(5)

Field Equations

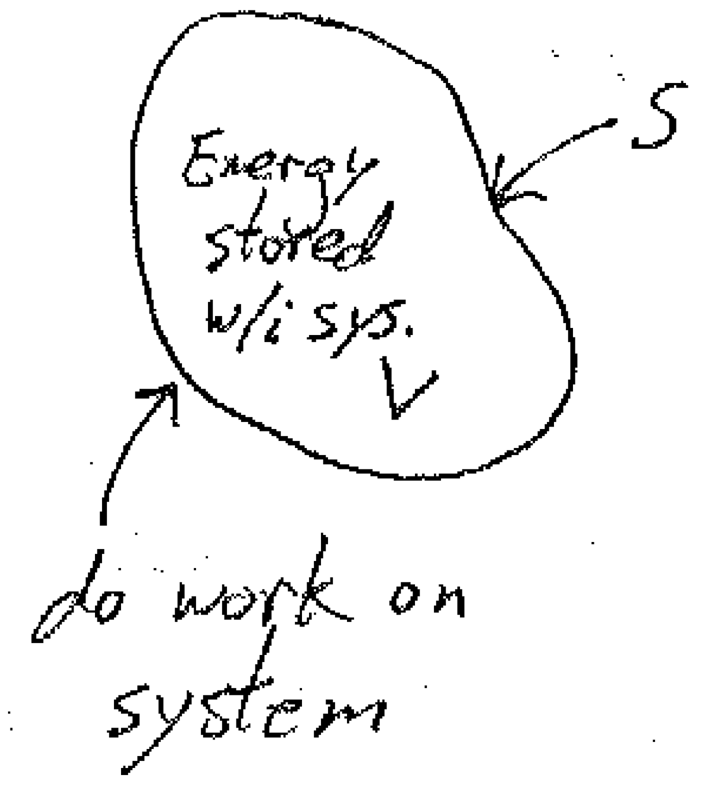
strain displacement: $\epsilon_{ij} = u_{i,j} + u_{j,i}$

compatibility: $\frac{\partial^2 \epsilon_{11}}{\partial x_2^2} + \frac{\partial^2 \epsilon_{22}}{\partial x_1^2} = 2 \frac{\partial^2 \epsilon_{12}}{\partial x_1 \partial x_2}$

Equilibrium: $\frac{\partial \sigma_{ji}}{\partial x_j} = 0$ or $\frac{\partial \sigma_{ij}}{\partial x_i} + \rho b_j = \rho \ddot{u}_j$

Constitutive law

Hooke's Law $\left.\begin{array}{l}\underline{\underline{\sigma}}(\underline{\underline{\epsilon}}) \\ \underline{\underline{\epsilon}}(\underline{\underline{\sigma}})\end{array}\right\}$ linear, isotropic, homogeneous



can look at $\dot{U}$ or dU ($\dot{W}$, dW)

$dW = dU$

$\dot{W} = \dot{U}$

ways to do work

mechanical $\dot{W} = F\dot{x}$

thermal $\dot{W} = \int q_i n_i \, dS$ ← heat flux vector

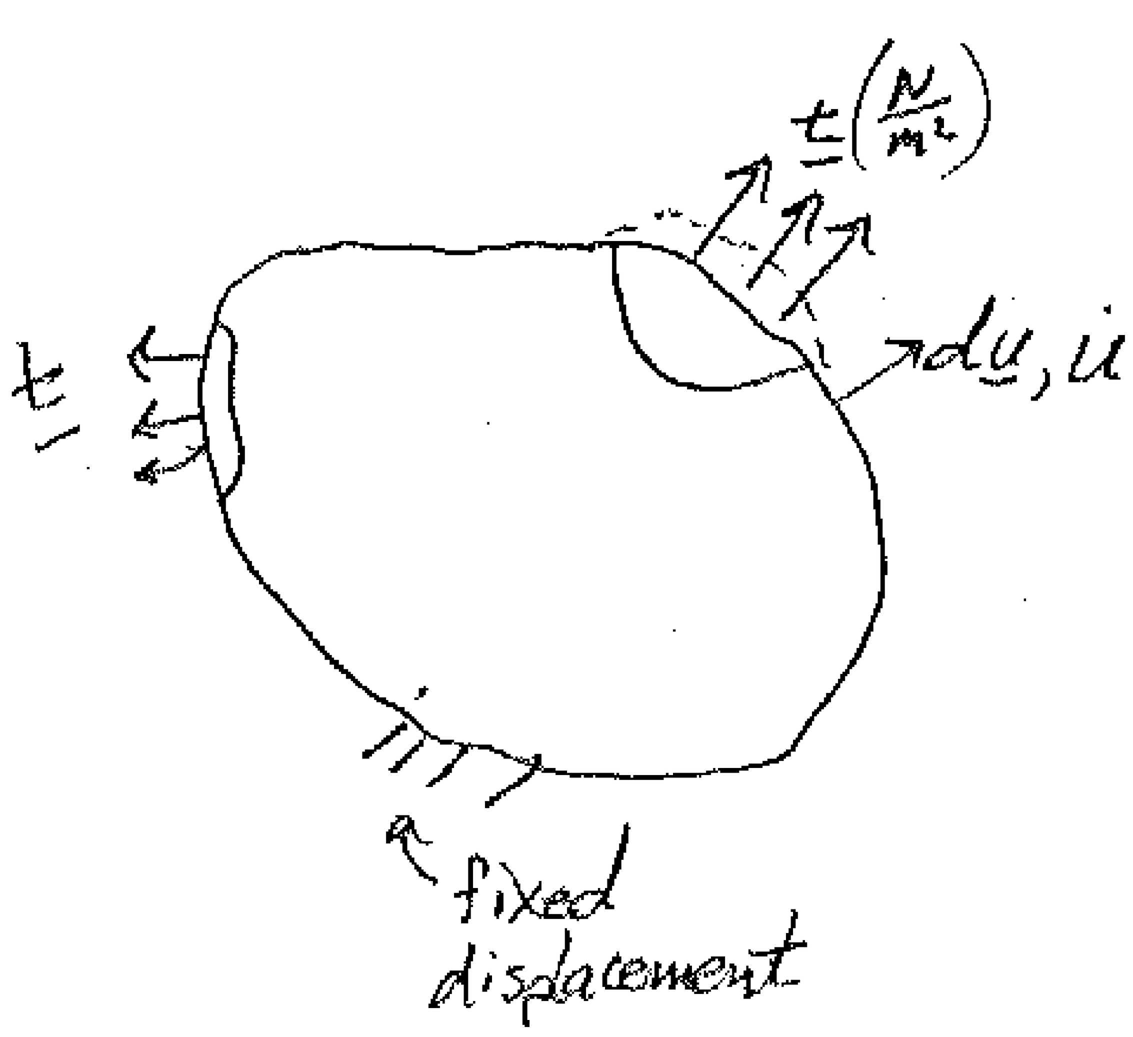
electrical $\dot{W} = V \cdot \dot{Q}$

magnetic $\dot{W} = B \cdot \dot{H}$

chemical . . .

Stored Energy

strain energy, thermal energy, surface energy, phase changes



$\underline{F} = \underline{t}\, dS$ ← surface area

$W = \int_S \int_0^{\underline{u}} \underline{t} \cdot d\underline{u} \, dS \qquad dW = \int_0^S \underline{t} \cdot d\underline{u} \, dS$

$\dot{W} = \int_0^S \underline{t} \cdot \dot{u} \, dS$

$\underline{t} = \underline{\underline{\sigma}} \cdot \hat{\underline{n}}$

$$\dot{W} = \int_S \underline{\underline{\sigma}} \cdot \hat{n} \cdot \dot{\underline{u}}\, dS = \int_S (\underline{\underline{\sigma}} \cdot \dot{u}) \cdot \hat{n}\, dS = \int_V \vec{\nabla} \cdot (\underline{\underline{\sigma}} \cdot \dot{\underline{u}})\, dV$$

$$= \int_S \sigma_{ji} n_j \dot{u}_i\, dS = \int_S (\sigma_{ji} \dot{u}_i) n_j\, dS = \int_V (\sigma_{ji} \dot{u}_i)_{,j}\, dV = \int_V \underbrace{\frac{\partial}{\partial x_j}(\sigma_{ji} \dot{u}_i)}\, dV$$

$$\frac{\partial}{\partial x_j}(\sigma_{ji} \dot{u}_i) = \underbrace{\sigma_{ji,j}} \dot{u}_i + \sigma_{ji} \dot{u}_{i,j}$$

$\frac{\partial \sigma_{ji}}{\partial x_j} = 0$ equilibrium assumed

$= \sigma_{ji} \dot{u}_{i,j}$

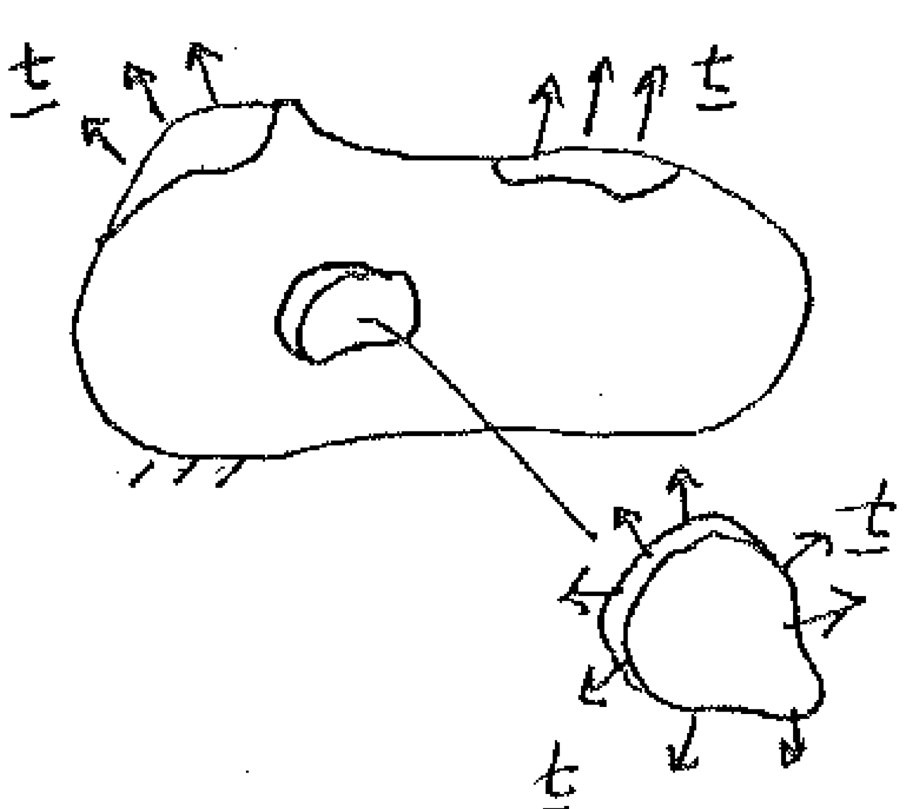
$= \sigma_{ji} (\dot{\epsilon}_{ij} + \dot{\omega}_{ij})$

$= \sigma_{ji} \dot{\epsilon}_{ij}$ product of symmetric and skew symmetric matrices is always zero

now,

$$\dot{W} = \int \sigma_{ij} \dot{\epsilon}_{ij}\, dV = \dot{U}$$

subvolumes can be applied to get strain energy



$\frac{\dot{U}}{dV} = \dot{u}$ strain energy density rate

$= \sigma_{ij} \dot{\epsilon}_{ij}$

$du = \sigma_{ij}\, d\epsilon_{ij}$

Inthereversible case, we have a perfect differential,

(1) $du = \sigma_{ij}\, d\epsilon_{ij}$

(2) $du = \frac{\partial u}{\partial \epsilon_{ij}}\, d\epsilon_{ij}$

$$\sigma_{ij} = \frac{\partial u}{\partial \epsilon_{ij}} = \sigma_{ij}(\epsilon_{ij})$$

Approximate the stress-strain relation using Taylor ~~sy~~ series

$\sigma_{ij} = \sigma_{ij}(\epsilon_{kl})$

stress component can be a function of all 9 strain components

$$\Delta\sigma_{ij} = \frac{\partial \sigma_{ij}}{\partial \epsilon_{kl}} \Delta\epsilon_{kl} + \frac{1}{2}\underbrace{\left(\frac{\partial^2 \sigma_{ij}}{\partial \epsilon_{kl}\partial \epsilon_{mn}} \Delta\epsilon_{kl}\Delta\epsilon_{mn}\right)}_{\text{non linear elasticity}} + \cdots$$

$\Delta\sigma_{ij} = \underbrace{C_{ijkl}}_{} \Delta\epsilon_{kl}$

coefficients of a 4th order stiffness tensor

let $\left.\begin{aligned}\Delta\sigma_{ij} &= \sigma_{ij} - 0 \\ \Delta\epsilon_{ij} &= \epsilon_{ij} - 0\end{aligned}\right\}$ zero reference state

$\sigma_{11} = C_{11kl}\epsilon_{kl}$ (9 components)

$= C_{1111}\epsilon_{11} + C_{1112}\epsilon_{12} + \cdots C_{1133}\epsilon_{33}$ (σ_{11} linear combination of 9 strain components)

$3\times3\times3\times3 = 81$ components in C_{ijkl} total

Thermodynamic relations

$du = \sigma_{ij}\, d\epsilon_{ij} + T ds$

$du = \frac{\partial u}{\partial \epsilon_{ij}} d\epsilon_{ij} + \frac{\partial u}{\partial s} ds$

$\sigma_{ij} = \frac{\partial u}{\partial \epsilon_{ij}}$, $T = \frac{\partial u}{\partial s}$ where $u = u(\sigma_{kl}, s)$

$\sigma_{ij} = \sigma_{ij}(\epsilon_{kl}, s)$

$T = T(\epsilon_{kl}, s)$

Legendre transformation

$h = u - Ts - \sigma_{ij}\epsilon_{ij}$

$dh = du - Tds - sdT - \sigma_{ij}d\epsilon_{ij} - \epsilon_{ij}d\sigma_{ij}$

~~$dh = du$~~

$dh = -sdT - \epsilon_{ij}d\sigma_{ij}$

$dh = +\left(\frac{\partial h}{\partial T}\right)dT + \left(\frac{\partial h}{\partial \sigma_{ij}}\right)d\sigma_{ij}$, $\frac{\partial h}{\partial T} = -s$, $\frac{\partial h}{\sigma_{ij}} = -\epsilon_{ij}$

18

$h = h(T, \sigma_{ij})$

$s = s(T, \sigma_{ij})$

$\epsilon_{ij} = \epsilon_{ij}(T, \sigma_{ij})$ (indices could be different)

Approximate

$s(T, \underline{\underline{\sigma}})$ and $\underline{\underline{\epsilon}}(T, \underline{\underline{\sigma}})$ as linear functions

$$\Delta\epsilon_{ij} = \left.\frac{\partial \epsilon_{ij}}{\partial \sigma_{kl}}\right|_T \Delta\sigma_{kl} + \left.\frac{\partial \epsilon_{ij}}{\partial T}\right|_{\underline{\underline{\sigma}}} dT$$

$$\Delta\epsilon_{ij} = S^T_{ijkl}\,\Delta\sigma_{kl} + \alpha^{\underline{\underline{\sigma}}}_{ij}\,dT$$

$$\Delta s = \underbrace{\left.\frac{\partial s}{\partial \sigma_{kl}}\right|_T}_{\alpha_{kl}} \Delta\sigma_{kl} + \underbrace{\left.\frac{\partial s}{\partial T}\right|_{\sigma}}_{\sim \text{heat capacity}} \Delta T$$

$$\frac{\partial \epsilon_{ij}}{\partial T} = \frac{\partial s}{\partial \sigma_{ij}} = \alpha_{ij} \quad \text{thermal expansion}$$

$$-\frac{\partial^2 h}{\partial \sigma_{ij}\partial T} = -\frac{\partial^2 h}{\partial T \partial \sigma_{ij}} = \alpha_{ij} \quad \text{Maxwell relation}$$

Ex) $PV = RT$

$P = \frac{RT}{V}$ ← stress: if gas must expand many terms of Taylor series needed to get $PV = RT$ because of $\frac{1}{V}$ term

Equilibrium $\vec{\nabla}\cdot\underline{\underline{\sigma}}=0$ or $\vec{\nabla}\cdot\underline{\underline{\sigma}}+\rho\underline{b}=\rho\ddot{\underline{u}}$

$$\sigma_{ji,j}=0$$

$$\sigma_{11,1}+\sigma_{21,2}+\sigma_{31,3}=0$$

$$\frac{\partial\sigma_{11}}{\partial x_1}+\frac{\partial\sigma_{21}}{\partial x_2}+\frac{\partial\sigma_{31}}{\partial x_3}=0$$

Compatibility (2-D)

$$\epsilon_{11,22}+\epsilon_{22,11}=2\epsilon_{12,12}$$

Constitutive law

$$\epsilon_{ij}=\frac{1+\nu}{E}\sigma_{ij}-\frac{\nu}{E}\sigma_{kk}\delta_{ij}$$

$$\sigma_{ij}=\lambda\epsilon_{kk}\delta_{ij}+2\mu\epsilon_{ij}$$

$$=\lambda \underbrace{e}_{\epsilon_{11}+\epsilon_{22}+\epsilon_{33}}+2\mu\epsilon_{ij}$$

$$\epsilon_{ij}=\frac{1+\nu}{E}\sigma_{ij}-\frac{\nu}{E}\sigma_{kk}\delta_{ij}$$

$$\epsilon_{kk}=\frac{1+\nu}{E}\sigma_{kk}-\frac{\nu}{E}\sigma_{kk}(3)$$

$$\epsilon_{kk}=\frac{1-2\nu}{E}\sigma_{kk}$$

$$\sigma_{kk}=\frac{E}{1-2\nu}\epsilon_{kk}$$

$$\epsilon_{ij}=\frac{1+\nu}{E}\sigma_{ij}-\frac{\nu}{E}\left(\frac{E}{1-2\nu}\right)\epsilon_{kk}\delta_{ij}$$

$$\left[\epsilon_{ij}+\frac{\nu}{1-2\nu}\epsilon_{kk}\delta_{ij}\right]\frac{E}{1+\nu}=\sigma_{ij}$$

$$\underbrace{\frac{E}{1+\nu}}_{2\mu}\epsilon_{ij}+\underbrace{\frac{E\nu}{(1-2\nu)(1+\nu)}}_{\lambda}\epsilon_{kk}\delta_{ij}=\sigma_{ij}$$