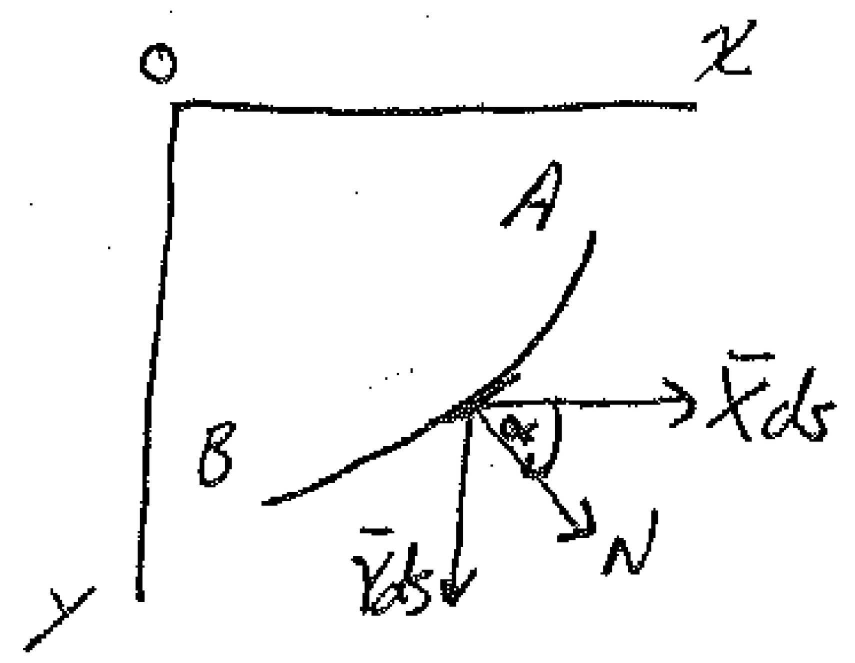
Section 59 (Timoshenko): Stress Resultant on a curve

$\bar{X}, \bar{Y}$ given (a)

also written as $t_i = \sigma_{ji} n_j$

$$\bar{X} = t_x = \sigma_{xx} n_x + \sigma_{yx} n_y$$



$\sin\alpha = -\frac{dx}{ds} = n_y$

$\cos\alpha = \frac{dy}{ds} = n_x$

$$\bar{X} = \frac{d}{ds}\left(\frac{\partial\phi}{\partial y}\right)$$
(c)
$$\bar{Y} = -\frac{d}{ds}\left(\frac{\partial\phi}{\partial x}\right)$$

Note: $\frac{d}{ds} = \frac{\partial}{\partial x}\frac{dx}{ds} + \frac{\partial}{\partial y}\frac{dy}{ds}$

$$M = \int_A^B x\bar{Y}ds - \int_A^B y\bar{X}ds$$

integrate by parts

$\int u\,dv = uv - \int v\,du$

$x = u$ $\quad d\left(\frac{\partial\phi}{\partial x}\right) = dv$

$du = dx$ $\quad v = \frac{\partial\phi}{\partial x}$

$$\Rightarrow \int_A^B \left[\frac{\partial\phi}{\partial x}dx + \frac{\partial\phi}{\partial y}dy\right] = \phi\Big|_A^B$$ ~~SINCE $\phi(x,y)$~~ ✱

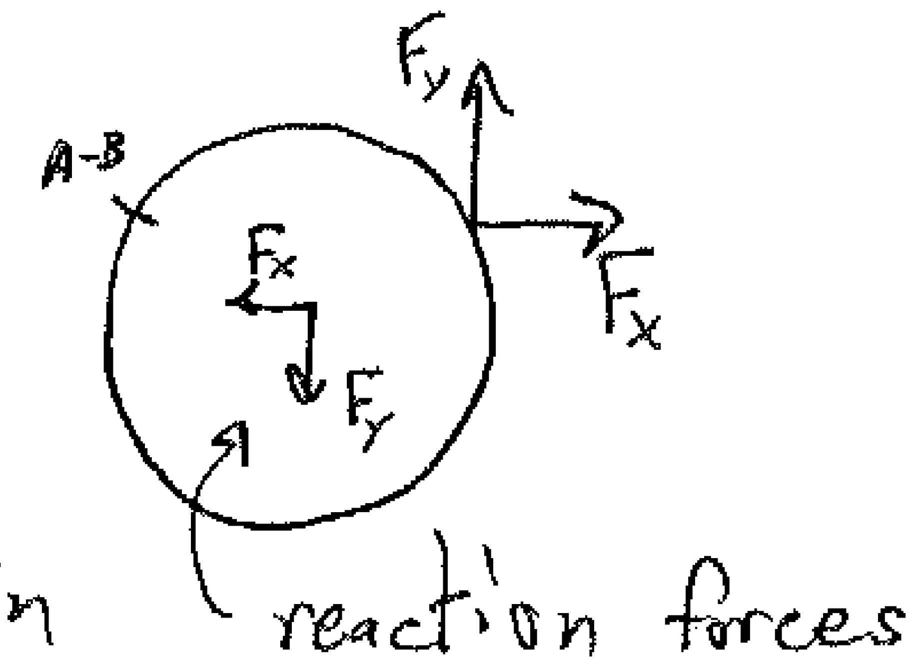
$$M = \phi\Big|_A^B - \left[x\frac{\partial\phi}{\partial x} + y\frac{\partial\phi}{\partial x}\right]\Big|_A^B$$

Prescribe loads on A-B specified by giving $\frac{\partial\phi}{\partial x}, \frac{\partial\phi}{\partial y}$

If AB forms closed circuit

$$F_y = \frac{\partial\phi}{\partial x}\Big|_A^B$$

$$F_x = -\frac{\partial\phi}{\partial y}\Big|_A^B$$



if no reaction at origin

$F_y = F_x = 0$

$\psi, \chi \to z^{\pm n}$ on a closed circuit

$$F_x + iF_y = 0$$

$$M = 0$$

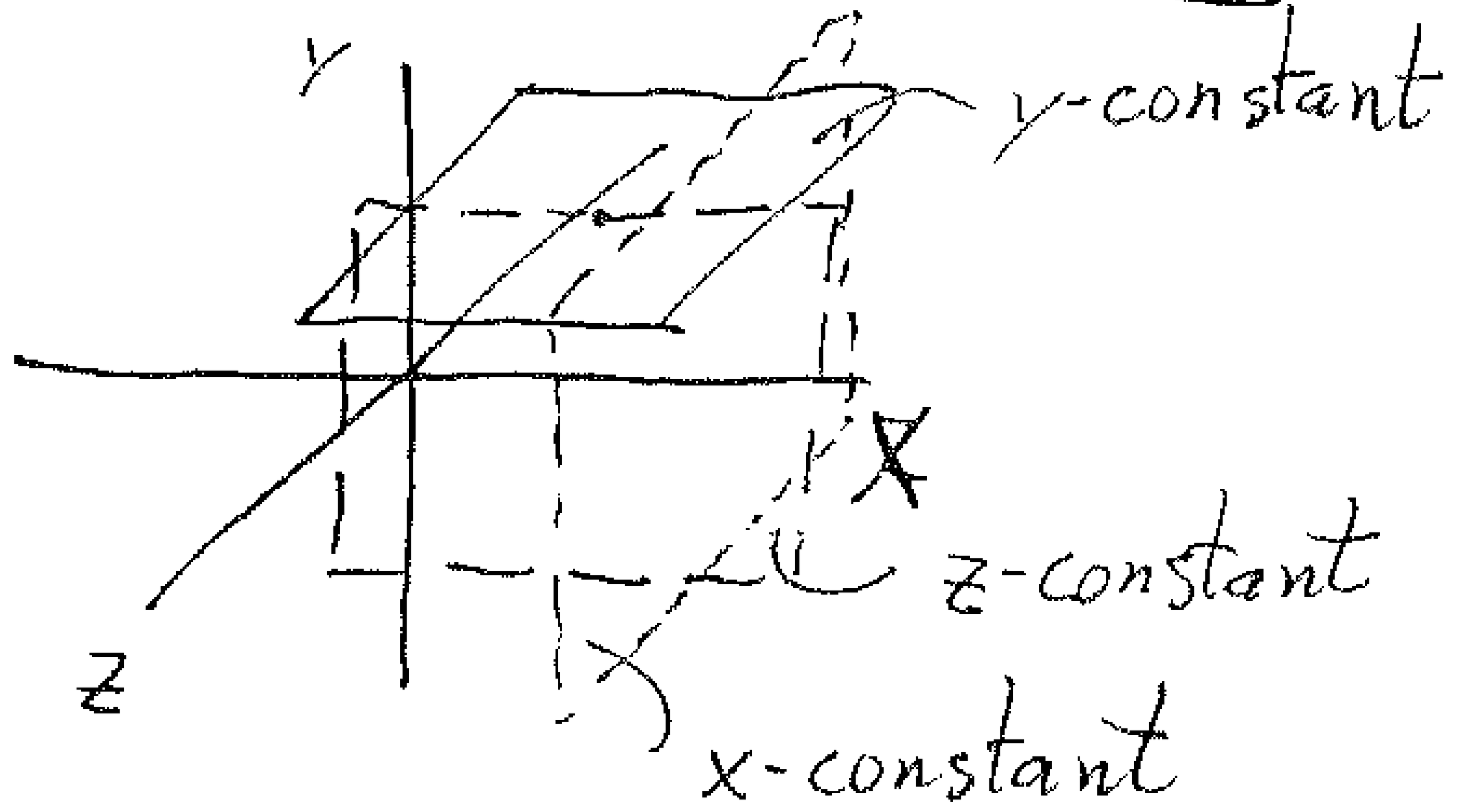
Loads at origin?

$$\log z \to \log r + i\theta$$

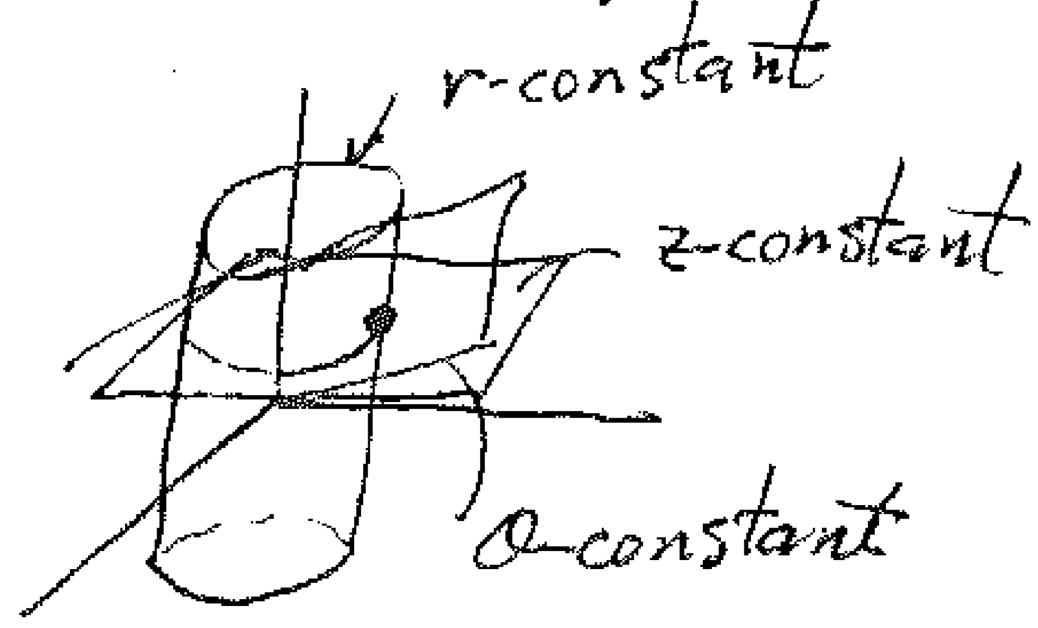
$$\psi = c \log z$$

$$\chi(z) = Dz\log(z)$$

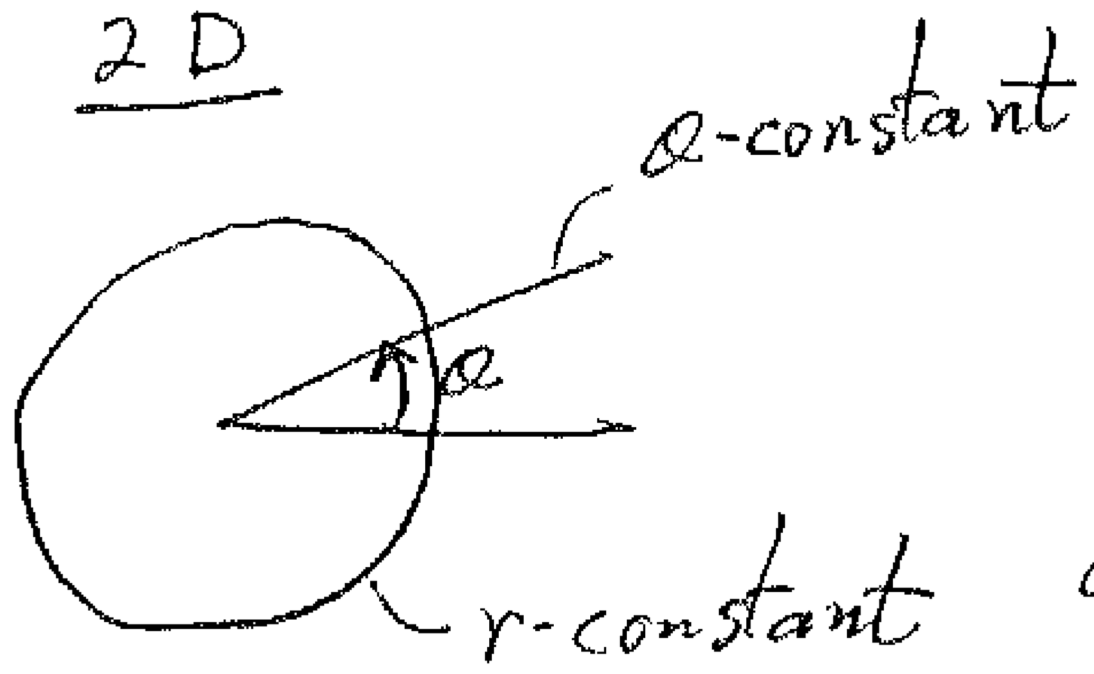
<u>60: Curvilinear coordinates</u>



look at intersection point of 3-planes



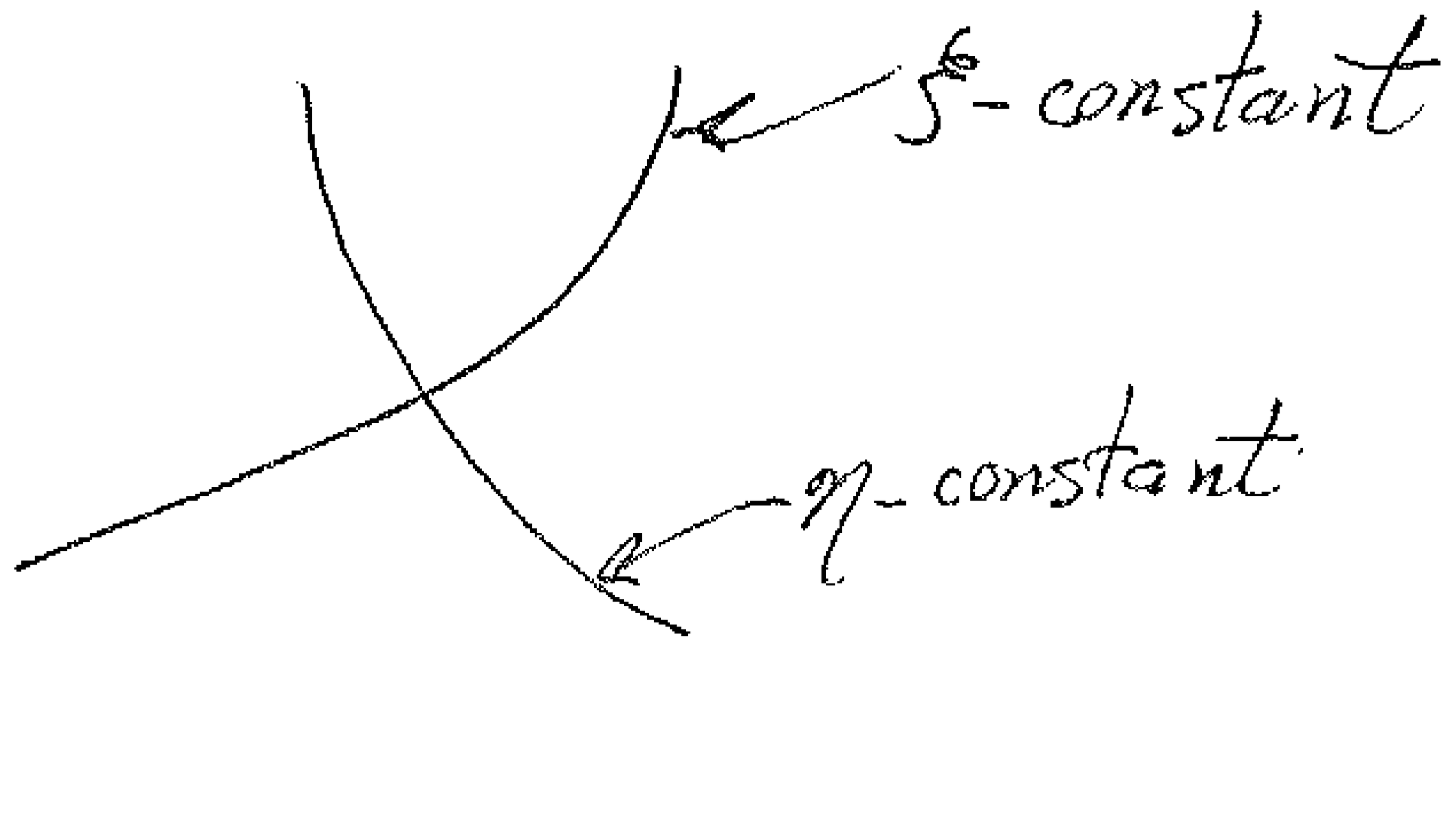
<u>2D</u>



$$r = \sqrt{x^2 + y^2}$$

$$\theta = \tan^{-1}\left(\frac{y}{x}\right)$$

Special case of $F_1(x,y) = \xi$, $F_2(x,y) = \eta$



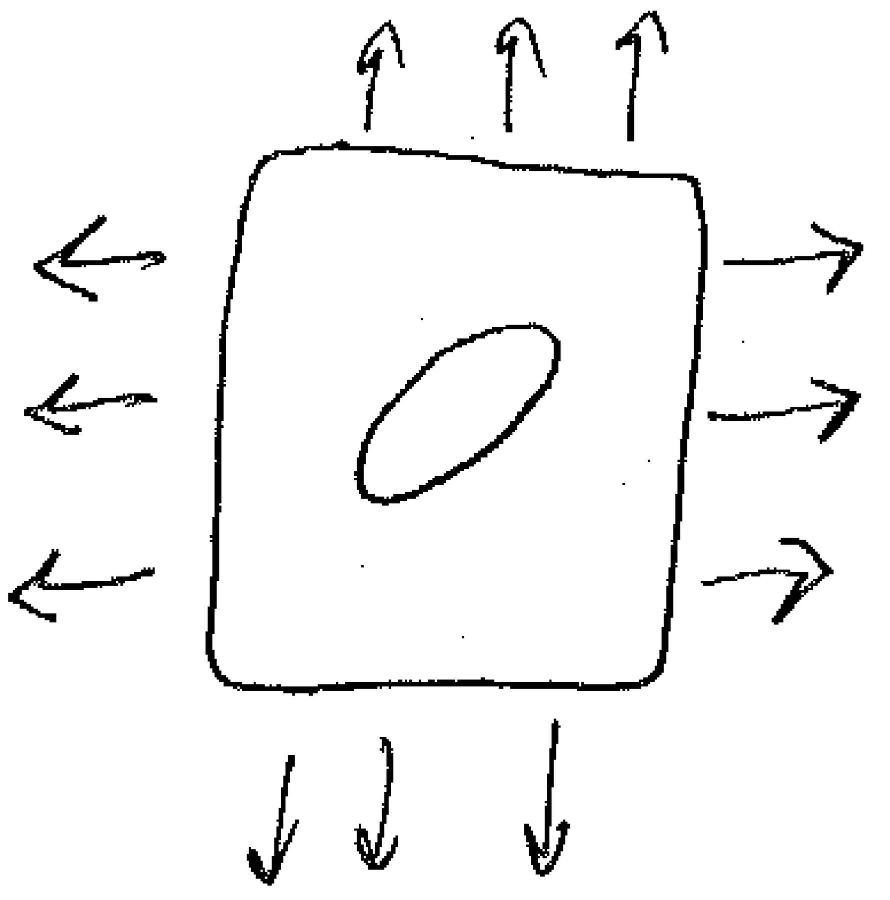
can invert function

$$x = f_1(\xi, \eta)$$

$$y = f_2(\xi, \eta)$$

*(79)

Elliptic hole problem



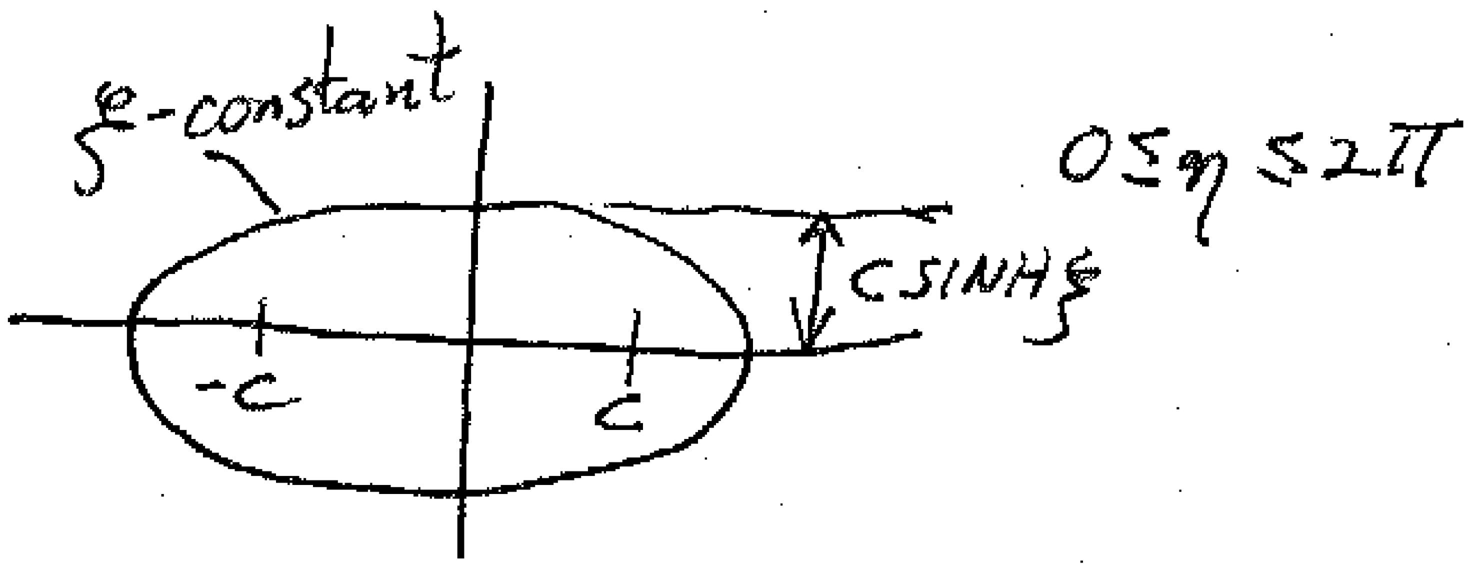
let: $x = c\cosh(\xi)\cos\eta$

$y = c\sinh(\xi)\sin\eta$

$$\frac{x^2}{c^2\cosh^2\xi} + \frac{y^2}{c^2\sinh^2\xi} = 1$$

if ξ = constant

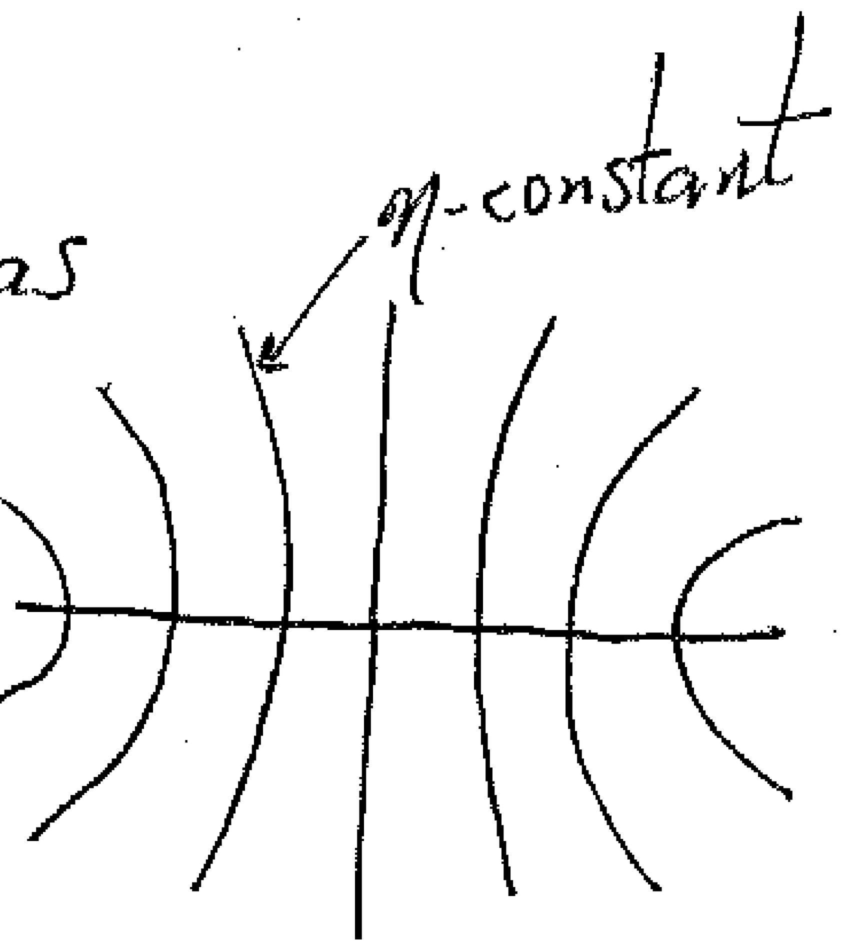
ellipse with foci at c



also can eliminate ξ

to get family of ~~hyperab~~ hyperbolas

$$\frac{x^2}{c^2\cos^2\eta} + \frac{-y^2}{c^2\sin^2\eta} = 1$$



Note: $x + iy = c\cosh(\xi + i\eta)$

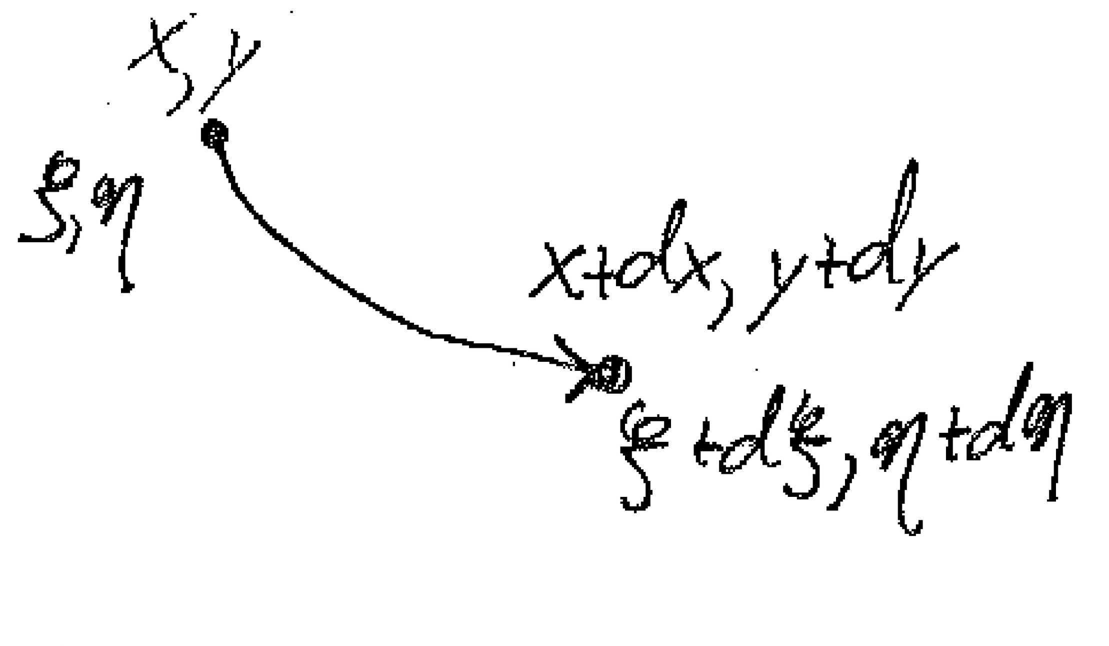
where $x = c\cosh\xi\cos\eta$

$y = c\sinh\xi\sin\eta$

$\to z = c\cosh\rho$

$\rho = \xi + i\eta$ special case of $z = f(\rho)$

ξ, η satisfy Cauchy-Riemann conditions

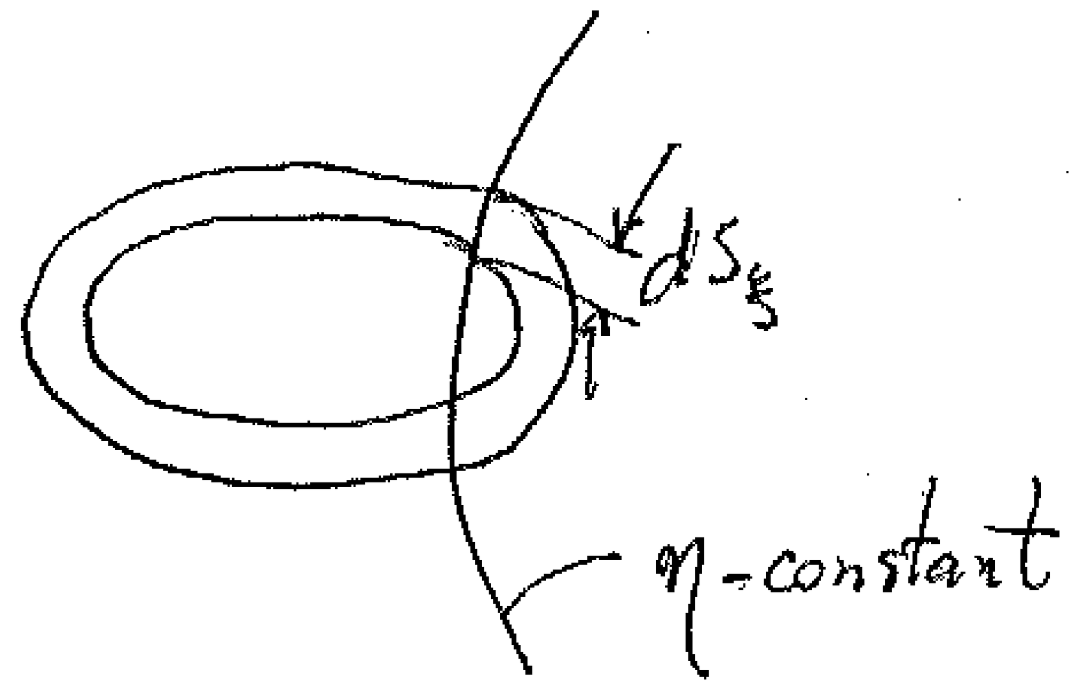


$$dx = \frac{\partial x}{\partial \xi}d\xi + \frac{\partial x}{\partial \eta}d\eta$$

Same for dy

let η = constant

vary ξ



$$dx = \frac{\partial x}{\partial \xi} d\xi$$

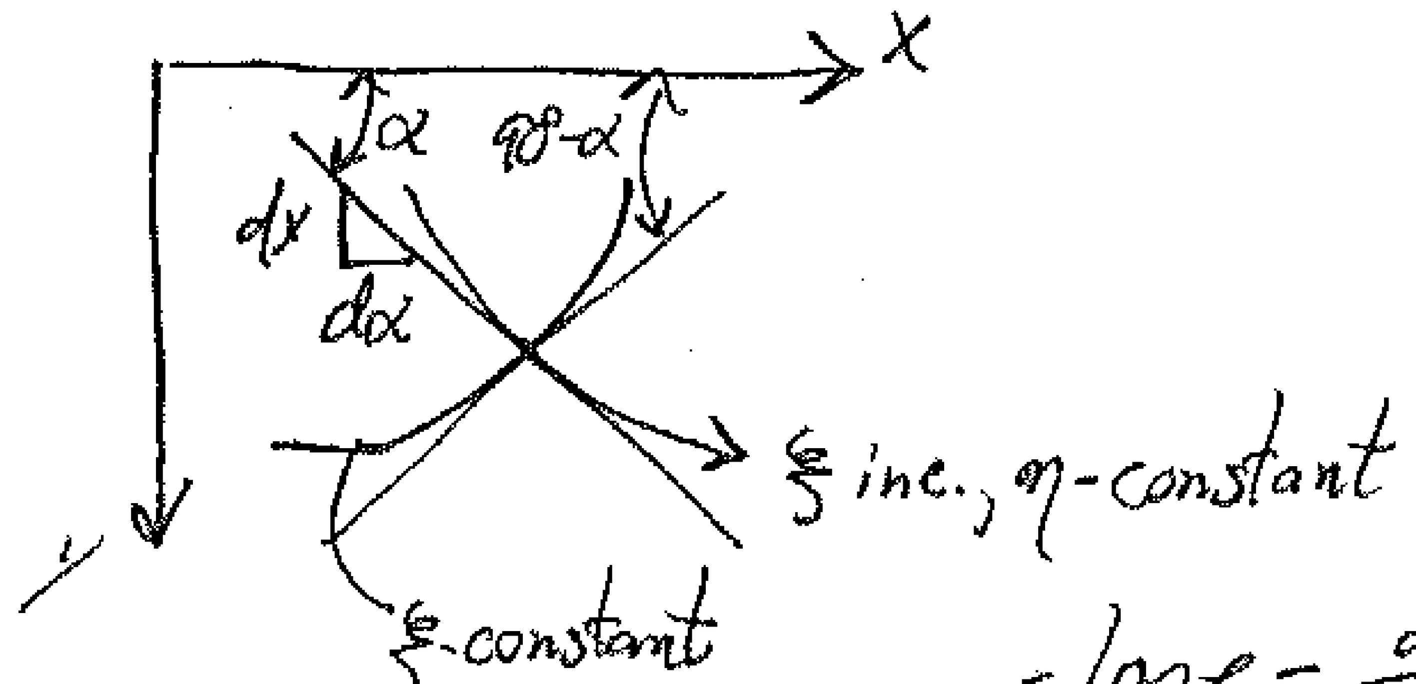
$$dy = \frac{\partial y}{\partial \xi} d\xi$$

$$dS_\xi^2 = dx^2 + dy^2 = \left[\left(\frac{\partial x}{\partial \xi}\right)^2 + \left(\frac{\partial y}{\partial \xi}\right)^2\right] d\xi^2$$

since $z = f(\rho)$

$$\rho = \xi + i\eta$$

$$\frac{\partial z}{\partial \xi} = \frac{\partial x}{\partial \xi} + i\frac{\partial y}{\partial \xi} = f'(\rho)$$



$$\text{slope} - \left.\frac{dy}{dx}\right|_\eta = \frac{\partial y/\partial \xi}{\partial x/\partial \xi} = \tan(\alpha)$$

$$\left.\frac{dy}{dx}\right|_\xi = -\cot(\alpha)$$

For the elliptical system

$$f'(\rho) = c\,\text{SINH}(\rho)$$

$$= \underbrace{c\,\text{SINH}\,\xi\,\text{COS}\,\eta}_{J\cos(\alpha)} + i\underbrace{c\,\text{COSH}\,\xi\,\text{SIN}\,\eta}_{J\,\text{SIN}\,\alpha} = Je^{i\alpha}$$

$$J^2 = c^2\left[\text{SINH}^2\xi\,\text{COS}^2\eta + \text{COSH}^2\xi\,\text{SIN}^2\eta\right]$$

see pg. 195 (Timoshenko)

Stress components in elliptical coordinates

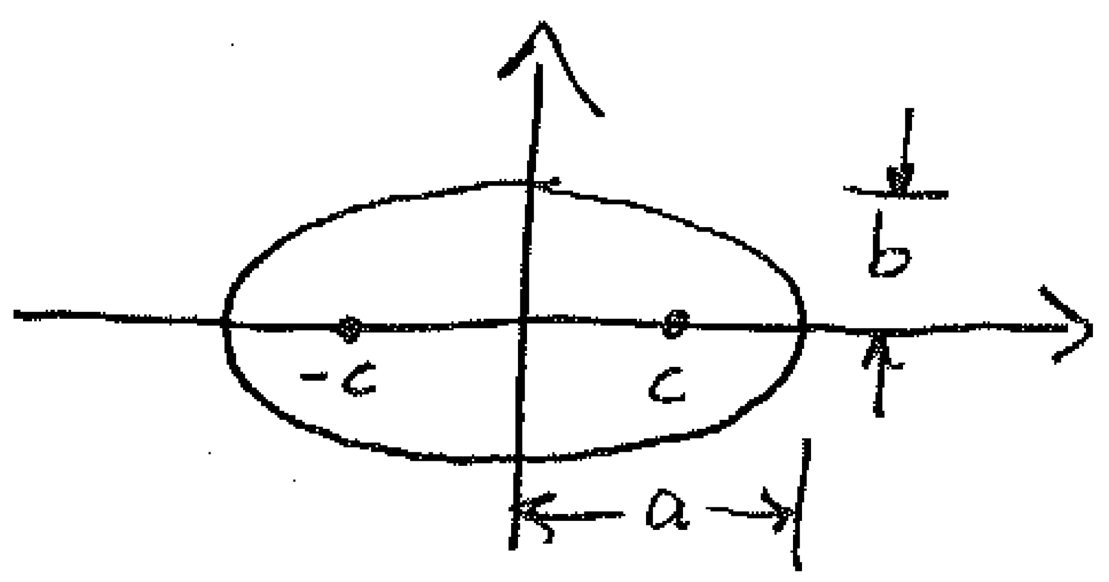
In curvilinear coordinates $\psi(\rho), \chi(\rho), z(\rho)$ needed

$\sigma_{xx}, \sigma_{yy}, \sigma_{xy}$ known in terms of $\psi(z), \chi(z)$

see figure 128

use coordinate transformation - p. 196

62 | Solution in Elliptical Coordinates (ξ, η)

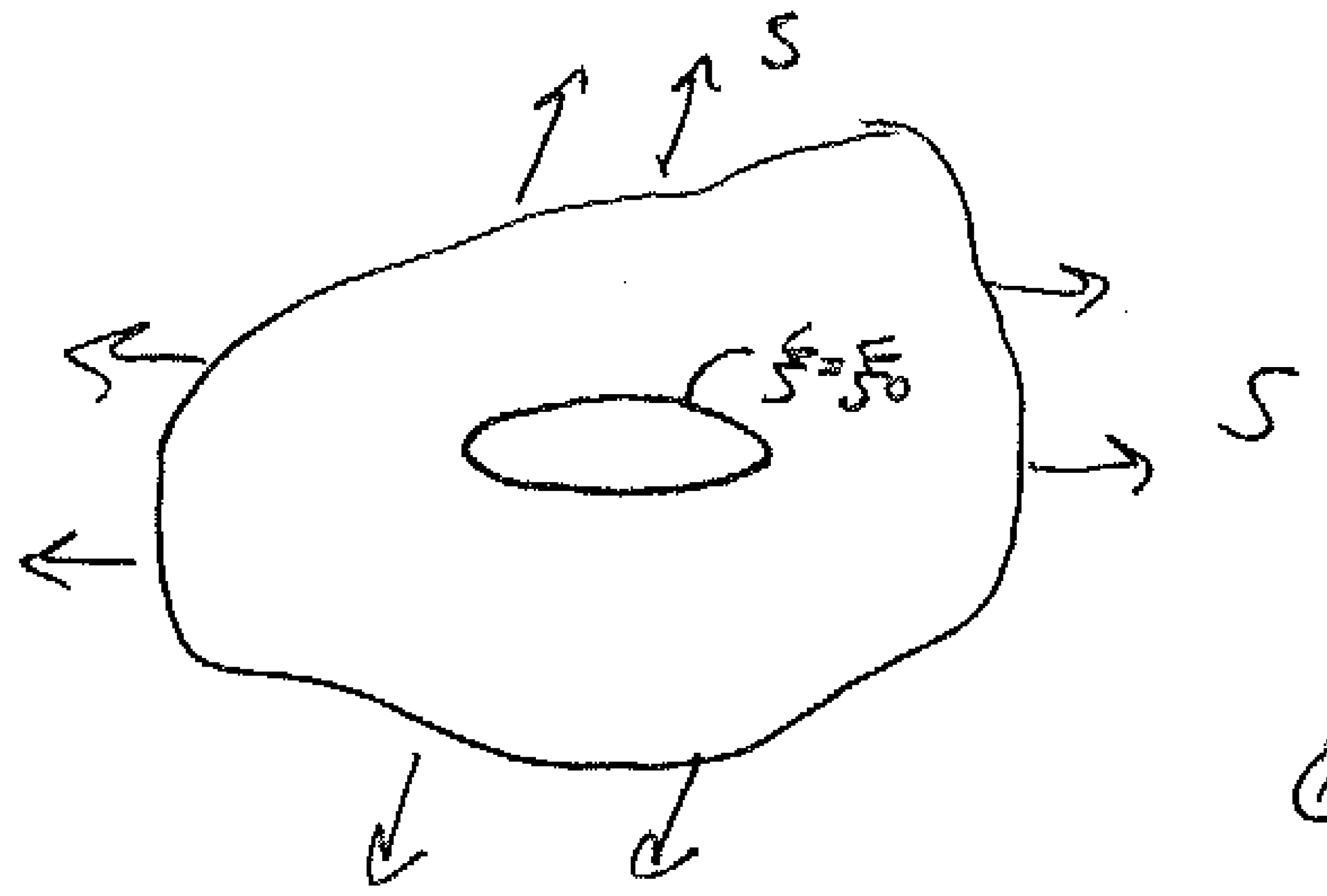


$a = c \cosh \xi_0$ } boundary of
$b = c \sinh \xi_0$ } hole

as $\xi_0 \to 0$ crack with length $2c$

as $\xi_0 \to 0$ infinitely large circle

Plate in tension

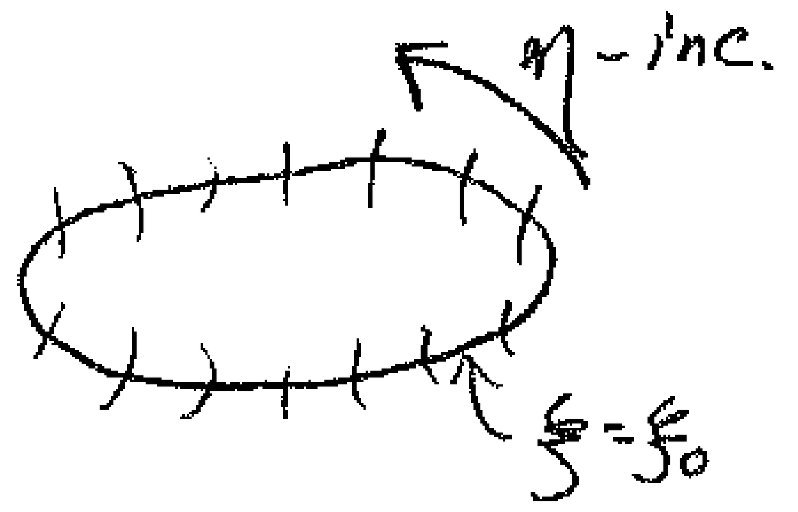


B.C.

① $\sigma_x = \sigma_y = S$ as $\xi \to \infty$

② $\sigma_\xi, \tau_{\xi\eta} = 0$ when $\xi = \xi_0$

① is satisfied by (l) and (m) p. 199



stress and displacement are periodic in η

must find functions that are periodic and compatible

$$\sigma_\xi + \sigma_\eta = \sigma_x + \sigma_y$$

$$\sigma_\eta - \sigma_\xi - 2i\tau_{\eta\xi} = e^{2i\alpha}(\sigma_y - \sigma_x + 2i\tau_{xy})$$

$$z = c\,\cosh\rho$$

$$\frac{dz}{d\rho} = c\,\sinh\rho$$

$$\frac{d\rho}{dz} = \frac{1}{c\,\sinh\rho}$$

$$\psi'(z) = A\cosh\rho = \frac{\partial\psi}{\partial\rho}\frac{d\rho}{dz}$$

recall:

$$2\,\mathrm{Re}\,\psi'(z) = S$$

$$\sinh z = \frac{1}{2}(e^{z} - e^{-z})$$

$$\cosh z = \frac{1}{2}(e^{z} + e^{-z})$$

as $\rho \to \infty$ $\sinh\rho = \cosh\rho = \frac{1}{2}e^{\rho}$

$\tanh\rho \to 1$

$$\therefore \sigma_x + \sigma_y = 4A = 2S$$

$$A = \frac{S}{2}$$

$$\psi'' = -\frac{A}{c}\frac{1}{\sinh^3\rho}$$

$$z = c\cosh\rho$$

$$\bar{z} = c\cosh\bar{\rho}$$

$$\bar{z}\psi''(z) = -\frac{A\cosh\bar{\rho}}{\sinh^3\rho}$$ vanishes at ∞

$$\sigma_\xi - i\tau_{\xi\eta} = \psi'(z) + \bar{\psi}'(\bar{z}) - e^{2i\alpha}[\bar{z}\psi''(z) + \chi''(z)]$$

at the boundary of the hole

$\xi = \xi_0$, $\rho + \bar{\rho} = 2\xi_0$, $\bar{\rho} = 2\xi_0 - \rho$

at $\xi = \xi_0 \to \sigma_\xi + i\tau_{\xi\eta} = \frac{1}{\sinh^3\rho\,\sinh\bar{\rho}}(A\cosh 2\xi_0 + B)\cosh\bar{\rho}$

(see (l)-(m)) p. 199