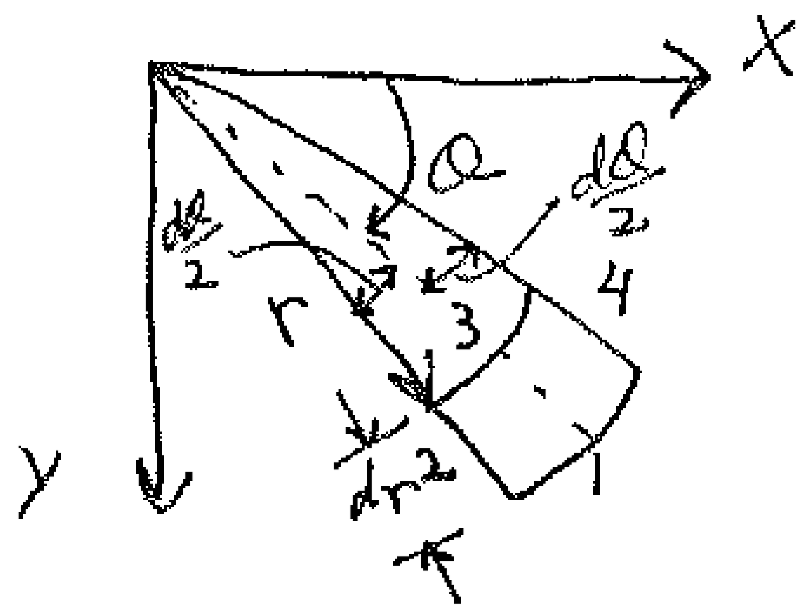
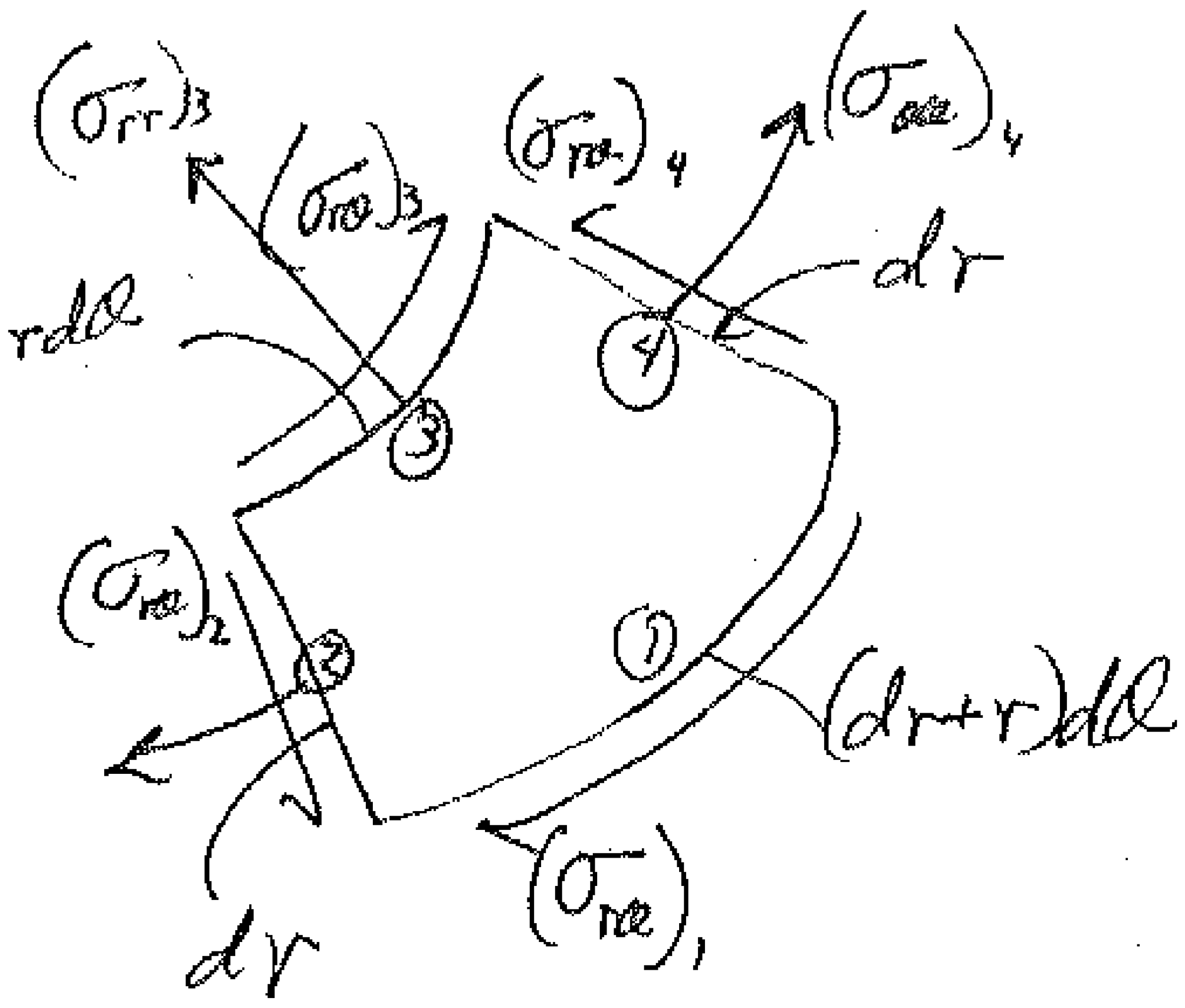
(35)

Polar coordinates: Airy stress analysis





$A_1 = (r+dr)d\theta dz$

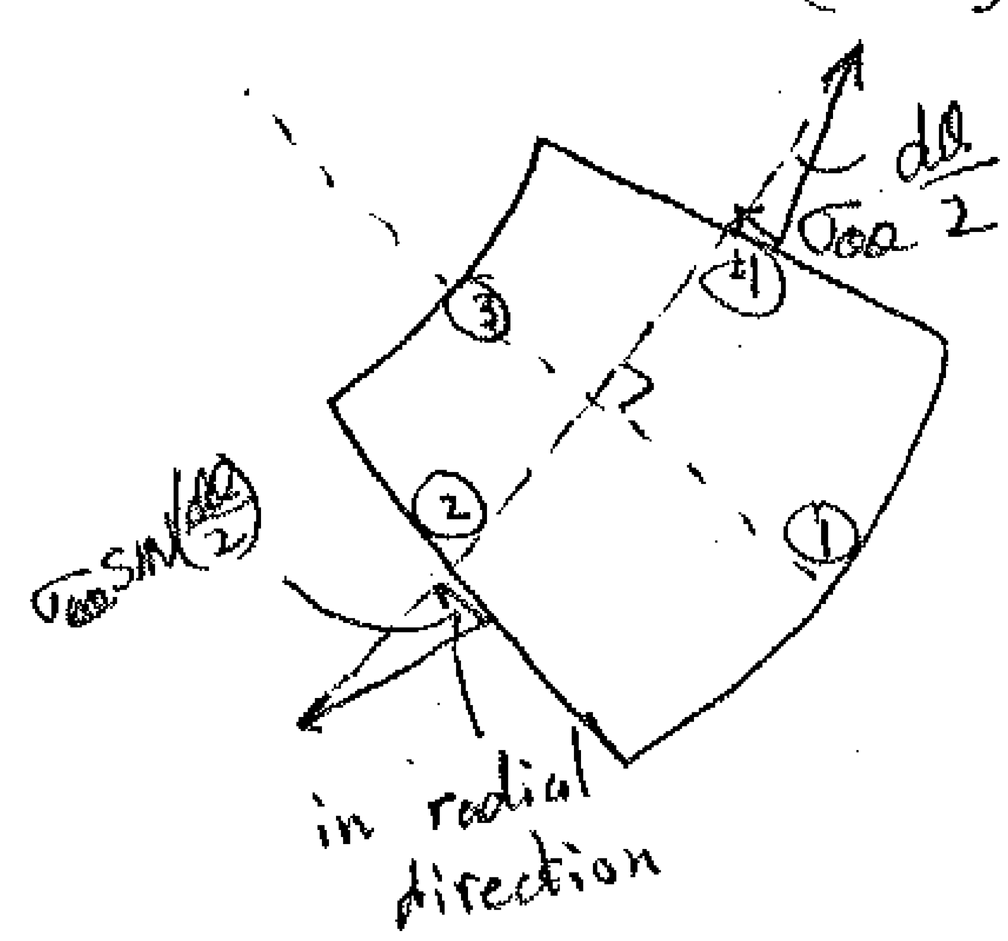
$A_2 = dr\,dz$

$A_3 = r\,d\theta dz$

$A_4 = dr\,dz$

$\sum F_r = 0$

$$(\sigma_{rr})_1(r+dr)d\theta dz - (\sigma_{rr})_3\, r\,d\theta dz + (\sigma_{r\theta})_2\, dr\,dz$$
$$-(\sigma_{r\theta})_4\, dr\,dz - (\sigma_{\theta\theta})_2 \sin\left(\frac{d\theta}{2}\right) dr\,dz - (\sigma_{\theta\theta})_4 \sin\left(\frac{d\theta}{2}\right) dr\,dz = 0$$



Divide by $dr\,d\theta$

$$\frac{(\sigma_{rr})_1(r+dr) - (\sigma_{rr})_3 r}{dr} + \frac{(\sigma_{r\theta})_2 - (\sigma_{r\theta})_4}{d\theta}$$
$$-\left((\sigma_{\theta\theta})_4 - (\sigma_{\theta\theta})_2\right)\frac{d\theta}{2}dr = 0$$

limit $dr, d\theta \to 0$

Equil. Eqn: $$\frac{\partial}{\partial r}(\sigma_{rr} r) + \frac{\partial \sigma_{r\theta}}{\partial \theta} - \sigma_{\theta\theta} = 0$$

$$\sum F_\theta \rightarrow \frac{1}{r}\frac{\partial \sigma_{\theta\theta}}{\partial \theta} + \frac{\partial \sigma_{r\theta}}{\partial r} + \frac{2\sigma_{r\theta}}{r} = 0$$

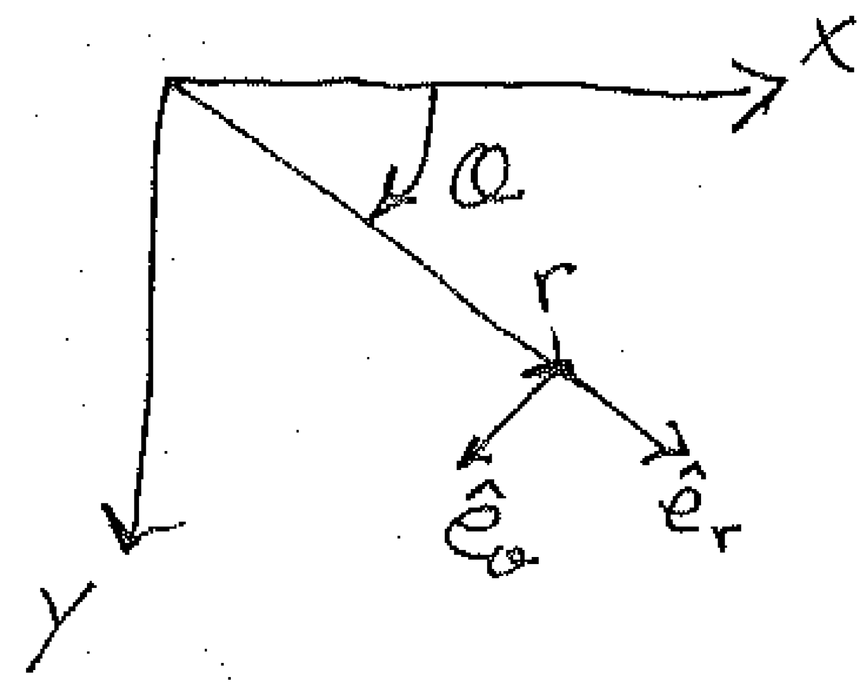
Cartesian form: $$\frac{\partial \sigma_{ji}}{\partial x_j} = 0$$

$\vec{\nabla} \cdot \underline{\underline{\sigma}} = 0$ general form, independent of coordinate system

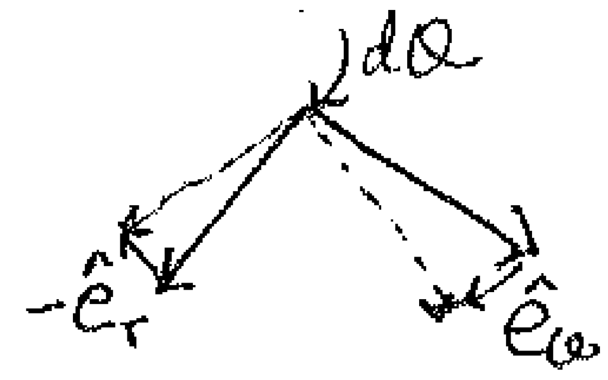
$$\vec{\nabla} = \frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}$$
$$\vec{\nabla} = \frac{\partial}{\partial r}\hat{e}_r + \frac{1}{r}\frac{\partial}{\partial \theta}\hat{e}_\theta + \frac{\partial}{\partial r}\hat{e}_z$$

stress tensor:

$$\underline{\underline{T}} = T_{rr}\hat{e}_r\hat{e}_r + T_{r\theta}\hat{e}_r\hat{e}_\theta + T_{rz}\hat{e}_r\hat{e}_z$$
$$T_{\theta r}\hat{e}_\theta\hat{e}_r + T_{\theta\theta}\hat{e}_\theta\hat{e}_\theta + T_{\theta z}\hat{e}_\theta\hat{e}_z$$
$$T_{zr}\hat{e}_z\hat{e}_r + T_{z\theta}\hat{e}_z\hat{e}_\theta + T_{zz}\hat{e}_z\hat{e}_z$$



$$\frac{\partial \hat{e}_r}{\partial \theta} = \hat{e}_\theta$$



$$\frac{\partial \hat{e}_\theta}{\partial \theta} = -\hat{e}_r$$ all others zero

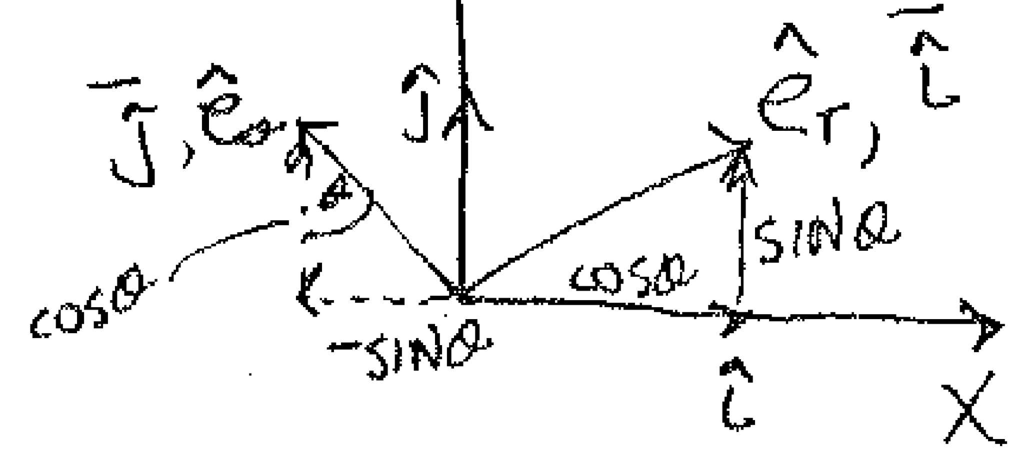
To operate $\vec{\nabla}\cdot\underline{\underline{T}} = 0$ you get the same result as the $\Sigma F_r = \Sigma F_\theta = 0$ result

$$\sigma_{xx} = \frac{\partial^2\phi}{\partial y^2}$$
$$\sigma_{yy} = \frac{\partial^2\phi}{\partial x^2}$$
$$\sigma_{xy} = \frac{\partial^2\phi}{\partial x\partial y}$$

$$\frac{\partial}{\partial x} = \vec{\nabla}\cdot\hat{i} = \left(\hat{e}_r\frac{\partial}{\partial r} + \frac{\hat{e}_\theta}{r}\frac{\partial}{\partial\theta}\right)\cdot\hat{i} \quad \Big| \quad \frac{\partial}{\partial y} = \vec{\nabla}\cdot\hat{j} = \left(\hat{e}_r\frac{\partial}{\partial r} + \frac{\hat{e}_\theta}{r}\frac{\partial}{\partial\theta}\right)\cdot\hat{j}$$

$$r^2 = x^2 + y^2$$
$$\theta = \tan^{-1}\left(\frac{y}{x}\right)$$

$$\frac{\partial}{\partial x} = \cos\theta\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial\theta}$$
$$\frac{\partial}{\partial y} = \left(\hat{e}_r\frac{\partial}{\partial r} + \frac{\hat{e}_\theta}{r}\frac{\partial}{\partial\theta}\right)\cdot\hat{j}$$
$$= \underbrace{\sin\theta}_{\frac{\partial r}{\partial y}}\frac{\partial}{\partial r} + \underbrace{\frac{\cos\theta}{r}}_{\frac{\partial\theta}{\partial y}}\frac{\partial}{\partial\theta}$$



	$\bar{i}$	$\bar{j}$
$\hat{i}$	cosθ	-sinθ
$\hat{j}$	sinθ	cosθ

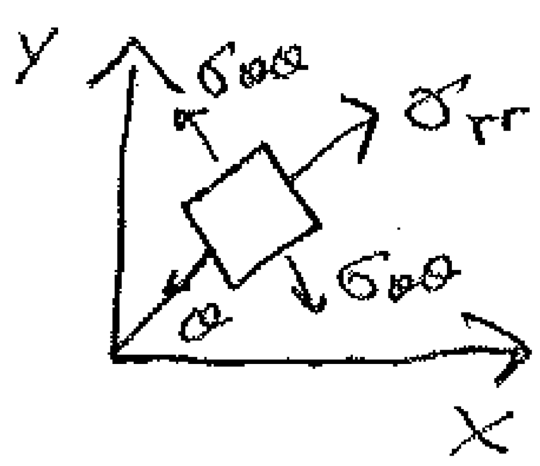
$$\sigma_{xx} = T_{xx} = \frac{\partial^2 \phi}{\partial x^2} = \left(\sin\theta \frac{\partial}{\partial r} + \frac{\cos\theta}{r}\frac{\partial}{\partial \theta}\right)\left(\sin\theta \frac{\partial \phi}{\partial r} + \frac{\cos\theta}{r}\frac{\partial \phi}{\partial \theta}\right)$$

$$= \sin^2\theta \frac{\partial^2 \phi}{\partial r} + \sin\theta\cos\theta \frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial \phi}{\partial \theta}\right) + \frac{\cos\theta}{r}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial \phi}{\partial r}\right)$$

$$+ \frac{\cos\theta}{r^2}\frac{\partial}{\partial \theta}\left(\cos\theta \frac{\partial \phi}{\partial \theta}\right)$$

$$\sigma_{xx} = \sin^2\theta \frac{\partial^2 \phi}{\partial r^2} + \sin\theta\cos\theta \frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial \phi}{\partial r}\right) + \frac{\cos\theta}{r}\left[\cos\theta \frac{\partial \phi}{\partial r} + \sin\theta \frac{\partial^2 \phi}{\partial r \partial \theta}\right]$$

$$+ \frac{\cos\theta}{r^2}\left[-\sin\theta \frac{\partial \phi}{\partial \theta} + \cos\theta \frac{\partial^2 \phi}{\partial \theta^2}\right]$$



$\theta = 0^\circ$

$\sigma_{xx} = \sigma_{rr}$

$\sigma_{yy} = \sigma_{\theta\theta}$

$$\boxed{\sigma_{rr} = \frac{1}{r}\frac{\partial \phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \phi}{\partial \theta^2}}$$

when $\theta = \frac{\pi}{2}$

$\sigma_{xx} = \sigma_{\theta\theta}$

$\sigma_{yy} = \sigma_{rr}$

$$\boxed{\sigma_{\theta\theta} = \frac{\partial^2 \phi}{\partial r^2}}$$

also:

$$\boxed{\sigma_{r\theta} = -\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial \phi}{\partial \theta}\right)}$$

$$\nabla^2 \phi = \vec{\nabla} \cdot \vec{\nabla}\phi$$

$$= \left(\hat{e}_r \frac{\partial}{\partial r} + \frac{\hat{e}_\theta}{r}\frac{\partial}{\partial \theta}\right) \cdot \left(\hat{e}_r \frac{\partial \phi}{\partial r} + \frac{\hat{e}_\theta}{r}\frac{\partial \phi}{\partial \theta}\right)$$

$$= \hat{e}_r \cdot \cancelto{0}{\frac{\partial}{\partial r}}\left(\hat{e}_r \frac{\partial \phi}{\partial r}\right) + \hat{e}_r \cdot \cancelto{0}{\frac{\partial}{\partial r}}\left(\frac{\hat{e}_\theta}{r}\frac{\partial \phi}{\partial \theta}\right)$$

$$+ \frac{\hat{e}_\theta}{r} \cdot \underbrace{\frac{\partial}{\partial \theta}\left(\hat{e}_r \frac{\partial \phi}{\partial r}\right)}_{\underbrace{\frac{\partial \hat{e}_r}{\partial \theta}}_{\hat{e}_\theta}\frac{\partial \phi}{\partial r} + \hat{e}_r \frac{\partial^2 \phi}{\partial \theta \partial r}} + \frac{\hat{e}_\theta}{r} \cdot \frac{\partial}{\partial \theta}\left(\frac{\hat{e}_\theta}{r}\frac{\partial \phi}{\partial \theta}\right)$$

$$\boxed{\nabla^2 \phi = \frac{\partial^2 \phi}{\partial r^2} + \frac{1}{r}\frac{\partial \phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \phi}{\partial \theta^2}}$$

$$\nabla^4\phi = 0 = \left[\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\right]\left[\frac{\partial^2\phi}{\partial r^2} + \frac{1}{r}\frac{\partial\phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2}\right]$$

$$\sigma_{rr} = \frac{1}{r}\frac{\partial\phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2}$$

$$\sigma_{\theta\theta} = \frac{\partial^2\phi}{\partial r^2}$$

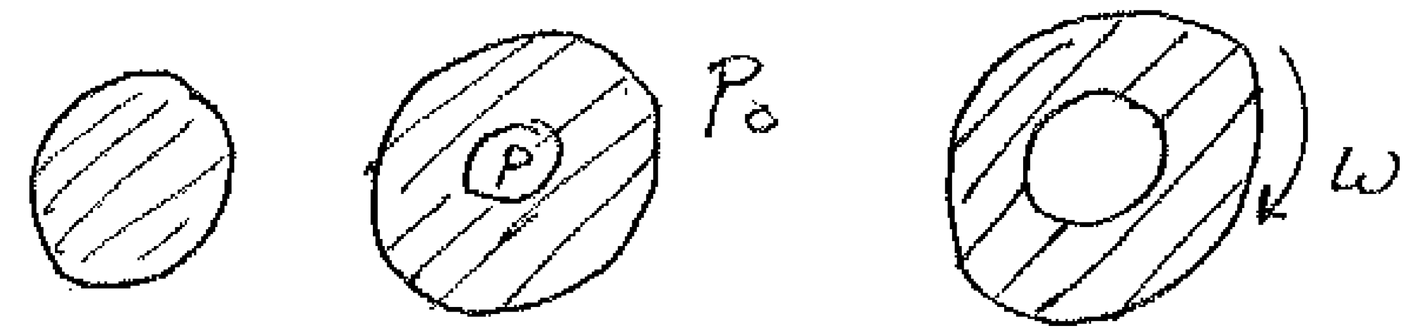
$$\sigma_{r\theta} = \frac{1}{r^2}\frac{\partial\phi}{\partial\theta} - \frac{1}{r}\frac{\partial^2\phi}{\partial r\,\partial\theta}$$

Also need constitutive law and strain/ displacement relation in polar coordinates

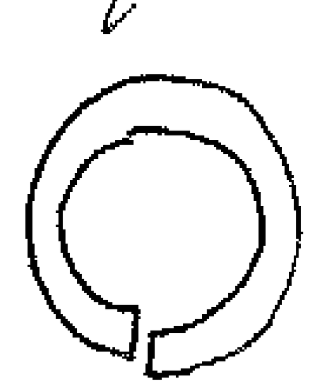
Need to integrate $\nabla^4\phi = 0$ to solve eqns.

Case 1: ϕ independent of θ

$\underline{\underline{\sigma}}$ independent of θ



Case 2: $\phi = f(r)\sin\theta$



dislocation stress field

① $\phi = \phi(r)$

$$\nabla^4\phi = 0 = \left[\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r}\right]\left[\frac{\partial^2\phi}{\partial r^2} + \frac{1}{r}\frac{\partial\phi}{\partial r}\right] = 0$$

Let $e^t = r$

$$\frac{d}{dr} = \frac{d}{d(e^t)} = \frac{1}{e^t}\frac{d}{dt} \quad\Big|\quad \frac{d^2}{dr^2} = \frac{d}{dr}\left[\frac{1}{e^t}\frac{d}{dt}\right] = \frac{1}{e^t}\frac{d}{dt}\left(\frac{1}{e^t}\frac{d}{dt}\right) = -\frac{1}{e^{2t}}\frac{d}{dt} + \frac{1}{e^{2t}}\frac{d^2}{dt^2}$$

$$\frac{d^3}{dr^3} = 2e^{-3t}\frac{d}{dt} - 3e^{-3t}\frac{d^2}{dt^2} + e^{-3t}\frac{d^3}{dt^3}$$

$$\frac{d^4}{dr^4} = -6e^{-4t}\frac{d}{dt} + 11e^{-4t}\frac{d^2}{dt^2} - 6e^{-4t}\frac{d^3}{dt^3} + e^{-4t}\frac{d^4}{dt^4}$$

$$\nabla^4\phi(r) = \frac{d^4\phi}{dr^4} + \frac{2}{r}\frac{d^3\phi}{dr^3} - \frac{1}{r^2}\frac{d^2\phi}{dr^2} + \frac{1}{r^3}\frac{d\phi}{dr} = 0$$

when substituting, each term has an e^{-4t} that cancells

$$\underbrace{\frac{d^4\phi}{dt^4} - 6\frac{d^3\phi}{dt^3} + 11\frac{d^2\phi}{dt^2} - 6\frac{d\phi}{dt}}_{\text{from 1st term}}$$

$$+ \underbrace{2\frac{d^3\phi}{dt^3} - 6\frac{d^2\phi}{dt^2} + 4\frac{d\phi}{dt}}_{\text{2nd term}}$$

$$\underbrace{-1\frac{d^2\phi}{dt^2} + 1\frac{d\phi}{dt}}_{\text{3rd}} + \underbrace{\frac{d\phi}{dt}}_{\text{4th}} = 0$$

$$\frac{d^4\phi}{dt^4} - 4\frac{d^3\phi}{dt^3} + 4\frac{d^2\phi}{dt^2} + 0\frac{d\phi}{dt} = 0$$

form of an ODE

Solve by using $\phi = e^{\lambda t}$

$\lambda^4 - 4\lambda^3 + 4\lambda^2 = 0$

$\lambda^2(\lambda^2 - 4\lambda + 4) = 0$

$\lambda = 0$

$(\lambda - 2)^2 = 0 \rightarrow$ double root $\lambda = 2$

$\phi = Ae^{2t} + Bte^{2t}$ (homogeneous solution)

need to add particular solution by direct integration

$$\frac{d^3\phi}{dt^3} - \frac{d^2\phi}{dt^2} + 4\frac{d\phi}{dt} = C_1$$

$$\frac{d^2\phi}{dt^2} - \frac{d\phi}{dt} + 4\phi = C_1 t + C_2$$

– 1st order approach

$$\left.\begin{aligned} 4\phi &= C_1 t + C_2 \\ \phi &= \frac{C_1 t}{4} + \frac{C_2}{4} \end{aligned}\right\} \rightarrow \frac{d\phi}{dt} = \frac{C_1}{4}$$

$$-4\frac{d\phi}{dt}+4\phi=-c_1+c_1t+c_2$$

$$\phi=Ae^{2t}+Bte^{2t}+c_1t+c_2$$

substitute

$$r=e^t$$

$$e^{2t}=r^2$$

$$\phi=Ar^2+Br^2\ln r-c+D\ln r$$

(in Timoshenko: $\phi=A\ln r+Br^2\ln r+Cr^2+D$)

$$\sigma_{rr}=\frac{1}{r}\frac{d\phi}{dr}+\frac{1}{r^2}\frac{d^2\phi}{d\theta}\to 0$$

$$\frac{\partial\phi}{\partial r}=2Cr+2Br\ln r+Br+\frac{A}{r}$$

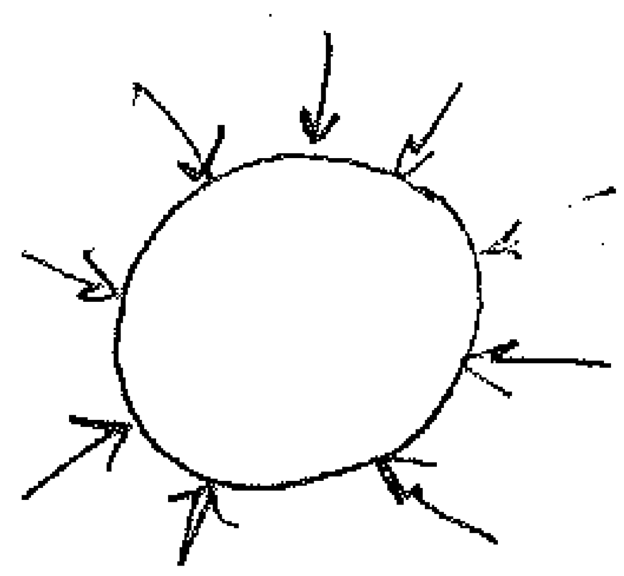
$$\sigma_{rr}=2C+2B\ln r+B+\frac{A}{r^2}$$

note: constants redefined for consistency w/ Timoshenko notation.

$$\sigma_{\theta\theta}=\frac{\partial^2\phi}{\partial r^2}=2C+2B\ln r+2B-\frac{A}{r^2}+B$$

$$=-\frac{A}{r^2}+B[3+2\ln r]+2C$$

$$\sigma_{r\theta}=\frac{1}{r^2}\frac{\partial\phi}{\partial\theta}-\frac{1}{r}\frac{\partial^2\phi}{\partial r\partial\theta}=0$$



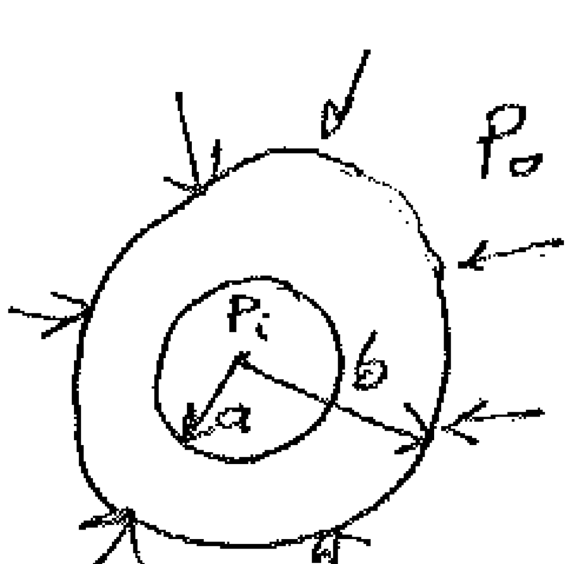
P_o (axsymmetric — no θ dependence)

from $\sigma_{rr}\to B+A=0$

σ_{rr} is finite at $r=0$

$$\sigma_{rr}=2C=-P_o$$

$$\sigma_{\theta\theta}=2C=-P_o$$



@ $r=a$ $\sigma_{rr}=-P_i$

@ $r=b$ $\sigma_{rr}=-P_o$

take $B=0$

$$\sigma_{rr}=\frac{A}{r^2}+2C$$

$$\sigma_{\theta\theta}=-\frac{A}{r^2}+2C$$

from B.C.'s

$$-P_i = \frac{A}{a^2} + 2C$$

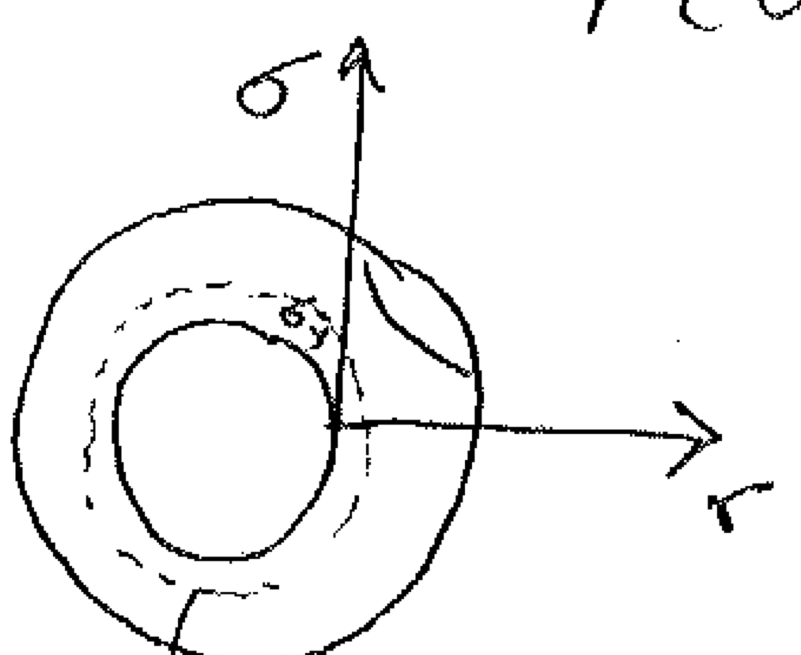
$$-P_o = -\frac{A}{b^2} + 2C$$

$$A = \frac{P_o - P_i}{-\frac{1}{b^2} + \frac{1}{a^2}} = \frac{a^2 b^2 (P_o - P_i)}{b^2 - a^2}$$

$$2C = \frac{P_i a^2 - P_o b^2}{b^2 - a^2}$$

$$\sigma_{rr} = \frac{a^2 b^2 (P_o - P_i)}{r^2 (b^2 - a^2)} + \frac{P_i a^2 - P_o b^2}{b^2 - a^2}$$

$$\sigma_{\theta\theta} = -\frac{a^2 b^2 (P_o - P_i)}{r^2 (b^2 - a^2)} + \frac{P_i a^2 - P_o b^2}{b^2 - a^2}$$



relate yield surface to pressure at given radius

when yielded, if pressure unloaded and vessel cut, it will spring back due to residual stress (compression inside/tension outside)