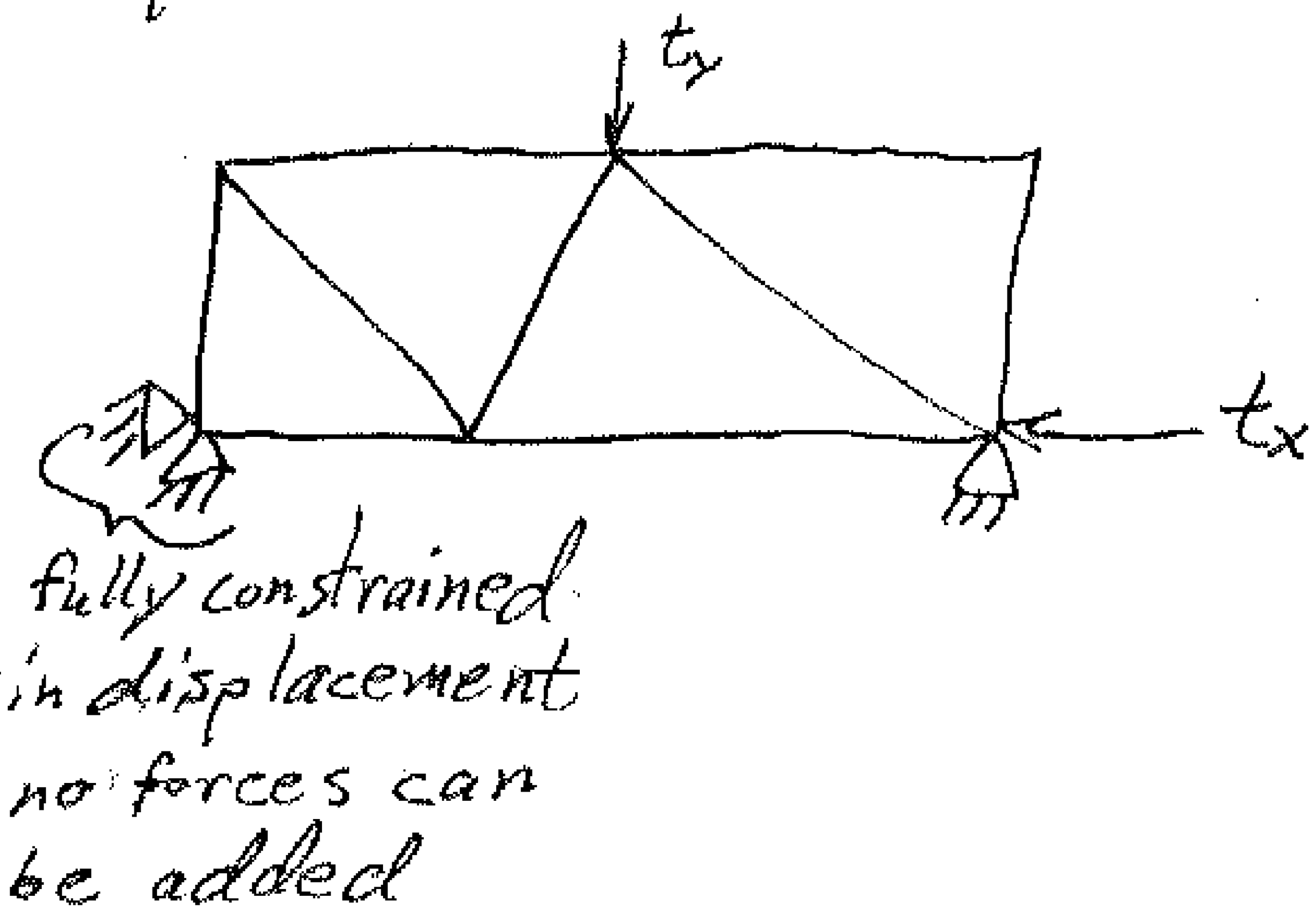
Problem of Elasticity

must satisfy all previous equations plus boundary conditions (B.C.s)

B.C.

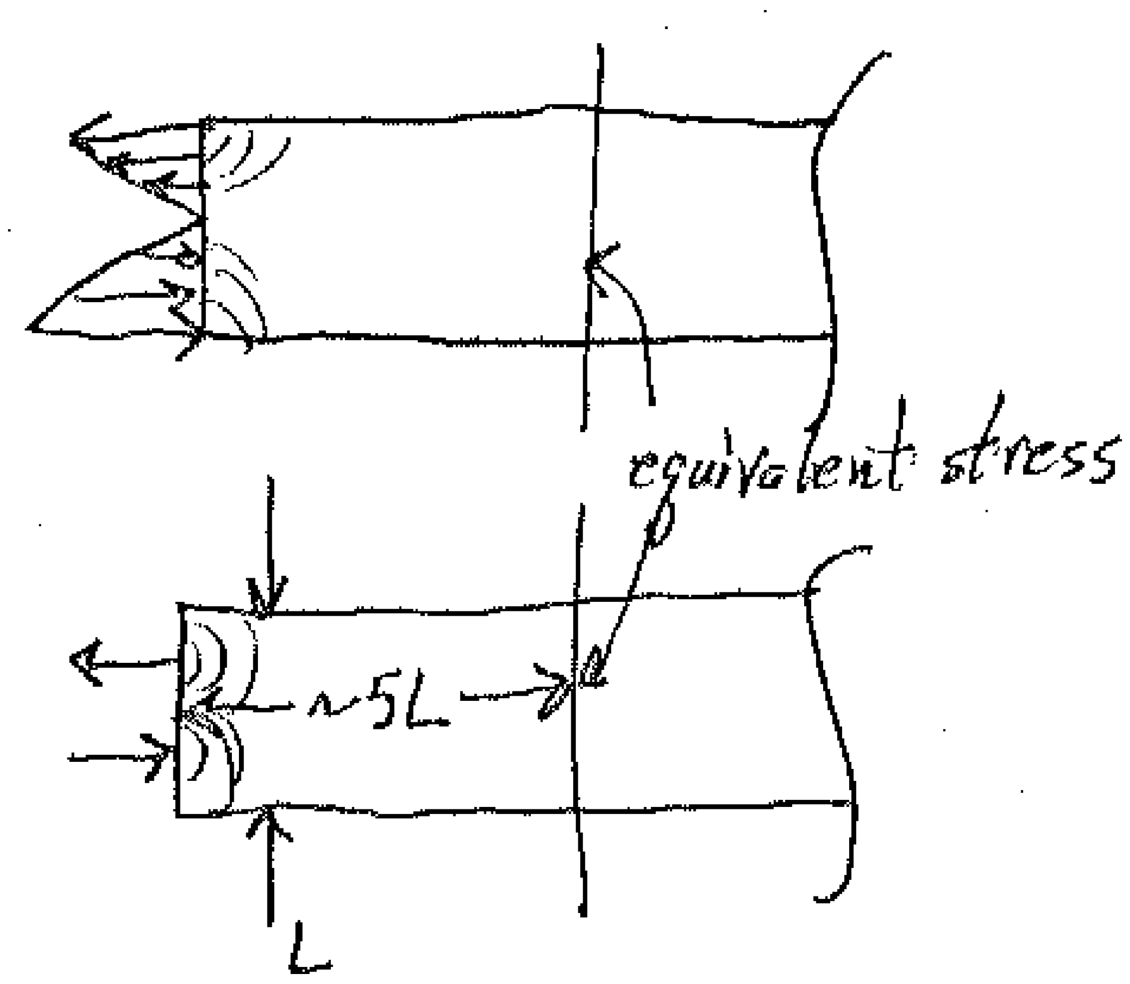
① traction, $\underline{t}$

② displacement, $\underline{u}(\underline{x})$



St. Venant

Replace exact force distribution with a statically equivalent set of forces



Uniqueness of solutions

$$u = \int \sigma_{ij} \, d\epsilon_{ij}$$

$$= \frac{1}{2} C_{ijkl} \epsilon_{ij} \epsilon_{kl}$$

Boresi and Chong go through this in detail

21

Displacement vector

$\underset{\sim}{q} = u_1\hat{i} + u_2\hat{j} + u_3\hat{k}$

from equilibrium

$$\frac{\partial\sigma_{ji}}{\partial x_j} = 0 = \frac{\partial\sigma_{11}}{\partial x_1} + \frac{\partial\sigma_{21}}{\partial x_2} + \frac{\partial\sigma_{31}}{\partial x_3}$$

$$\left\{\begin{array}{l}\sigma_{11} = \lambda e + 2\mu\epsilon_{11}\\ \sigma_{21} = 2\mu\epsilon_{21}\\ \sigma_{31} = 2\mu\epsilon_{31}\end{array}\right.$$

sub into equilibrium eqn.

$$\lambda e_{,i} + 2\mu\epsilon_{11,1} + 2\mu\epsilon_{21,2} + 2\mu\epsilon_{31,3} = 0$$

$$\epsilon_{11} = u_{1,1}$$
$$\epsilon_{21} = \tfrac{1}{2}(u_{2,1} + u_{1,2})$$
$$\epsilon_{31} = \tfrac{1}{2}(u_{3,1} + u_{1,3})$$

$$= \lambda e_{,i} + 2\mu u_{1,11} + \mu(u_{2,12} + u_{1,22}) + \mu(u_{3,13} + u_{1,33})$$

$$= \lambda e_{,i} + \mu(u_{1,11} + u_{1,22} + u_{1,33}) + \mu(u_{1,11} + u_{2,12} + u_{3,13})$$

$$(e_{,i} = \epsilon_{11,1} + \epsilon_{22,1} + \epsilon_{33,1} = u_{1,11} + u_{2,21} + u_{3,31}$$

$$= \underbrace{(\lambda+\mu)e_{,1} + \mu\nabla^2 u_1}_{\beta_x} = 0$$

recall

$\nabla = \partial_x\hat{i} + \partial_y\hat{j} + \partial_z\hat{k}$

$\nabla^2 = \vec{\nabla}\cdot\vec{\nabla} = \partial_{xx} + \partial_{yy} + \partial_{zz}$

$$(\lambda+\mu)e_{,2} + \mu\nabla^2 u_2 = 0$$

$$(\lambda+\mu)e_{,3} + \mu\nabla^2 u_3 = 0$$

$$\beta_x\hat{i} + \beta_y\hat{j} + \beta_z\hat{k} = 0$$

$$(\lambda+\mu)\nabla\nabla\cdot\underset{\sim}{q} + \mu\nabla^2\underset{\sim}{q} = 0$$

see p. 245 Boresi

$\tau_{xy} = 2G\epsilon_{xy}$ $(G = \mu)$

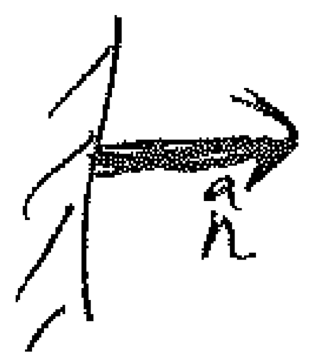
$= 2\mu\epsilon_{21}$

2.1

$\underline{t} = \underline{\underline{\sigma}} \cdot \hat{n}$

internal tractions keep material together

can also be applied externally



$\underline{t} = 0$

$t_i = \sigma_{ji} n_j$

$t_1 = 0 = \sigma_{11} n_1 + \sigma_{21} n_2 + \sigma_{31} n_3$

$t_2 = 0 = \sigma_{12} n_1 + \sigma_{22} n_2 + \sigma_{32} n_3$

$t_3 = 0 = \sigma_{13} n_1 + \sigma_{23} n_2 + \sigma_{33} n_3$

$n_2 = n_3 = 0$

$n_1 = 1$

$\therefore \quad \sigma_{11} = \sigma_{12} = \sigma_{13} = 0$

all others undetermined except

$\sigma_{21} = \sigma_{12} = 0$

$\sigma_{13} = \sigma_{31} = 0$

$\sigma_{22}, \sigma_{33}, \sigma_{23}$ unknown

$t_1 = \sigma_{11} n_1 + \sigma_{21} n_2 + \sigma_{31} n_3$

$\sigma_{11} = \lambda e + 2\mu \epsilon_{11}$

$\sigma_{21} = 2\mu \epsilon_{21}$

$\sigma_{31} = 2\mu \epsilon_{31}$

$t_1 = \lambda e n_1 + 2\mu \epsilon_{11} n_1 + 2\mu \epsilon_{21} n_2 + 2\mu \epsilon_{31} n_3$

sub in displacements

$t_1 = \lambda e n_1 + 2\mu u_{1,1} n_1 + \mu(u_{2,1} + u_{1,2}) n_2 + \mu(u_{3,1} + u_{1,3}) n_3$

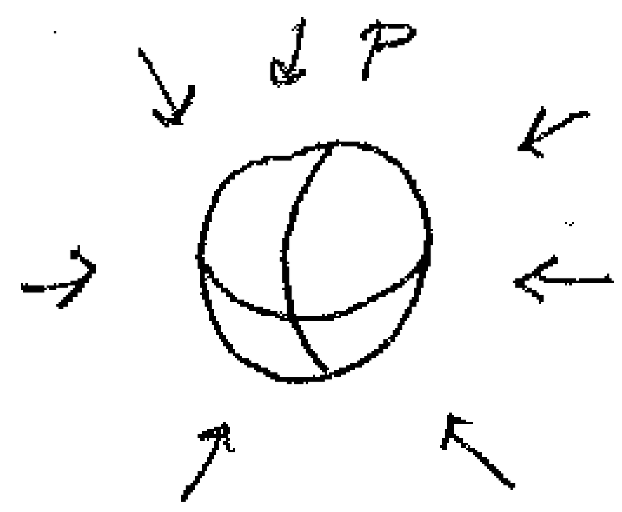
$t_1 = \lambda e n_1 + \mu(u_{1,1} n_1 + u_{1,2} n_2 + u_{1,3} n_3) + \mu(u_{1,1} n_1 + u_{2,1} n_2 + u_{3,1} n_3)$

$\lambda e = \lambda(\epsilon_{11} + \epsilon_{22} + \epsilon_{33}) = \lambda(u_{1,1} + u_{2,2} + u_{3,3})$

$$\boxed{\underline{t} = \lambda \hat{n} \nabla \cdot \underset{\sim}{u} + 2\mu\left(n_1 \nabla \underset{\sim}{u_1} + n_2 \nabla \underset{\sim}{u_2} + n_3 \nabla \underset{\sim}{u_3} + \hat{n} \cdot \nabla \underset{\sim}{u}\right)}$$

Semi-Inverse Method

1) Guess at solution (stress field)
2) Check to see if it matches B.C.'s.
   If yes, it is the solution



hydrostatic loading on sphere

$$[\sigma] = \begin{bmatrix} -p & 0 & 0 \\ 0 & -p & 0 \\ 0 & 0 & -p \end{bmatrix}$$

$$\epsilon_{ij} = \frac{1+\nu}{E}\sigma_{ij} - \frac{\nu}{E}\sigma_{kk}\delta_{ij}$$

$$\epsilon_{11} = \frac{1+\nu}{E}(-p) - \frac{\nu}{E}(-p)\,3$$

$$\epsilon_{11} = \frac{1-2\nu}{E}(-p) = \epsilon_{22} = \epsilon_{33}$$

$$\epsilon_{12} = \epsilon_{13} = \epsilon_{23} = 0$$

$$\frac{\Delta V}{V_0} = \epsilon_{kk} = \epsilon_{11} + \epsilon_{22} + \epsilon_{33} = e$$

$$= \frac{-3P(1-2\nu)}{E}$$

$$P = -Ke$$

$$K = \frac{E}{3(1-2\nu)}$$

if $\frac{\Delta V}{V_0} = e = 0$ it's incompressible

$$\therefore\ 1-2\nu = 0$$

$$\boxed{\nu = \frac{1}{2}}$$

Find displacements

$$\epsilon_{11} = \frac{\partial u_1}{\partial x_1} = \frac{-P(1-2\nu)}{E}$$

$$u_1 = -\frac{P(1-2\nu)}{E} x_1 + f(x_2, x_3)$$

$$u_2 = -\frac{P(1-2\nu)}{E} x_2 + g(x_1, x_3)$$

$$u_3 = -\frac{P(1-2\nu)}{E} x_3 + h(x_1, x_2)$$

$$\epsilon_{12} = 0 = \frac{1}{2}\left(\frac{\partial f}{\partial x_2} + \frac{\partial g}{\partial x_1}\right)$$

$$\epsilon_{13} = 0 = \frac{1}{2}\left(\frac{\partial f}{\partial x_3} + \frac{\partial h}{\partial x_1}\right)$$

$$\epsilon_{23} = 0 = \frac{1}{2}\left(\frac{\partial g}{\partial x_3} + \frac{\partial h}{\partial x_2}\right)$$

$$\frac{\partial f}{\partial x_2} = -\frac{\partial g}{\partial x_1} = a$$

$$f = a x_2 + f_1(x_3)$$

$$g = -a x_1 + f_2(x_3)$$

through other two equations

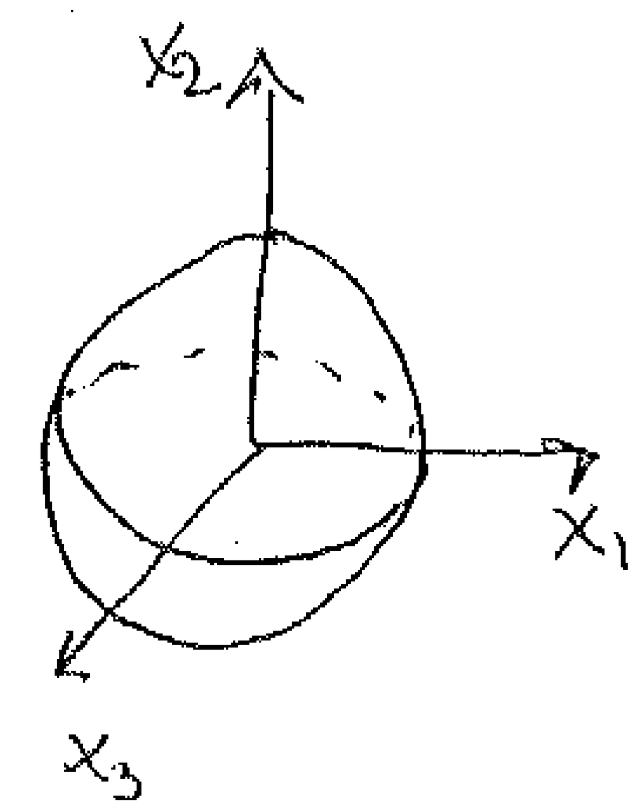
$$f = a x_2 + b x_3 + c$$

⋮

$$u_1 = -\frac{P(1-2\nu)}{E} x_1 + a x_2 + b x_3 + C$$

$$u_2 = -\frac{P(1-2\nu)}{E} x_2 - a x_1 - d x_3 + j$$

$$u_3 = -\frac{P(1-2\nu)}{E} x_3 - b x_1 - d x_2 + k$$



@ $(x_1, x_2, x_3) = (0, 0, 0)$

displacements are zero and $c = j = k = 0$

~~not~~ no rotations either $\omega_{12} = \omega_{13} = \omega_{23} = 0$

$$\therefore a, b, d = 0$$