stress Functions

vector field $\underline{v} = \vec{\nabla}\phi + \vec{\nabla}\times\underline{\psi}$

$$\vec{\nabla}\cdot(\vec{\nabla}\times\underline{\psi}) = 0 \quad \textit{identity}$$

$$\partial_k\hat{e}_k\cdot(\partial_m\hat{e}_m\times\psi_q\hat{e}_q) = 0 = \psi_{q,mk}\hat{e}_k\cdot(\hat{e}_m\times\hat{e}_q)$$

$$= \psi_{q,mk}\hat{e}_k\cdot\hat{e}_s e_{smq}$$

$$= \psi_{q,mk}\delta_{ks}e_{smq}$$

$$= \psi_{q,mk}e_{kmq}$$

$$= \psi_{3,21} - \psi_{2,31} + \psi_{1,32} - \psi_{3,12} + \psi_{2,13} - \psi_{1,32} = 0 \checkmark$$

if $\vec{\nabla}\cdot\underline{v} = 0$

then $\underline{v} = \vec{\nabla}\times\underline{\psi}$ can be found

also, $\vec{\nabla}\cdot\underline{\underline{\sigma}} = 0$

$$\vec{\nabla}\cdot(\vec{\nabla}\times\underline{\underline{\phi}}\times\overleftarrow{\nabla}) = 0$$

$$\vec{\nabla}\times\underline{\underline{\phi}}\times\overleftarrow{\nabla} \neq \vec{\nabla}\times(\vec{\nabla}\times\underline{\underline{\phi}})$$

$$\partial_m\hat{e}_m\times\phi_{pq}\hat{e}_p\hat{e}_q\times\partial_r\hat{e}_r = \phi_{pq,mr}(\hat{e}_m\times\hat{e}_p)(\hat{e}_q\times\hat{e}_r)$$

$$\vec{\nabla}\cdot(\vec{\nabla}\times\underline{\underline{\phi}}\times\overleftarrow{\nabla}) = \phi_{pq,mr}\,\partial_s\hat{e}_s\cdot(\hat{e}_m\times\hat{e}_p)(\hat{e}_q\times\hat{e}_r) = 0$$

Let,

$$\underline{\underline{\sigma}} = \vec{\nabla} \times \underline{\underline{\phi}} \times \overleftarrow{\nabla} = \phi_{pq,mr} (\hat{e}_m \times \hat{e}_p)(\hat{e}_q \times \hat{e}_r)$$

$$= \phi_{pq,mr}\, e_{smp}\, e_{tqr}\, \hat{e}_s \hat{e}_t$$

$$\sigma_{11} = \phi_{33,22} + \phi_{22,33} - 2\phi_{23,23}$$

$$\sigma_{22} = \phi_{33,11} + \phi_{11,33} - 2\phi_{13,13}$$

$$\sigma_{33} = \phi_{22,11} + \phi_{11,22} - 2\phi_{12,12}$$

$$\sigma_{23} = -\phi_{11,23} + \partial_1 [-\phi_{23,1} + \phi_{31,2} + \phi_{12,3}]$$

$$\sigma_{31} = -\phi_{22,31} + \partial_2 [\phi_{23,1} - \phi_{31,2} + \phi_{12,3}]$$

$$\sigma_{12} = -\phi_{33,12} + \partial_3 [\phi_{23,1} + \phi_{31,2} - \phi_{12,3}]$$

2D problems

$\sigma_{11}(x_1, x_2)$, $\sigma_{22}(x_1, x_2)$, $\sigma_{12}(x_1, x_2)$

$\phi_{33}(x_1, x_2)$: airy stress function (only like $\underline{\underline{\phi}}$ in $\sigma_{11}, \sigma_{22}, \sigma_{12}$)

torsion σ_{23}, σ_{31} use $\phi_{23,1} = \phi$

note ~~that~~ $\phi_{23,13}$ in σ_{12} is zero $\therefore$ no coupling, ϕ_{31} & ϕ_{12} couple w/ σ_{12}

Compatibility conditions

$$\underline{\underline{\epsilon}} = \frac{1}{2}(\underline{u}\overleftarrow{\nabla} + \vec{\nabla}\underline{u})$$

$$\vec{\nabla}\times(\vec{\nabla}\phi) = 0$$

$$\vec{\nabla}\times(\underbrace{\underline{u}\overleftarrow{\nabla} + \vec{\nabla}\underline{u}}_{\text{gradients}})\times\overleftarrow{\nabla} = 0$$

conversely, if $\vec{\nabla}\times\underline{\underline{\epsilon}}\times\overleftarrow{\nabla} = 0 \rightarrow \vec{\nabla}\cdot(\vec{\nabla}\times\underline{\underline{\epsilon}}\times\overleftarrow{\nabla}) = 0$

Bianchi conditions

sub stress into compatibility

$$\underline{\underline{\epsilon}} = \underline{\underline{\underline{\underline{s}}}} : \underline{\underline{\sigma}}$$

$$\epsilon_{ij}\hat{e}_i\hat{e}_j = s_{ijkl}\hat{e}_i\hat{e}_j\hat{e}_k\hat{e}_l : \sigma_{mn}\hat{e}_m\hat{e}_n$$

$$= s_{ijkl}\hat{e}_i\hat{e}_j\sigma_{mn}\delta_{km}\delta_{ln}$$

$$= s_{ijkl}\sigma_{kl}\hat{e}_i\hat{e}_j$$

$$\vec{\nabla}\times\underline{\underline{\epsilon}}\times\overleftarrow{\nabla} = 0 = \vec{\nabla}\times(\underline{\underline{\underline{\underline{s}}}} : \underline{\underline{\sigma}})\times\overleftarrow{\nabla}$$

$$= \partial_r\hat{e}_r\times(s_{ijkl}\sigma_{kl}\hat{e}_i\hat{e}_j)\times\partial_s\hat{e}_s$$

homogeneous case (s_{ijkl} not spatially varying)

$$0 = s_{ijkl}\sigma_{kl,rs}(\hat{e}_r\times\hat{e}_i)(\hat{e}_j\times\hat{e}_s)$$

$$= s_{ijkl}\sigma_{kl,rs}\,\epsilon_{pri}\,\epsilon_{qjs}\,\hat{e}_p\hat{e}_q$$

isotropic case

$$s_{ijkl} = \alpha\,\delta_{ij}\delta_{kl} + \beta(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk})$$

$$\alpha = -\frac{\nu}{E}$$

$$\beta = \frac{1+\nu}{2E}$$

(4)

11 component $- p = q = 1$

$$\alpha(\sigma_{kk,11} - \sigma_{kk,mm}) + 2\beta(-\sigma_{33,22} - \sigma_{22,33} + \underbrace{2\sigma_{23,23}}_{\text{eliminate using equilibrium}}) = 0$$

$$\sigma_{11,1} + \sigma_{21,2} + \sigma_{31,3} = 0$$

$$\partial_2[\sigma_{12,1} + \sigma_{22,2} + \sigma_{32,3}] = 0$$

$$\partial_3[\sigma_{13,1} + \sigma_{23,2} + \sigma_{33,3}] = 0$$

$$\sigma_{12,12} + \sigma_{22,22} + \sigma_{32,32} = 0$$

$$+\ \sigma_{13,13} + \sigma_{23,23} + \sigma_{33,33} = 0$$

$$\underbrace{\sigma_{12,12} + \sigma_{13,13}}_{} + \sigma_{22,22} + \sigma_{33,33} = -2\sigma_{23,23}$$

can be eliminated
using equilibrium equations
2 more times

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Elastic solutions for anisotropic beams

compatibility $\vec{\nabla}\times\underline{\underline{\epsilon}}\times\overleftarrow{\nabla}=0=\underline{\underline{S}}$ (symmetric)

$$-S_{11}=\epsilon_{33,22}+\epsilon_{22,33}-2\epsilon_{23,23}$$

$$\underline{\underline{\epsilon}}=\underline{\underline{\underline{\underline{s}}}}:\underline{\underline{\sigma}}$$

$$=s_{ijkl}\hat{e}_i\hat{e}_j\hat{e}_k\hat{e}_l:\sigma_{mn}\hat{e}_m\hat{e}_n$$

$$=s_{ijkl}\hat{e}_i\hat{e}_j\sigma_{mn}\delta_{km}\delta_{ln}$$

$$\epsilon_{ij}\hat{e}_i\hat{e}_j=s_{ijkl}\sigma_{kl}\hat{e}_i\hat{e}_j \quad \text{9 eqns.}$$

$$=s_{11kl}\sigma_{kl}\hat{e}_1\hat{e}_1+s_{12kl}\sigma_{kl}\hat{e}_1\hat{e}_2+\dots \quad \text{81 coef.}$$

$$s_{ijkl}=s_{ijlk}$$

$$s_{ijkl}=s_{jikl}$$

$$\vec{\nabla}\times(\underline{\underline{\underline{\underline{s}}}}:\underline{\underline{\sigma}})\times\overleftarrow{\nabla}=0$$

$$\vec{\nabla}\times(\underline{\underline{\underline{\underline{s}}}}:\underline{\underline{\sigma}})=(\vec{\nabla}\times\underline{\underline{\underline{\underline{s}}}}):\underline{\underline{\sigma}}+\underline{\underline{\underline{\underline{s}}}}:(\vec{\nabla}\times\underline{\underline{\sigma}})$$

$$\partial_m\hat{e}_m\times(s_{ijkl}\hat{e}_i\hat{e}_j\hat{e}_k\hat{e}_l:\sigma_{pq}\hat{e}_p\hat{e}_q)$$

$$\partial_m\hat{e}_m\times(s_{ijkl}\sigma_{kl}\hat{e}_i\hat{e}_j)$$

$$(s_{ijkl}\sigma_{kl})_{,m}\underbrace{(\hat{e}_m\times\hat{e}_i)\hat{e}_j}_{e_{smi}\hat{e}_s\hat{e}_j}$$

$$(s_{ijkl,m}\sigma_{kl}+s_{ijkl}\sigma_{kl,m})e_{smi}\hat{e}_s\hat{e}_i \quad \text{(expand)}$$

Cartesian form

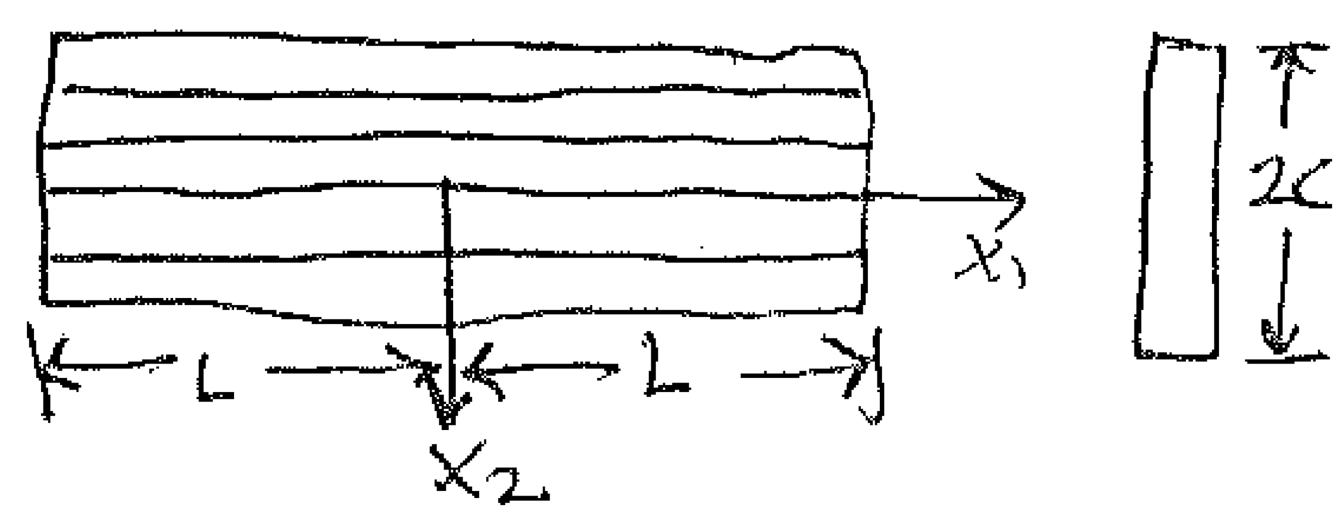
$$S_{rs}\hat{e}_r\hat{e}_s=0=s_{ijkl}\sigma_{kl,mq}e_{rmi}e_{sjq}\hat{e}_r\hat{e}_s$$

11 component, r=s=3

$$S_{33}=s_{ijkl}\sigma_{kl,mq}e_{3mi}e_{3jq}$$

		$j=1,\ q=2$	$j=2,\ q=1$
$m=1$ $i=2$	$s_{2jkl}\sigma_{kl,1q}e_{3jq}$	$s_{21kl}\sigma_{kl,12}$	$-s_{22kl}\sigma_{kl,11}$
$m=2$ $i=1$	$-s_{1jkl}\sigma_{kl,2q}e_{3jq}$	$-s_{11kl}\sigma_{kl,22}$	$s_{12kl}\sigma_{kl,21}$

$$S_{33}=0=-s_{11kl}\sigma_{kl,22}+2s_{12kl}\sigma_{kl,12}-s_{22kl}\sigma_{kl,11}$$



Plane stress

$\sigma_{11},\ \sigma_{12},\ \sigma_{22}$

$s_{1111},\ s_{2222},\ s_{1212}$

$s_{1211},\ s_{1222}=0$ due to composite symmetry

Note: Symmetry of composite critical when finding Airy stress function. If symmetric no coupling terms.

Airy stress function

$$\vec{\nabla}\times\underline{\underline{\phi}}\times\overleftarrow{\nabla}=\underline{\underline{\sigma}}$$

$$\sigma_{11}=\phi_{33,22}=\phi_{,22}$$

$$\sigma_{22}=\phi_{33,11}=\phi_{,11}$$

$$\sigma_{12}=-\phi_{33,12}=-\phi_{,12}$$

$$-S_{33}=s_{11kl}\sigma_{kl,22}-2s_{12kl}\sigma_{kl,12}+s_{22kl}\sigma_{kl,11}$$

$$=s_{1111}\sigma_{11,22}-2s_{1212}\sigma_{12,12}+s_{2211}\sigma_{11,11}$$

$$s_{1122}\sigma_{22,22}-2s_{1221}\sigma_{21,12}+s_{22,22}\sigma_{22,11}$$

$$+2\cancelto{0}{s_{1112}}\sigma_{12,22}$$

sub in Airy stress function

$$=s_{1111}\phi_{,2222}+4s_{1212}\phi_{,1212}+s_{2211}\phi_{,2211}$$

$$+s_{1122}\phi_{,1122}+s_{2222}\phi_{,1111}$$

$$=s_{1111}\phi_{,2222}+4s_{1212}\phi_{,1212}+2s_{1122}\phi_{,1122}+s_{2222}\phi_{,1111}=0$$

Consider polynomial solution in x, y

$\phi_2 = A_2 x^2 + B_2 xy + C_2 y^2$ equilibrium and compatibility guaranteed

$\sigma_{xx} = \phi_{,yy} = C_2$

$\sigma_{yy} = \phi_{,xx} = A_2$ gives uniform stress in beam

$\sigma_{xy} = -\phi_{,xy} = -B_2$

$\epsilon_{ij} = a_{ijkl}\sigma_{kl}$ compliances are different than isotropic case

$\phi_3 = \frac{A_3}{6}x^3 + B_3 x^2 y + C_3 x y^2 + \frac{D_3}{6}y^3$ satisfies governing equations but compliance depends on anisotropy

$\phi_4 = \frac{A_4}{12}x^4 + \frac{B_4}{6}x^3 y + \frac{C_4}{2}x^2 y^2 + \frac{D_4}{6}x y^3 + \frac{E_4}{12}y^4$

$\phi_{4,yyyy} = 2E_4$

$\phi_{4,xxxx} = 2A_4$

$\phi_{4,xyxy} = 2C_4$

$0 = a_{1111} 2E_4 + 2(2C_4)(a_{1122} + 2a_{1212}) + 2A_4 a_{2222}$

if isotropic: $0 = A_4 + 2C_4 + E_4$

if anisotropic its weighted based on compliance components

$\sigma_{xx} = \phi_{4,yy} = C_4 x^2 + D_4 xy + E_4 y^2$

$\sigma_{yy} = \phi_{4,xx} = A_4 x^2 + B_4 xy + C_4 y^2$

$\sigma_{xy} = -\phi_{4,xy} = -\left[\frac{B_4}{2}x^2 + 2C_4 xy + \frac{D_4}{2}y^2\right]$

must determine constants from boundary conditions