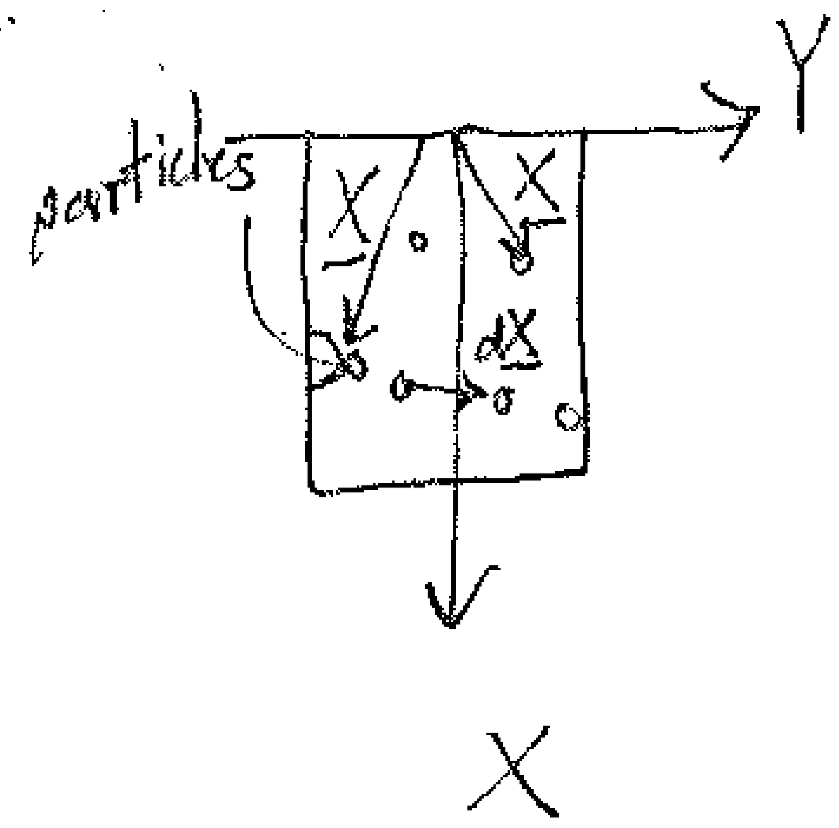
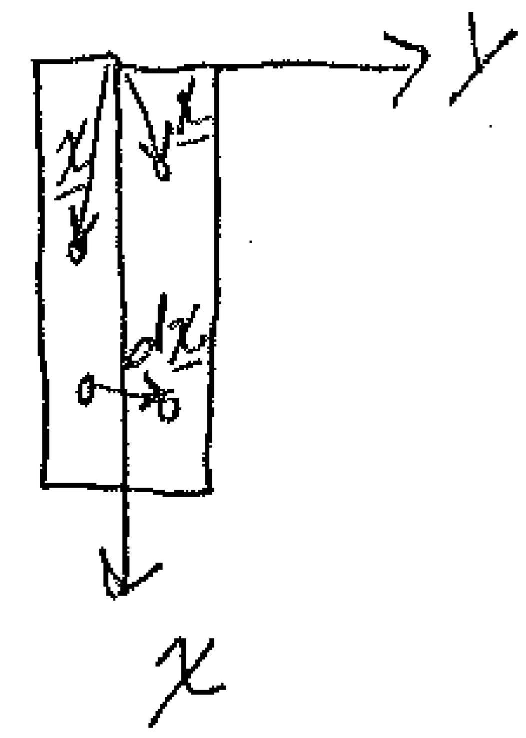




$$d\underline{X} = \underline{X}_B - \underline{X}_A$$
$$d\underline{x} = \underline{x}_b - \underline{x}_a$$

$$\underline{X} = X\hat{i} + Y\hat{j} + Z\hat{k}$$
$$\underline{x} = x\hat{i} + y\hat{j} + z\hat{k}$$

$x = kX$ (maps components in x-direction)

$y = \frac{1}{2}kY$ ($\frac{1}{2}$ from Poisson effect)

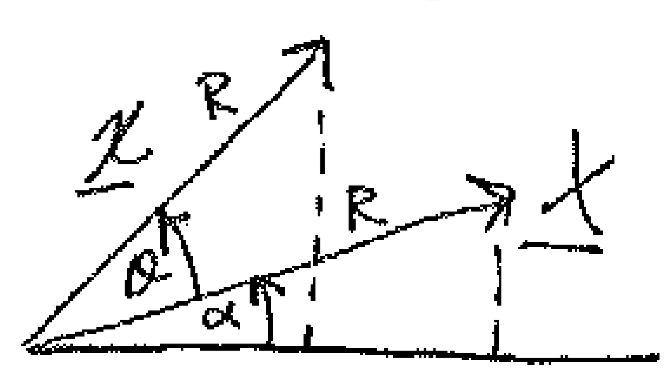
$z = \frac{1}{2}kZ$

$dx = k\,dX$

$dy = \cancel{\frac{1}{2}k\,dy}\ \frac{1}{2}k\,dY$

$dz = \frac{1}{2}k\,dZ$

Rotation (rigid with constant magnitude)



$$\underline{X} = \underbrace{R\cos\alpha}_{X}\,\hat{i} + \underbrace{R\sin\alpha}_{Y}\,\hat{j}$$

$$\underline{x} = R\cos(\alpha+\theta)\,\hat{i} + R\sin(\alpha+\theta)\,\hat{j}$$

$$\underline{x} = R[\cos\alpha\cos\theta - \sin\alpha\sin\theta]\,\hat{i} + R[\sin\alpha\cos\theta + \cos\alpha\sin\theta]\,\hat{j}$$
$$= [X\cos\theta - Y\sin\theta]\,\hat{i} + [X\sin\theta + Y\cos\theta]\,\hat{j}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

2

$$d\underline{x} = \underline{\underline{F}} \cdot d\underline{X}$$

$$F_{ij} = \frac{\partial x_i}{\partial X_j} = \begin{bmatrix} K & 0 & 0 \\ 0 & \frac{1}{2}K & 0 \\ 0 & 0 & \frac{1}{2}K \end{bmatrix} = \underline{\underline{R}} \cdot \underline{\underline{U}}$$

deformation

rotation

multiplicative deformation

Compatibility

must integrate ϵ_{ij} to get u_i

6 independent components

3 independent components

overspecification likely

$$\vec{\nabla} \times \underline{\underline{\epsilon}} \times \overleftarrow{\nabla} = 0$$

$$\frac{\partial^2 \epsilon_{xx}}{\partial y^2} + \frac{\partial \epsilon_{yy}^2}{\partial x^2} = 2 \frac{\partial^2 \epsilon_{xy}}{\partial x \partial y}$$

ensures u_x and u_y can be found from $\epsilon_{xx}, \epsilon_{yy}, \epsilon_{xy}$

$$\epsilon_{xx} = \frac{\partial u_x}{\partial x}, \quad \epsilon_{yy} = \frac{\partial u_y}{\partial y}$$

$$\epsilon_{xy} = \frac{1}{2}\left(\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}\right)$$

$$\frac{\partial^3 u_x}{\partial y^2 \partial x} + \frac{\partial^3 u_y}{\partial x^2 \partial y} = \frac{\partial^3 u_x}{\partial y^2 \partial x} + \frac{\partial^3 u_y}{\partial x^2 \partial y}$$

Integration of partial derivatives

consider $\underline{\underline{\epsilon}} = 0$

$$\epsilon_{xx} = \frac{\partial u_x}{\partial x} = 0$$

$$\epsilon_{xy} = \frac{1}{2}\left(\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}\right) = 0$$

$$\epsilon_{yy} = \frac{\partial u_y}{\partial y} = 0$$

$$\epsilon_{yz} = \frac{1}{2}\left(\frac{\partial u_y}{\partial z} + \frac{\partial u_z}{\partial y}\right) = 0$$

$$\epsilon_{xz} = \frac{1}{2}\left(\frac{\partial u_x}{\partial z} + \frac{\partial u_z}{\partial x}\right) = 0$$

$$\epsilon_{zz} = \frac{\partial u_z}{\partial z} = 0$$

$$\frac{\partial}{\partial z}\left(\frac{\partial u_x}{\partial y}+\frac{\partial u_y}{\partial x}\right)=0$$

$$\frac{\partial}{\partial y}\left(\frac{\partial u_x}{\partial z}+\frac{\partial u_z}{\partial x}\right)=0$$

$$\frac{\partial}{\partial x}\left(\frac{\partial u_y}{\partial z}+\frac{\partial u_z}{\partial y}\right)=0$$

common

$$\frac{\partial^2 u_x}{\partial z\partial y}=-\frac{\partial^2 u_y}{\partial z\partial x}$$

$$\frac{\partial^2 u_x}{\partial y\partial z}=-\frac{\partial^2 u_z}{\partial y\partial x}$$

$$\left.\begin{aligned}\frac{\partial^2 u_y}{\partial z\partial x}&=\frac{\partial^2 u_z}{\partial x\partial y}\\ \frac{\partial^2 u_y}{\partial x\partial z}&=\frac{\partial^2 u_z}{\partial x\partial y}\end{aligned}\right\}=0$$

similar arguments made for other components

$$\frac{\partial^2 u_y}{\partial x\partial z}=\frac{\partial^2 u_x}{\partial y\partial z}=\frac{\partial^2 u_z}{\partial x\partial y}=0$$

for $u_x \rightarrow \dfrac{\partial u_x}{\partial x}=\dfrac{\partial^2 u_x}{\partial y\partial z}=0$

for $u_y \rightarrow \dfrac{\partial u_y}{\partial y}=\dfrac{\partial^2 u_y}{\partial x\partial z}=0$

for $u_z \rightarrow \dfrac{\partial u_z}{\partial z}=\dfrac{\partial^2 u_z}{\partial x\partial y}=0$

$$\frac{\partial^2 u_x}{\partial y\partial z}=0$$

$$\frac{\partial u_x}{\partial y}=\int 0\,dz=A'(y)$$

since $\dfrac{\partial u_x}{\partial x}=0$ (not a function of x)

$$\frac{\partial u_x}{\partial z}=\int \cancelto{0}{\frac{\partial^2 u_x}{\partial y\partial z}}\,dy=B'(z)$$

$$u_x=\int\frac{\partial u_x}{\partial y}dy=A(y)+G(z)+D$$

$$u_x=\int\frac{\partial u_x}{\partial z}dz=B(z)+H(y)+E$$

$$\rightarrow u_x=A(y)+B(z)+C$$

similarly,

$$u_y = D(z) + E(x) + F$$
$$u_z = G(x) + H(y) + J$$

from earlier condition,

$$\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x} = 0$$

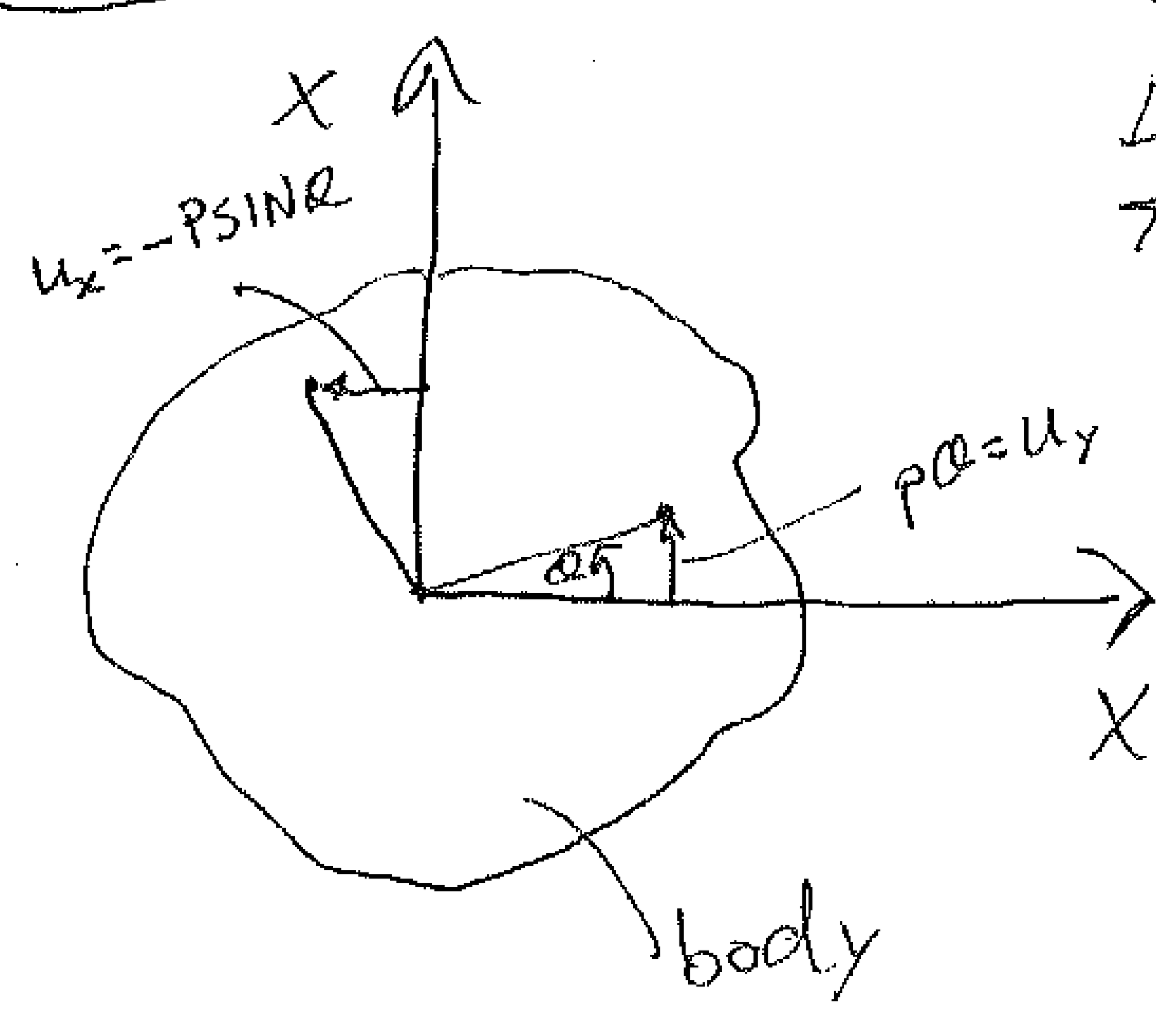
$$\left.\begin{aligned} A'(y) + E'(x) &= 0 \\ D'(z) + H'(y) &= 0 \\ G'(x) + B'(z) &= 0 \end{aligned}\right\} \rightarrow \left.\begin{aligned} A'(y) &= -E'(x) = L \\ D'(z) &= -H'(y) = M \\ G'(x) &= -B'(z) = N \end{aligned}\right\}$$

careful with signs to get correct displacement

$$\frac{dA}{dy} = L \rightarrow A = Ly + R$$
$$\frac{dD}{dz} = M \rightarrow D = Mz + S$$
$$\frac{dG}{dx} = N \rightarrow G = Nx + T$$

$$\boxed{\begin{aligned} u_x &= Ly - Nz + R \\ u_y &= Mz - Lx + S \\ u_z &= Nx - My + T \end{aligned}}$$

SINCE STRAIN IS ZERO WE MUST HAVE RIGID BODY MOTION OR NO MOTION AT ALL. LINEAR COMPONENT OF DISPLACENT DOES NOT CONTRIBUTE TO STRESS/STRAIN. ($\sin\theta \simeq \theta$)

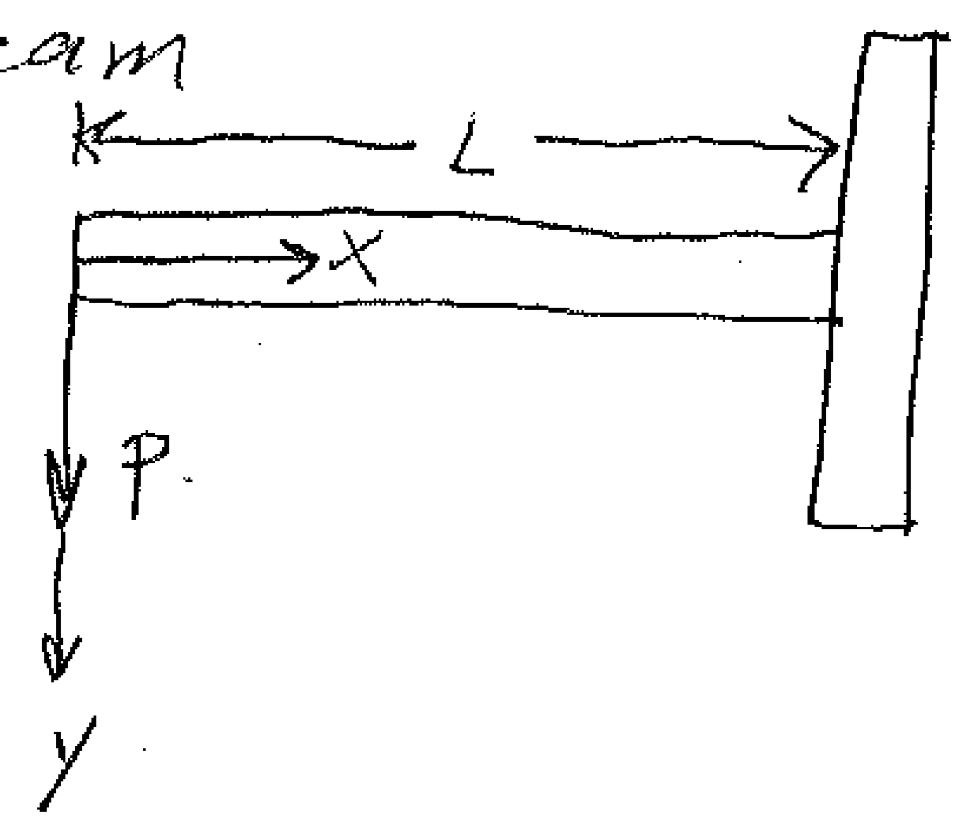


(5)

Example problem

Significant background information introduced without proof. Focus on integrating strain to obtain displacement components. Other information provides how strains initially determined. Techniques used to integrate eqns. can be applied to homework. This example is from Timoshenko and Goodier

Consider the beam



For plane elasticity, a combination of equilibrium, compatibility and linear isotropic constitutive behavior leads to LaPlace's eqns. for stress.

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)\left(\sigma_{xx}+\sigma_{yy}\right)=0$$

or

$$\nabla^2\left(\sigma_{xx}+\sigma_{yy}\right)=0$$

Airy's stress function solves this set of equations and results in a strain field that can be integrated to get displacements. There are no body forces present ∴

$$\sigma_{xx}=\frac{\partial^2\phi}{\partial y^2}=\phi_{,yy}$$

$$\sigma_{yy}=\frac{\partial^2\phi}{\partial x^2}=\phi_{,xx}$$

$$\sigma_{xy}=-\frac{\partial^2\phi}{\partial x\partial y}=\phi_{,xy}$$

Details of these eqns. will be described in coming lectures. For now it is sufficient to note that substitution of the stress function into LaPlace's eqn. for stress leads to a biharmonic eqn.

$$\nabla^2\phi = 0$$

$$\frac{\partial^4\phi}{\partial x^4} + 2\frac{\partial^4\phi}{\partial x^2\partial y^2} + \frac{\partial^4\phi}{\partial y^4} = 0$$

and this can be specified by certain polynomials.

For the case of bending of a cantilever beam by loading on the end, the stress distribution is

$$\sigma_{xx} = -\frac{Pxy}{I}$$

$$\sigma_{yy} = 0$$

$$\sigma_{xy} = -\frac{P}{I}\frac{1}{2}(c^2 - y^2)$$

☆ This is the stress distribution for either plane stress or plane strain. The strain and displacement distribution differ for the two cases.

For plane stress, the strain is

$$\epsilon_{xx} = -\frac{Pxy}{EI}$$

$$\epsilon_{yy} = \nu\frac{Pxy}{EI}$$

$$\gamma_{xy} = 2\epsilon_{xy} = -\frac{P}{I}\frac{1}{2G}(c^2 - y^2)$$

Integration of ϵ_{xx} and ϵ_{yy} gives

$$u_x = \int -\frac{Pxy}{EI}dx = -\frac{1}{2}\frac{Px^2y}{EI} + f(y)$$

$$u_y = \int \nu\frac{Pxy}{EI}dy = \frac{1}{2}\nu\frac{Pxy^2}{EI} + g(x)$$

These expressions for displacement are substituted into the equation for shear strain to obtain

$$-\frac{P}{I}\frac{1}{2G}(c^2-y^2) = \frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}$$

which upon substitution of the displacements expressions gives

$$-\frac{P}{I}\frac{1}{2G}(c^2-y^2) = -\frac{1}{2}\frac{Px^2}{EI} + \frac{df(y)}{dy} + \frac{1}{2}\frac{Py^2}{EI} + \frac{dg(x)}{dx}$$

The term in this equation can be grouped into terms containing constants only, terms with function of x only, and terms containing functions of y only. These are denoted $F(x)$, $G(y)$, K. The result is

$$F(x) + G(y) = K$$

$$F(x) = \frac{-1}{2}\frac{Px^2}{EI} + \frac{dg(x)}{dx}$$

$$G(y) = \frac{1}{2}\nu\frac{Py^2}{EI} + \frac{df(y)}{dy} - \frac{Py^2}{I2G}$$

$$K = -\frac{P}{I}\frac{c^2}{2G}$$

This eqn. means that the functions of x must be some constant and the functions of y must be some constant, otherwise y could be held constant and x varied, violating the eqn. This results in,

$$e + d = K$$

where $$d = \frac{-1}{2}\frac{Px^2}{EI} + \frac{dg(x)}{dx}$$

$$e = \frac{1}{2}\nu\frac{Py^2}{EI} + \frac{df(y)}{dy} - \frac{Py^2}{I2G}$$

These two eqns. are now integrated

$$\frac{dg(x)}{dx} = \frac{1}{2}\frac{Px^2}{EI} + d$$

$$\frac{df(y)}{dy} = e - \frac{\nu P y^2}{2EI} + \frac{Py^2}{2IG}$$

$$f(y) = ey - \frac{\nu P y^3}{6EI} + \frac{Py^3}{6IG} + g$$

$$g(x) = \frac{Px^3}{6EI} + dx + h$$

The functions of x and y are now substituted into the expression for displacement to obtain,

$$u_x = \int -\frac{Pxy}{EI}dx = -\frac{1}{2}\frac{Px^2y}{EI} - \frac{\nu P y^3}{6EI} + \frac{Py^3}{6IG} + ey + g$$

$$u_y = \int \nu \frac{Pxy}{EI}dy = \frac{1}{2}\nu\frac{Pxy^2}{EI} + \frac{Px^3}{6EI} + dx + h$$

where, from previous discussion, it is recognized that the linear terms and constants represent rigid body rotations and displacements

The problem was set-up with a coordinate system at the left end and the beam fixed at a rigid wall on the right. The boundary conditions for displacement are

$$u_x(L,0) = 0 = g$$

$$u_y(L,0) = 0 = \frac{PL^3}{6EI} + dL + h$$

and depending on how rotation is fixed, either

$$u_{x,y}(L,0) = 0$$

or

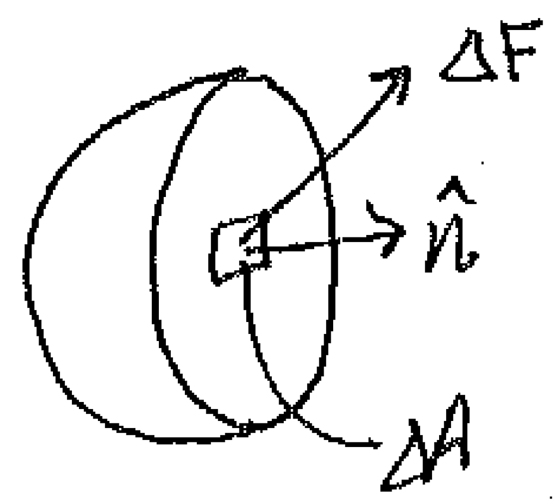
$$u_{y,x}(L,0) = 0$$

These boundary conditions provide sufficient constraints to determine the constants.

<u>stress vector</u>

$\underline{t}$, t_i (3 components)

- Stress tensor (9 components)
- Body forces - inertia, electromagnetic
- surface force - proportional to area



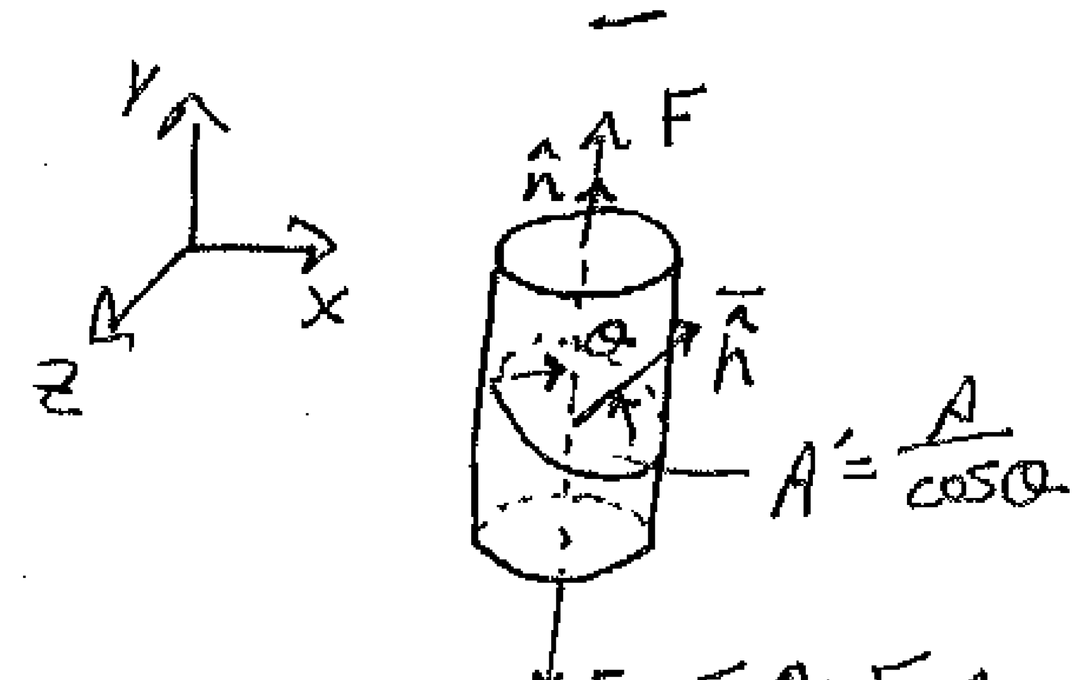
Definition of traction

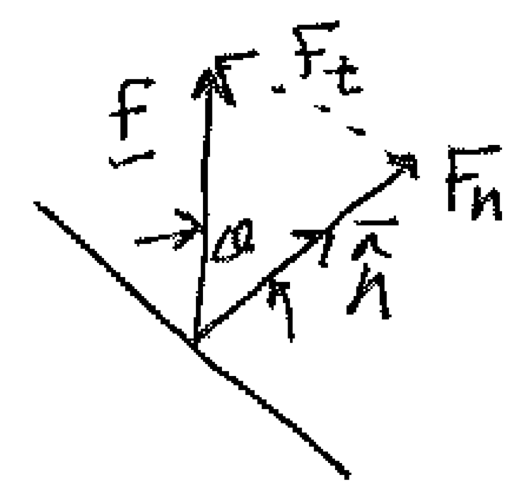
$$\underline{t} = \lim_{\Delta A \to 0} \frac{\Delta F}{\Delta A}$$

$$\underline{t} = \frac{F}{A}$$

t_i is the component of the stress vector on any plane defined by $\hat{n}$

$\underline{t} \to$ surface traction



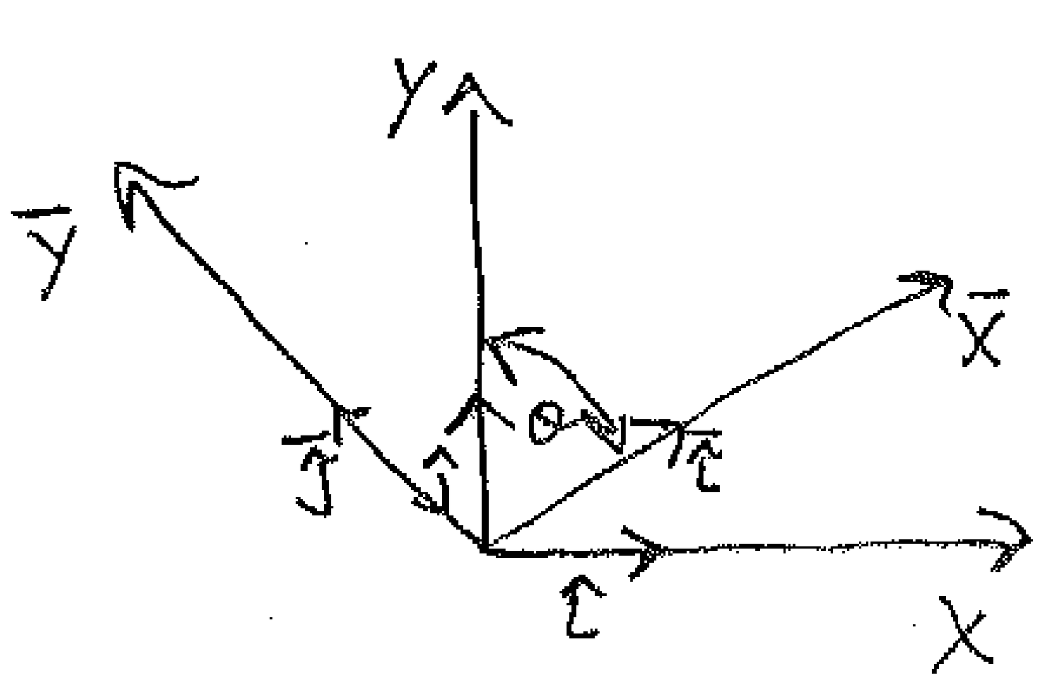


$\underline{F} = F_x \hat{i} + F_y \hat{j}$

$\hat{n} = n_x \hat{i} + n_y \hat{j}$ on top and bottom of ~~cylined~~ cylinder

$n_x = 0$

$F \cdot \hat{n} = F_y$



	$\hat{i}$	$\hat{j}$
$\bar{i}$	$\sin\theta$	$\cos\theta$
$\bar{j}$	$-\cos\theta$	$\sin\theta$

$\bar{i} = \sin\theta\,\hat{i} + \cos\theta\,\hat{j}$

$\bar{j} = -\cos\theta\,\hat{i} + \sin\theta\,\hat{j}$

$F_n = F \cdot \bar{i} = F\cos\theta$

$F_t = F \cdot \bar{j} = F\sin\theta$

recall $t = \frac{F}{A}$

$$t_n = \frac{F_n}{A'} = \frac{F\cos\theta}{\frac{A}{\cos\theta}}$$

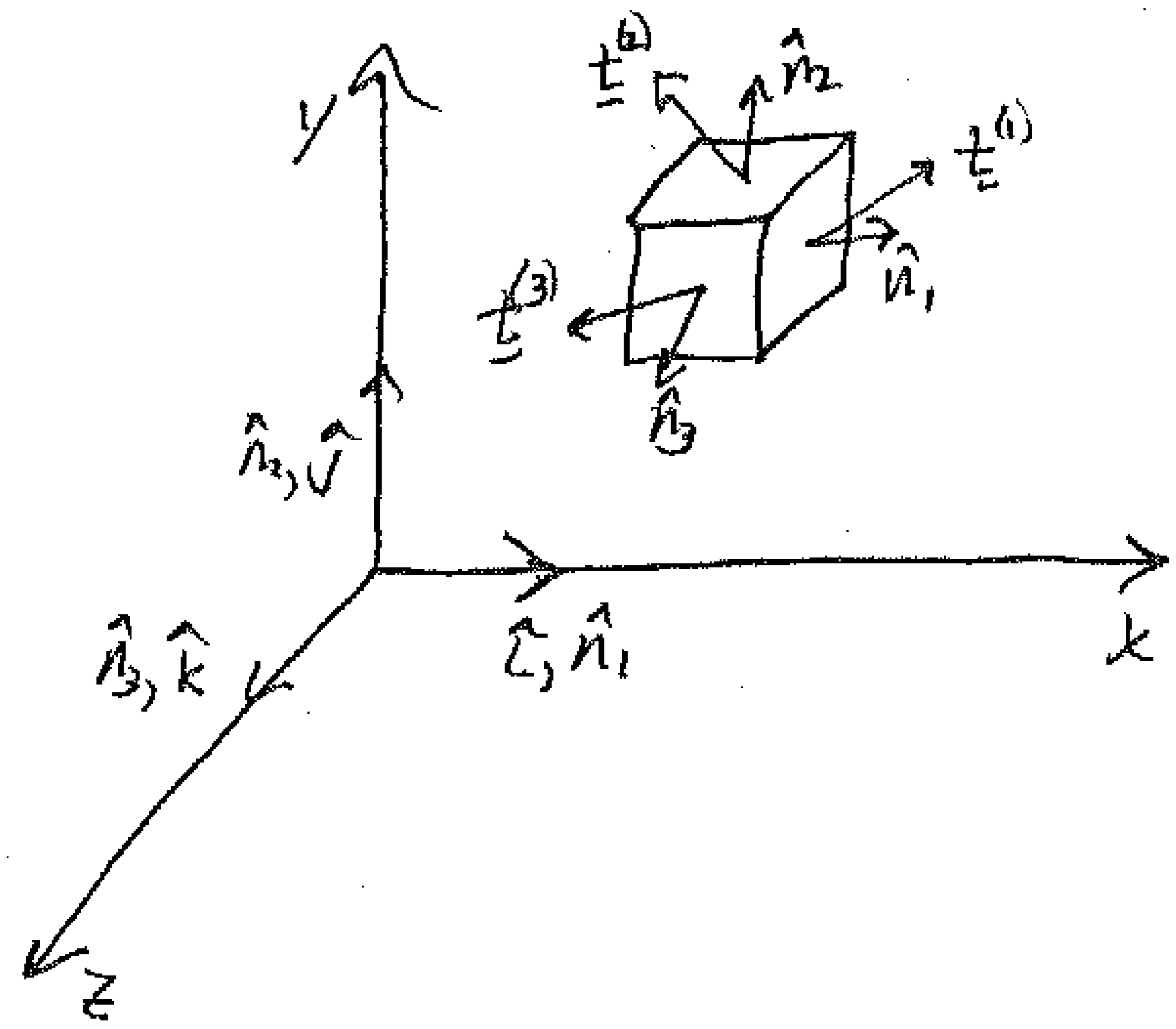
$$\sigma_n = t_n = \frac{F}{A}\cos^2\theta$$

$$t_t = \frac{F_t}{A'} = \frac{F\sin\theta}{\frac{A}{\cos\theta}} = \frac{F}{A}\cos\theta\sin\theta$$

There exists an angle in which when the plane is rotate to this angle, the shear stress in zero.

~~Princip~~
Principal direction: corresponding normal stress to this plane is called the principal stress

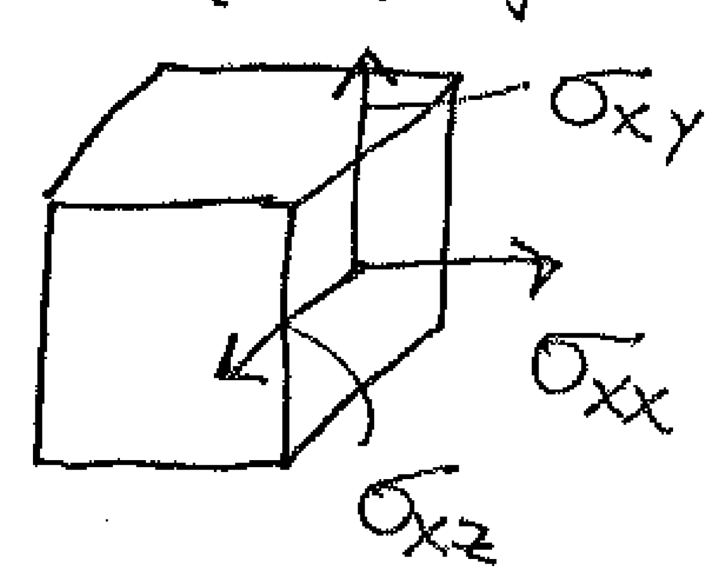
---



$$\underline{t}^{(1)} = t_1^{(1)}\hat{n}_1 + t_2^{(1)}\hat{n}_2 + t_3^{(1)}\hat{n}_3$$

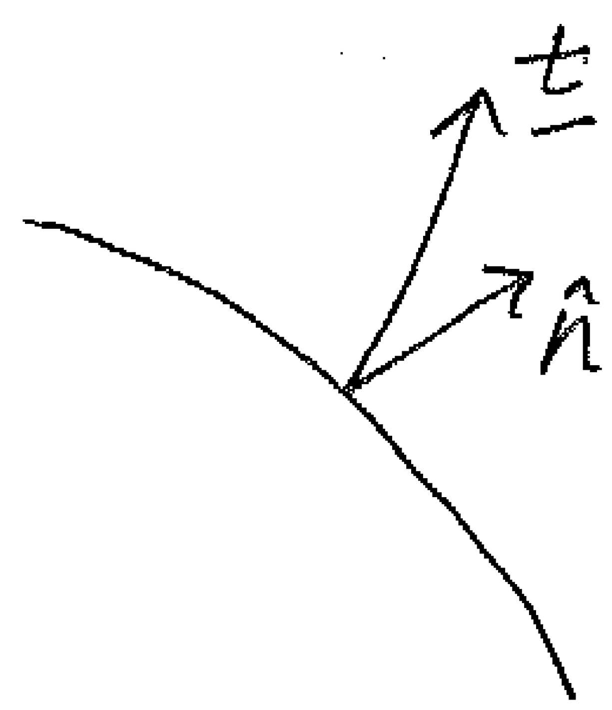
stress tensor

$$t_i = \sigma_{ji} n_j$$



σ_{ij} direction

plane acting on



$t_i = \sigma_{ji} n_j$ (boundary conditions on surface)

stress tensor has 9 components

if no body or surface couples, the off diagonal terms are zero

(11)

$$\underline{t} = (\sigma_{11} n_1 + \sigma_{21} n_2 + \sigma_{31} n_3)\hat{i}$$
$$+ (\sigma_{21} n_1 + \sigma_{22} n_2 + \sigma_{23} n_3)\hat{j}$$
$$+ (\sigma_{31} n_1 + \sigma_{32} n_2 + \sigma_{33} n_3)\hat{k}$$

traction depends on normal of surface and stress components at the point

stress transformation

new coord. old coord.

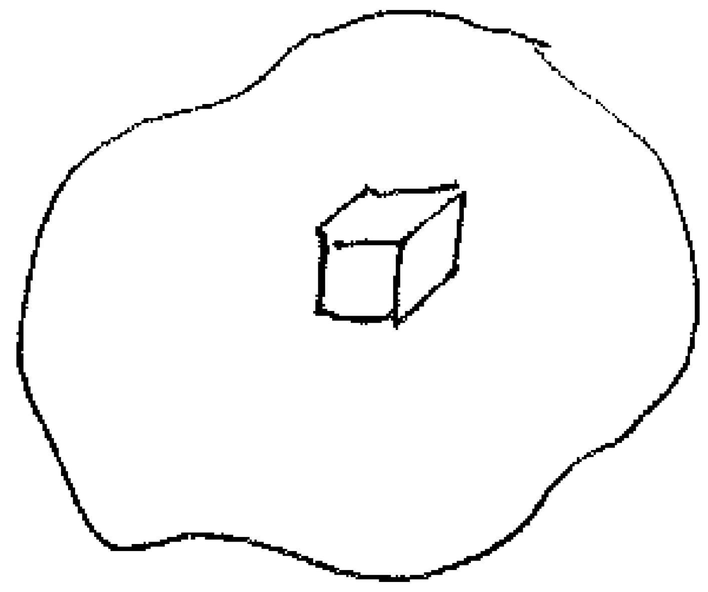
$$\bar{\sigma}_{ij} = a_{ki} a_{lj} \sigma_{kp}$$

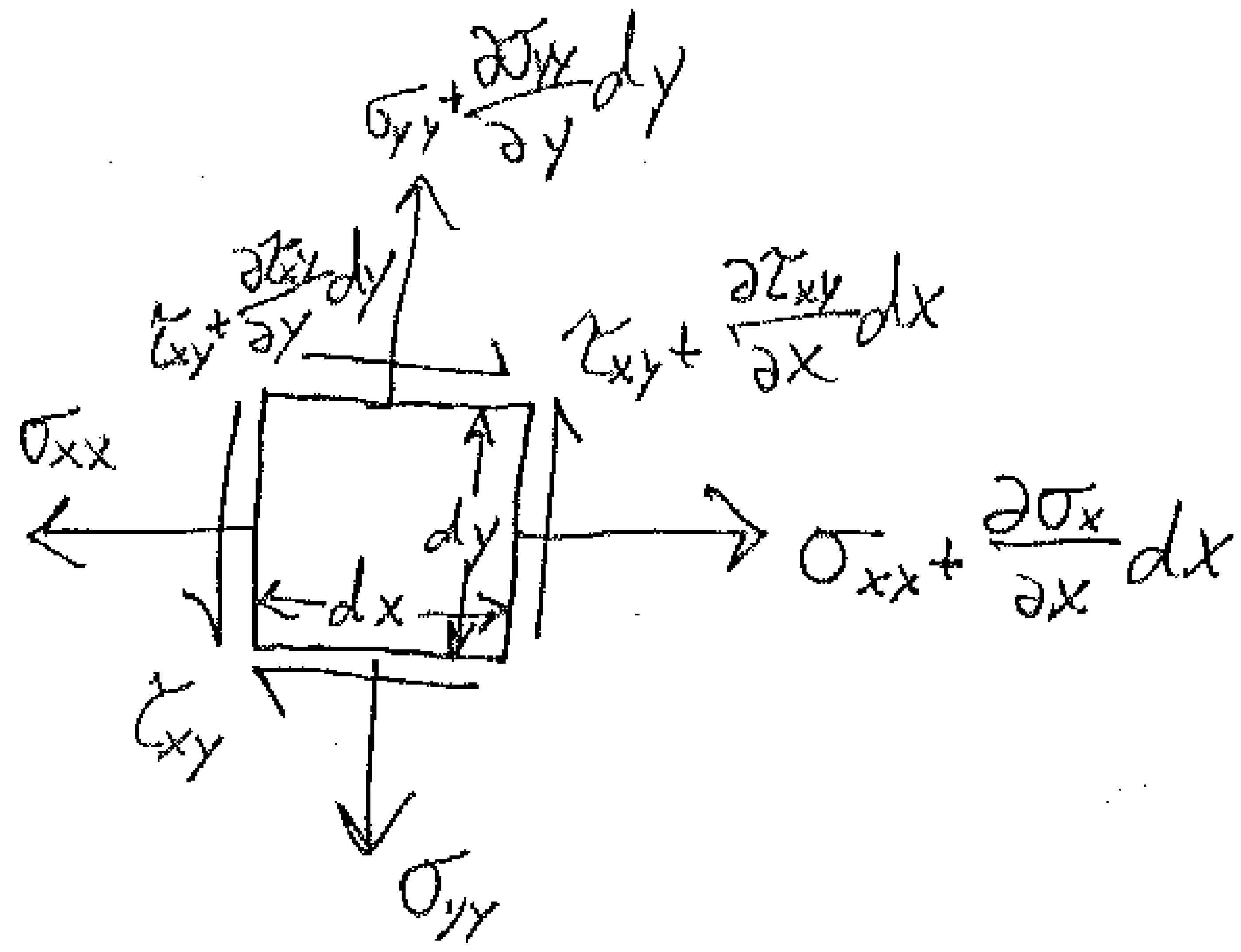
$$\underline{\underline{\bar{\sigma}}} = \underline{\underline{A}}^T \underline{\underline{\sigma}} \, \underline{\underline{A}}$$

other notation used,

$$\bar{\sigma}_{ij} = a_i^k a_j^p \sigma_{kp}$$

Equilibrium





$$\Sigma F = 0$$

$$\left[\sigma_{xx}-\left(\sigma_{xx}+\frac{\partial\sigma_{xx}}{\partial x}dx\right)\right]dydz+\left[\tau_{yx}-\left(\tau_{yx}+\frac{\partial\tau_{yx}}{\partial y}dy\right)\right]dxdz$$

$$+\left[\tau_{zx}-\left(\tau_{zx}+\frac{\partial\tau_{zx}}{\partial z}dz\right)\right]dxdy+F_x dxdydz=0$$

body force in x-direction

$$\frac{\partial\sigma_{xx}}{\partial x}+\frac{\partial\tau_{yx}}{\partial y}+\frac{\partial\tau_{zx}}{\partial z}+F_x=0 \quad \text{static equilibrium}$$

$$\boxed{\vec{\nabla}\cdot\underline{\underline{\sigma}}+\underline{f}=0}$$

$$\vec{\nabla}=\frac{\partial}{\partial x}\hat{i}+\frac{\partial}{\partial y}\hat{j}+\frac{\partial}{\partial z}\hat{k}$$

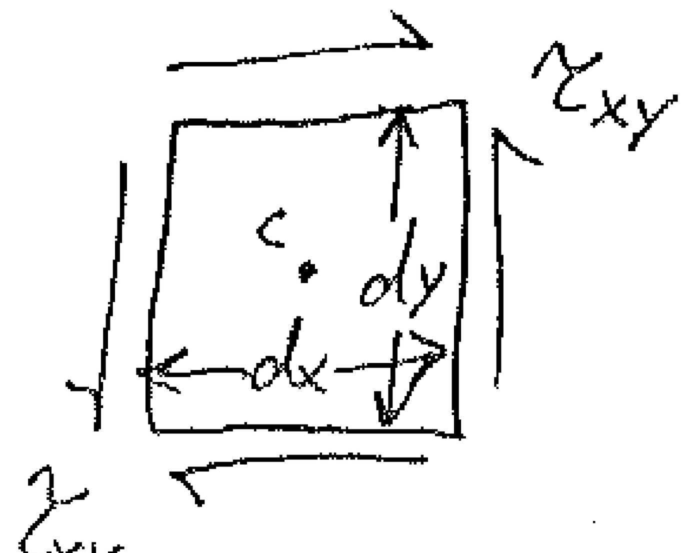
$\vec{\nabla}\cdot\underline{v} \rightarrow$ scalar (divergence)

$\vec{\nabla}\times\underline{v} \rightarrow$ vector (curl)

$\vec{\nabla}\underline{v} \rightarrow$ tensor (gradient)

$$\frac{\partial v_x}{\partial x}+\frac{\partial v_y}{\partial y}+\frac{\partial v_z}{\partial z}=\frac{\partial v_i}{\partial x_i}=v_{i,i}$$

Moments



partial terms will cancel, not shown

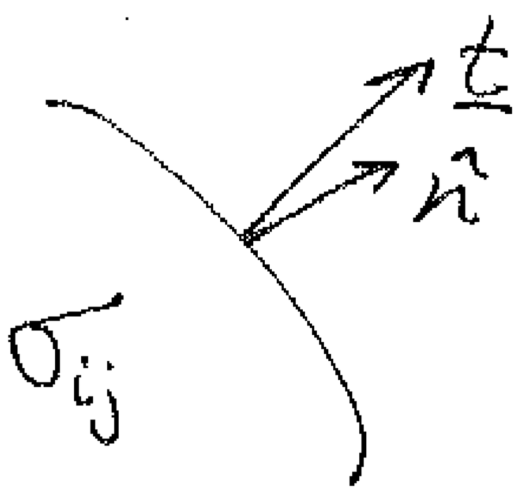
See page 165 in Boresi & Chong result after summing moments is $\sigma_{ij}=\sigma_{ji}$ (six ~~ind~~ independent components in stress tensor)

③

stress components on surface of body must be in equilibrium with the external forces acting on the surface

$\underline{t} = \underline{\underline{\sigma}} \cdot \hat{n}$

$t_i = \sigma_{ji} n_j$



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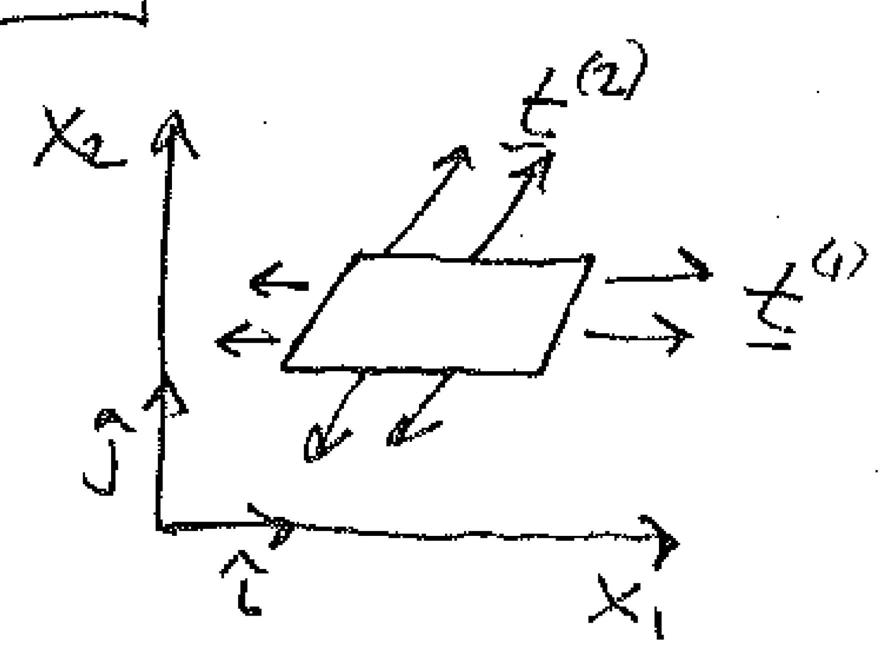
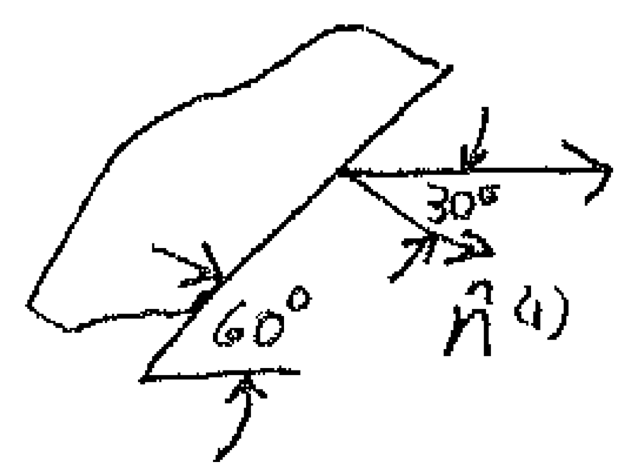




$t^{(1)} = 100 psi$

$\hat{n} = n_x \hat{i} + n_y \hat{j}$

$\underline{t}^{(1)} = t_x \hat{i} + \cancel{t_y}^{\,0} \hat{j} = 100 \hat{i}$

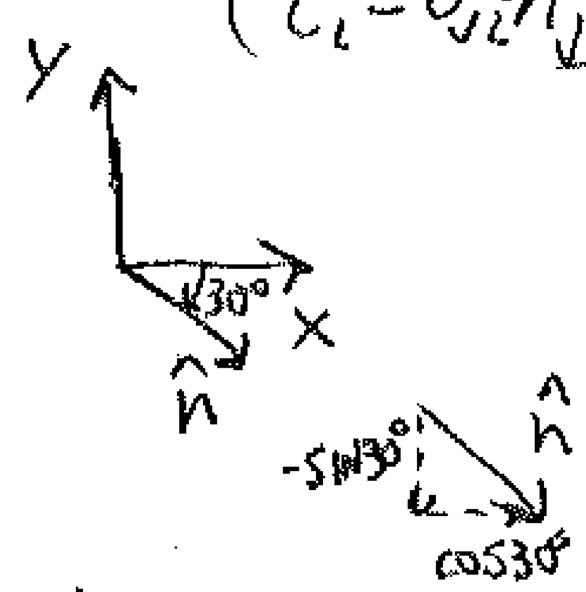
$(t_i = \sigma_{ji} n_j)$

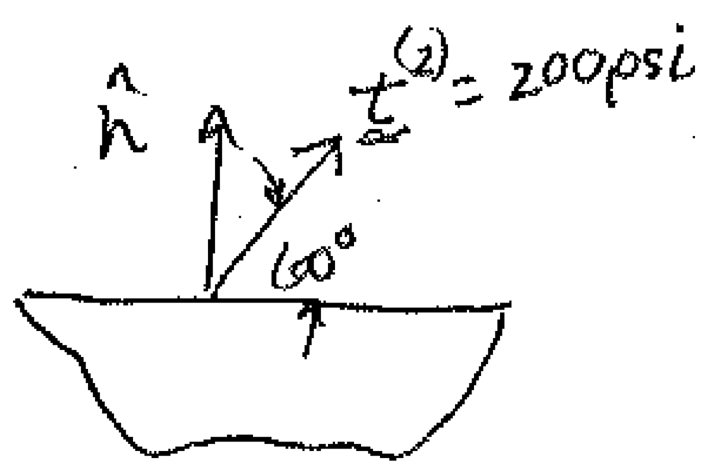
$t_x = \sigma_{xx} n_x + \sigma_{yx} n_y = 100$

$t_x = \sigma_{xx}(\cos 30°) + \sigma_{yx}(-\sin 30°)$

$t_y = 0 = \sigma_{xy} \cos(30°) - \sigma_{yy} \sin 30°$

$(n_x) \qquad (n_y)$





$\hat{n} = n_x \hat{i} + n_y \hat{j}$

$\hat{n} = \hat{j}$

$\underset{\sim}{t}^{(2)} = |t| \cos(60°) \hat{i} + |t| \sin(60°) \hat{j}$

$= 200 \cos(60°) \hat{i} + 200 \sin(60°) \hat{j}$

$t_x = \sigma_{xx} \cancel{n_x}^{\,0} + \sigma_{yx} n_y$

$t_x = \sigma_{yx} n_y$

$\sigma_{yx} = 100 psi$

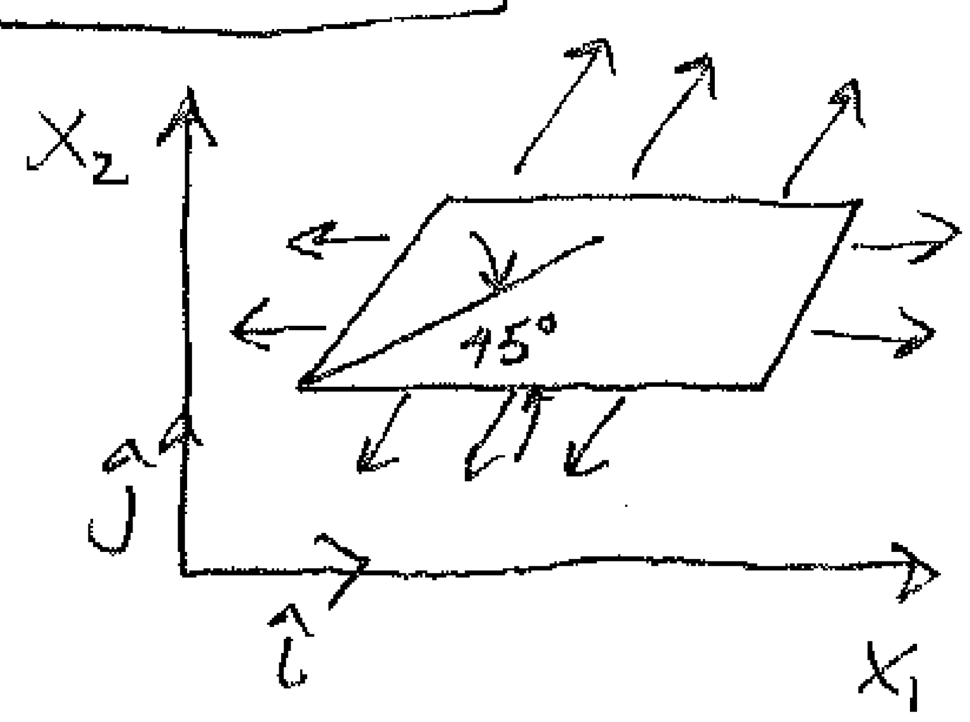
$t_y = 200 \sin 60°$

$t_y = \sigma_{xy} \cancel{n_x}^{\,0} + \sigma_{yy} n_y = 173.2 psi$

$t_x = 100 = \sigma_{xx} n_x + \sigma_{yx} \cancel{n_y}^{\,0} = \sigma_{xx} \cos(30°)$

$= 173.2 psi$

p. 169 #8 cont.



stress at 45°?

coordinate transformation

$$\underset{\text{new}}{\overline{\sigma_{ij}}} = a_{ki} a_{\ell j} \underset{\text{old}}{\sigma_{k\ell}}$$

$$\overline{\sigma_{ij}} = A^T \underline{\underline{\sigma}} A$$

$$\begin{array}{c|cc} & \hat{i} & \hat{j} \\ \hline \bar{\hat{i}} & \cos 45^\circ & \sin 45^\circ \\ \bar{\hat{j}} & -\sin 45^\circ & \cos 45^\circ \end{array}$$

$$\overline{\sigma_{ij}} = \begin{bmatrix} 73.2 & 0 \\ 0 & 23.1 \end{bmatrix}$$

Eigenvalues

$$\begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \end{bmatrix}$$

$$\left.\begin{aligned} \sigma_1 &= \sigma_{11} - \lambda \\ \sigma_2 &= \sigma_{22} - \lambda \\ \sigma_3 &= \sigma_{33} - \lambda \end{aligned}\right\} ?$$

$$[\underline{\underline{\sigma}} - \lambda \underline{\underline{I}}][n] = 0$$

direction COSINES
(eigenvectors)

$$\hat{n} \Rightarrow n_1^2 + n_2^2 + n_3^2 = 1$$

$$\det \left| \underline{\underline{\sigma}} - \lambda \underline{\underline{I}} \right| = 0$$