependent on θ

Rotating disk

hole optional



body force $R = \rho\omega^2 r$ radial

$S = 0$ tangential

$$\vec{\nabla} \cdot \underline{\underline{\sigma}} + \underline{F} = 0$$

$$\frac{\partial \sigma_{rr}}{\partial r} + \frac{1}{r}\frac{\partial \sigma_{r\theta}}{\partial \theta} + \frac{\sigma_{rr} - \sigma_{\theta\theta}}{r} + R = 0$$

$$\frac{1}{r}\frac{\partial \sigma_{\theta\theta}}{\partial \theta} + \frac{\partial \sigma_{r\theta}}{\partial r} + \frac{2\sigma_{r\theta}}{r} + \cancel{S}^{\,0} = 0$$

solve problem using displacement approach

not θ dependence

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use

$$\begin{cases} \epsilon_{rr} = \dfrac{\partial u_r}{\partial r} \\ \epsilon_{\theta\theta} = \dfrac{u_r}{r} \end{cases}$$

no θ dependence ~~sot~~ so displacement can be solved directly

$$\sigma_{rr} = \frac{E}{1-\nu^2}\left(\epsilon_{rr} + \nu\epsilon_{\theta\theta}\right)$$

$$\sigma_{\theta\theta} = \frac{E}{1+\nu^2}\left(\epsilon_{\theta\theta} + \nu\epsilon_{rr}\right)$$

compatibility automatically satisfied if u_r found

$$\sigma_{r\theta} = 0$$

$$\cancel{\frac{\partial\sigma_{rr}}{\partial r} + \frac{\sigma_{rr} - \sigma_{\theta\theta}}{r} + \rho\omega^2 r = 0}$$

$$\cancel{\frac{1}{r^2}\frac{d}{dr}\left(r^2\sigma_{rr}\right) - \sigma_{\theta\theta} + \rho\omega^2 r = 0}$$

$$\cancel{\frac{d\sigma_{rr}}{dr} + \frac{\sigma_{rr} - \sigma_{\theta\theta}}{r} + \rho\omega^2 =}$$

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\sigma_{rr}\right) - \left(\frac{\sigma_{rr} + \sigma_{\theta\theta}}{r}\right) + \rho r\omega^2 = 0$$

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\sigma_{r\theta}\right) = 0$$

$$\frac{2}{r}\sigma_{rr} + \frac{d\sigma_{rr}}{dr} - \left(\frac{\sigma_{rr} + \sigma_{\theta\theta}}{r}\right) + \rho r\omega^2 = 0$$

$$\frac{d\sigma_{rr}}{dr} + \frac{\sigma_{rr} - \sigma_{\theta\theta}}{r} + \rho r\omega^2 = 0 \quad \checkmark$$

$$r^2\frac{du_r}{dr^2} + r\frac{du_r}{dr} - u_r = -\frac{1-\nu^2}{E}\rho\omega^2 r^2$$

$$u_r = \frac{1}{E}\left[(1-\nu)Cr - (1+\nu)\frac{C_1}{r} - \frac{1-\nu^2}{8}\rho\omega^2 r^2\right]$$

$$\sigma_{rr} = C + \frac{C_1}{r^2} - \frac{3+\nu}{8}\rho\omega^2 r^2$$

$$\sigma_{\theta\theta} = C - \frac{C_1}{r^2} - \frac{1+3\nu}{8}\rho\omega^2 r^2$$

$$\sigma_{r\theta} = 0$$

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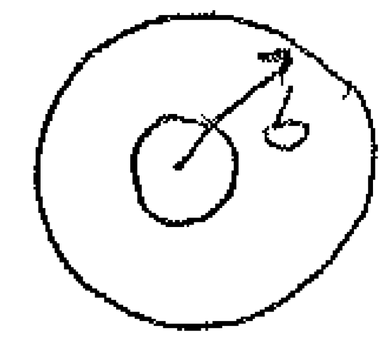
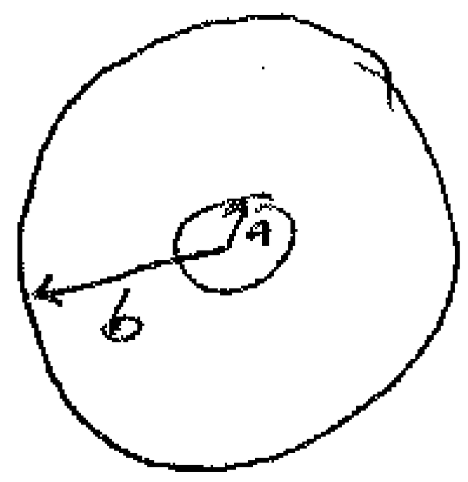
if $P_{outer} = 0$

$$C = \frac{3+\nu}{8}\rho\omega^2 b^2 \quad (\sigma_{rr} = 0 \text{ at } r = b)$$



if no hole $C_1 = 0$

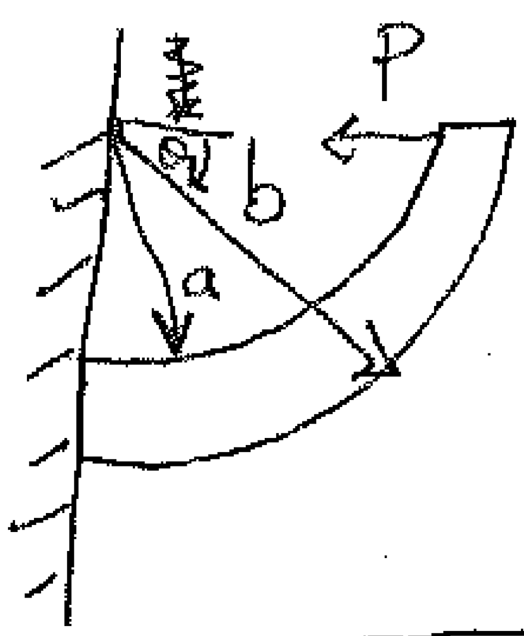


if hole included

$\sigma_{rr} = 0$ at $r = a, b$

a tiny hole in the center ~~doubl~~ doubles the stress due to concentrations

Bending of a curved bar by force at end



Note: P is distributed to match internal shear stress at $\theta = 0$

We can see $\sigma = \sigma(\theta)$ also r

try stress function ~~$f(r)\sin\phi = \phi$~~

$f(r)\sin\theta = \phi$

put in $\nabla^4\phi = 0$

integrate $f(r) = Ar^3 + \frac{B}{r} + Cr + Dr\ln r$

$$\sigma_{rr} = \left(2Ar - \frac{2B}{r^3} + \frac{D}{r}\right)\sin\theta$$

$$\sigma_{\theta\theta} = \left(6Ar + \frac{2B}{r^3} + \frac{D}{r}\right)\sin\theta$$

$$\sigma_{r\theta} = -\left(2Ar - \frac{2B}{r^3} + \frac{D}{r}\right)\cos\theta$$

B.C.'s:

$\sigma_{rr} = 0$ at $r = a, b$

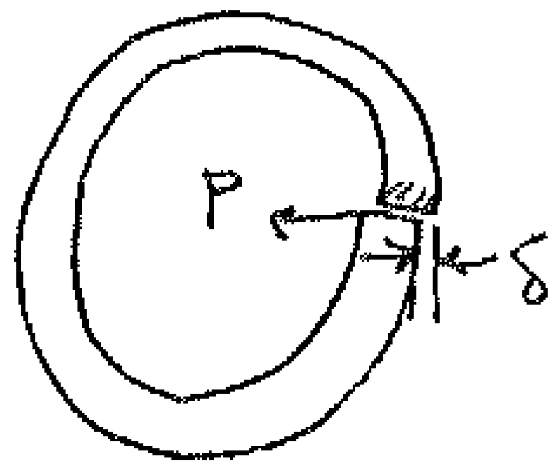
$A = \frac{P}{2N}$, $B = -\frac{Pa^2b^2}{2N}$

(see Timoshenko p. 85)

if $f(r)\cos\theta = \phi$

$f(r)$ is different

if bar is a full circle



$\theta = 0$ $\quad u_r = 0$

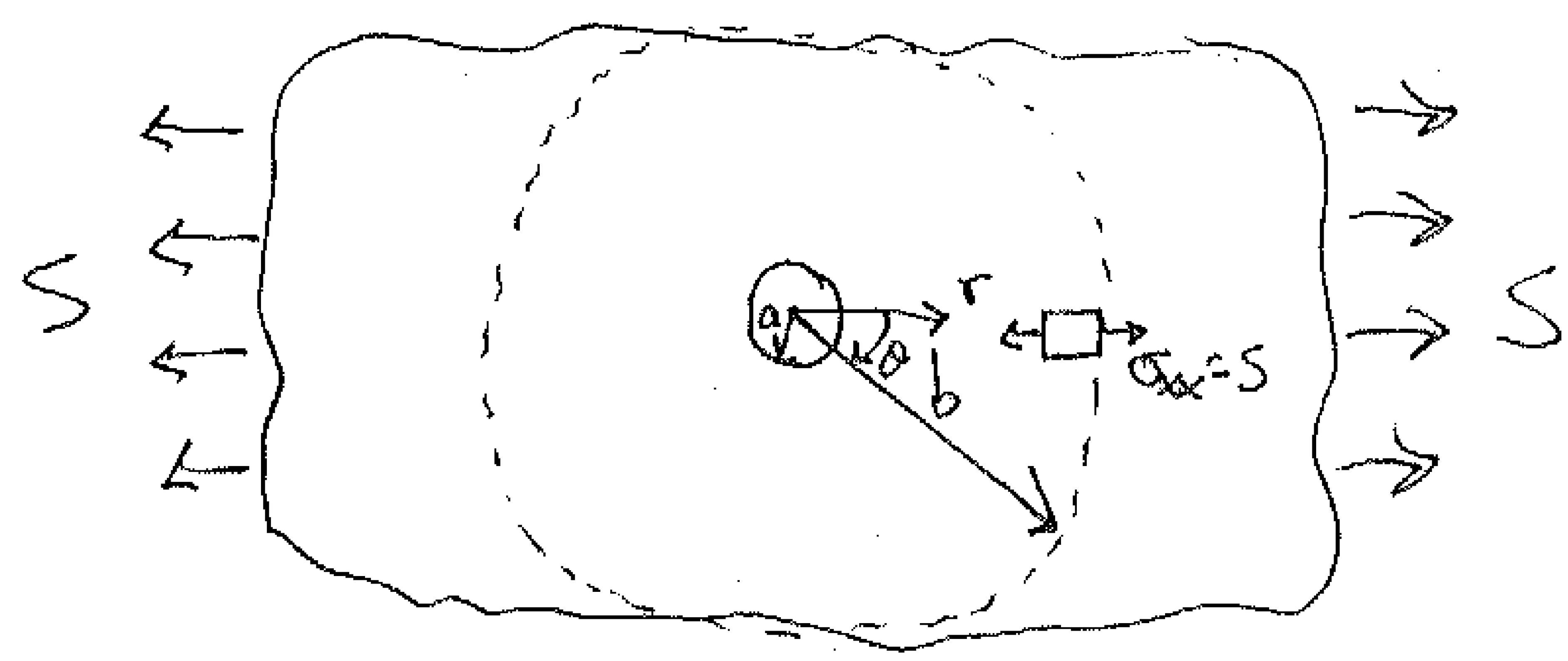
$\theta = 2\pi$ $\quad u_r = \delta$

This gives the elastic stress field assumed to arise at the core of an edge dislocation.

$$\delta = -\frac{2D}{E}2\pi$$

$$D = -\frac{P}{N}(a^2 + b^2)$$

Infinite plate



$r = a$

$\sigma_{rr} = \sigma_{\theta r} = 0$

$r = b$

$b \gg a$ but $\ll$ plate dim.

$\theta = 0 \quad \sigma_{rr} = S$

$\theta = \frac{\pi}{2} \quad \sigma_{\theta\theta} = S$

$$\sigma_{rr}\Big|_{r=b} = S\cos^2\theta = \frac{1}{2}(1+\cos 2\theta)S$$

$$\sigma_{r\theta}\Big|_{r=b} = -\frac{S}{2}\sin 2\theta$$

$$\phi = f(r)\cos(2\theta)$$

$$\nabla^4 \phi = 0$$

integrate,

$f = Ar^2 + Br^4 + \frac{C}{r^2} + D$, apply B.C's to get $A = -\frac{S}{4}$

$B = 0$

$C = -\frac{a^4}{4}$

$D = \frac{a^2}{2}S$

$$\sigma_{rr} = \frac{S}{2}\left(1 - \frac{a^2}{r^2}\right) + \frac{S}{2}\left(1 + \frac{3a^4}{r^4} - \frac{4a^2}{r^2}\right)\cos(2\theta)$$

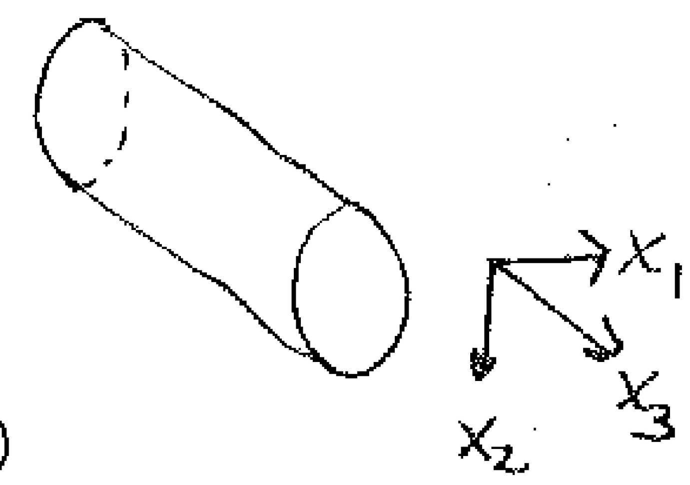
$$\sigma_{\theta\theta} = \frac{S}{2}\left(1 + \frac{a^2}{r^2}\right) - \frac{S}{2}\left(1 + \frac{3a^4}{r^4}\right)\cos(2\theta)$$

$$\sigma_{r\theta} = -\frac{S}{2}\left(1 - \frac{3a^4}{r^4} + \frac{2a^2}{r^2}\right)\sin(2\theta)$$

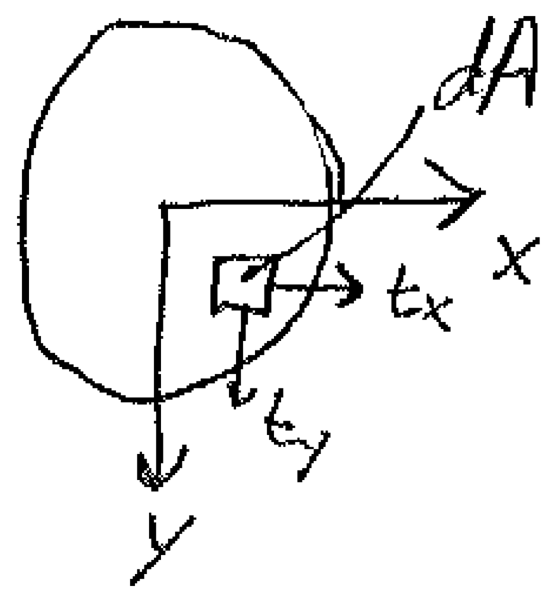
$t_1 = \sigma_{11} n_1 + \sigma_{21} n_2 + \sigma_{31} n_3$

$t_2 = \sigma_{12} n_1 + \sigma_{22} n_2 + \sigma_{32} n_3$

$t_3 = \sigma_{13} n_1 + \sigma_{23} n_2 + \cancelto{0}{\sigma_{33}} n_3 = 0$



$\hookrightarrow \; -G\theta y \frac{x}{r} + G\theta x \frac{y}{r} = 0$



$M = -\iint_A (t_x y - t_y x)\, dx\, dy$

Note: If cross section is not circular, $w \neq 0$

now, $w = \theta \psi(x,y)$

↑ warping function

$\epsilon_{xz} = \frac{1}{2}\left(-y\theta + \theta \frac{\partial \psi}{\partial x}\right)$

$\epsilon_{yz} = \frac{1}{2}\left(x\theta + \theta \frac{\partial \psi}{\partial y}\right)$

$\tau_{xz} = G\gamma_{xz} = G\theta\left(\frac{\partial \psi}{\partial x} - y\right)$

$\tau_{yz} = G\gamma_{yz} = G\theta\left(\frac{\partial \psi}{\partial y} + x\right)$

Equilibrium

$\tau_{xz,z} = \tau_{yz,z} = 0$

$\tau_{xz,x} + \tau_{yz,y} = 0 \rightarrow \frac{\partial \tau_{xz}}{\partial x \partial y} = -\frac{\partial \sigma_{yz}}{\partial y \partial x}$

Prandtl introduced a stress function, ϕ

$\tau_{xz} = \frac{\partial \phi}{\partial y}, \quad \tau_{yz} = -\frac{\partial \phi}{\partial x}$ see pg. 506 (Boresi & Chong)

$\frac{\partial \phi}{\partial y} = G\theta\left(\frac{\partial \psi}{\partial x} - y\right)$

$-\frac{\partial \phi}{\partial x} = G\theta\left(\frac{\partial \psi}{\partial y} + x\right)$