Complex Variables: Application of curvelinear coordinates

Complex variables advantageous for problems with boundaries that are ellipses, hyperbolas, nonconcentric circles, etc.

recall $\sigma_{xx} + \sigma_{yy} = \nabla^2\phi = P$

$$\nabla^2 P = \nabla^4 \phi = 0$$

$$\phi = r^2 f + g$$

radius in polar coordinates

complex variables

difficult to integrate in new curvilinear coord.
better to get stress and displacement

- Functions of complex variables

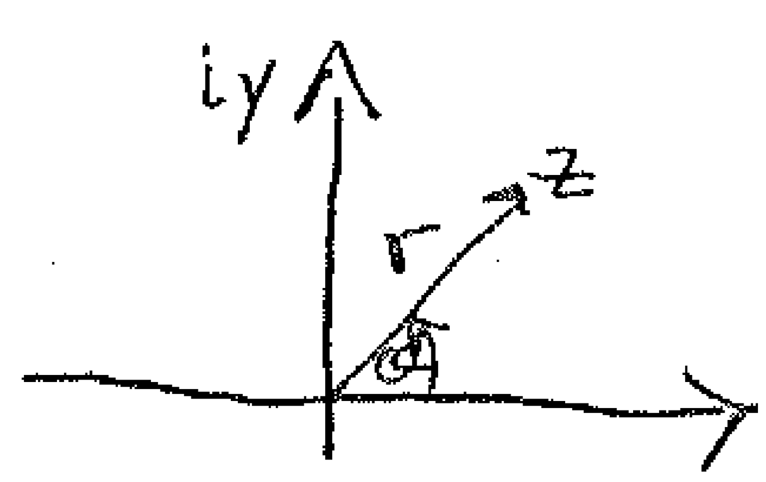
$$z = x + iy$$

$$f(z): \cancel{z^2 = x^2 + 2ixy - y^2}$$

$$z^2 = x^2 + 2ixy - y^2$$

$$f(z) = \alpha + i\beta$$

$$= \underbrace{x^2 - y^2}_{\alpha} + i\underbrace{2xy}_{\beta}$$



$$x + iy = r(\cos\theta + i\sin\theta)$$
$$= re^{i\theta}$$

$$f(z) = \frac{1}{z} = \frac{1}{x+iy}\,\frac{x-iy}{x-iy}$$

$$= \frac{x-iy}{x^2+y^2}$$

Separate $\sinh(z)$ into Re and Im

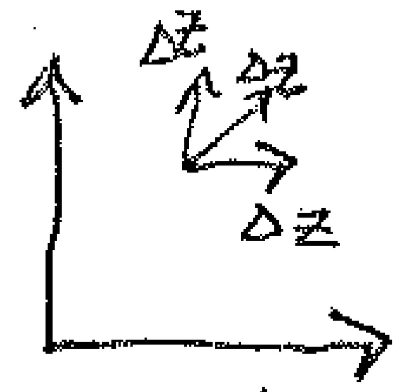
$$\sinh(z) = \frac{1}{2}(e^{z} - e^{-z})$$

(69)

check cosH(z) derivation

$$\frac{d}{dz}(z^2) = 2z$$

$$\frac{d}{dx}(\sin z) = \frac{d}{d(iy)}\sin(z) \rightarrow \text{analytic function}$$



independent of direction

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial (iy)} = \frac{df}{dz}$$

$$f = \alpha + i\beta$$

$$\frac{\partial f}{\partial x} = \frac{\partial \alpha}{\partial x} + i\frac{\partial \beta}{\partial x}$$

$$\frac{\partial f}{\partial (iy)} = \frac{1}{i}\frac{\partial f}{\partial y} = \frac{1}{i}\left(\frac{\partial \alpha}{\partial y} + i\frac{\partial \beta}{\partial y}\right)\cdot\frac{i}{i}$$

$$= -i\frac{\partial \alpha}{\partial y} + \frac{\partial \beta}{\partial y}$$

$$\boxed{\frac{\partial \alpha}{\partial x} = \frac{\partial \beta}{\partial y}\ ; \quad \frac{\partial \alpha}{\partial y} = -\frac{\partial \beta}{\partial x}} \leftarrow \text{Cauchy-Riemann conditions}$$

$$\frac{\partial^2 \alpha}{\partial x^2} + \frac{\partial^2 \alpha}{\partial y^2} = 0 \qquad \Bigg| \qquad \frac{\partial^2 \beta}{\partial x^2} + \frac{\partial^2 \beta}{\partial y^2} = 0$$

$$\frac{\partial^2 \beta}{\partial x \partial y} - \frac{\partial^2 \beta}{\partial y \partial x} = 0 \qquad \Bigg| \qquad -\frac{\partial^2 \alpha}{\partial y \partial x} + \frac{\partial^2 \alpha}{\partial y \partial x} = 0$$

Re(f) or Im(f) satisfies LaPlace's eqn.

∴ f is a harmonic eqn.

α and β are conjugate harmonic functions

$$\nabla^2(x\psi) = 2\frac{\partial\psi}{\partial x}$$

$$\nabla^2(x\psi) = 2\,\nabla^2\left(\frac{\partial\psi}{\partial x}\right) = 2\frac{\partial}{\partial x}(\nabla^2\psi) = 0$$

Similarly $\nabla^2(y\psi) = 2\frac{\partial\psi}{\partial y}$

$$\nabla^4(y\psi) = 0$$

$$\nabla^2(x^2\psi) \rightarrow \nabla^4(y^2\psi) = 0$$

$$\nabla^4[(x^2+y^2)\psi] = 0$$

$$\nabla^4(r^2\psi) = 0$$

See Timoshenko p. 53

$$\phi = \sin(\alpha x)[C_1\cosh(\alpha y) + C_2\sinh(\alpha y) + C_3 y\cosh(\alpha y) + C_4\sinh(\alpha y) + \cdots]$$

General stress functions

$$\nabla^2\phi = 0$$

$$P = \nabla^2\phi = \sigma_{xx} + \sigma_{yy}$$

P has a conjugate Q

$$\frac{\partial P}{\partial x} = \frac{\partial Q}{\partial y} \quad \Big| \quad \frac{\partial P}{\partial y} = -\frac{\partial Q}{\partial x}$$

P+iQ is analytic

$f(z) = P + iQ$

Suppose

$$\int (P+iQ)\,dz = 4\psi(z)$$ integration also analytic

$$\psi(z) = p + iq = \frac{1}{4}\int f(z)\,dz$$

$$\psi' = \frac{1}{4}f(z)$$

$$\frac{\partial\psi}{\partial x} = \psi'(z)\frac{\partial z}{\partial x} = \frac{\partial p}{\partial x} + i\frac{\partial q}{\partial x}$$

$$\left.\begin{array}{l}\frac{\partial p}{\partial x} = \frac{P}{4}\\ \frac{\partial q}{\partial x} = \frac{Q}{4}\end{array}\right\}$$ p and q are conjugate

$$\frac{\partial p}{\partial x} = \frac{\partial q}{\partial y} = \frac{P}{4}$$

Aside:

$f(z) = P + iQ$

$$\frac{\partial f}{\partial x} = \frac{\partial P}{\partial x} + i\frac{\partial Q}{\partial x}$$

$$\frac{\partial f}{\partial (iy)} = \frac{1}{i}\left(\frac{\partial P}{\partial y} + i\frac{\partial Q}{\partial y}\right)\cdot\frac{i}{i}$$

$$= -i\frac{\partial P}{\partial y} + \frac{\partial Q}{\partial y}$$

SINCE $\frac{\partial f}{\partial x} = \frac{\partial f}{\partial iy}$

then $\frac{\partial P}{\partial x} = \frac{\partial Q}{\partial y} \;\Big|\; \frac{\partial P}{\partial y} = -\frac{\partial Q}{\partial x}$

consider

$\phi - xp - yq$

$\nabla^2(\phi - xp - yq) = P - 2\frac{\partial p}{\partial x} - 2\frac{\partial q}{\partial y} = P - 4\frac{\partial p}{\partial x} = P - 4\frac{\partial q}{\partial y}$

$= P - \frac{P}{2} - \frac{P}{2} = 0$

recall $\nabla^2(x\psi) = 2\frac{\partial \psi}{\partial x} + x\cancel{\nabla^2\psi}^{\,0}$

Let $p_1 = \phi - xp - yq$

$\phi = p_1 + xp + yq$

Recall: Analytic Functions $\frac{d}{dz} = \frac{d}{dx} = \frac{d}{d(iy)}$ in complex plane

harmonic functions: $\nabla^2 F = 0$

given $f(z) = \alpha + i\beta$

if $f(z)$ is analytic, α and β are harmonic

Cauchy-Riemann conditions

$\frac{\partial \alpha}{\partial x} = \frac{\partial \beta}{\partial y}$ / $\frac{\partial \beta}{\partial x} = -\frac{\partial \alpha}{\partial y}$

Stress function, ϕ

$\nabla^2\phi = \nabla^2 P = 0$

$\therefore$ P is harmonic

conjugate harmonic function, Q

$f(z) = P + iQ$

$Re[f(z)] = P = \sigma_x + \sigma_y$

$\nabla^2\phi - P = 0$

put in form $\nabla^2(\phi - x) = 0$

~~##~~ We did this by integrating

$f(z) \rightarrow \psi(z) = p + iq = \frac{1}{4}\int f(z)\,dz$

This leads to $\nabla^2(\underbrace{\phi - xp - yq}_{\text{harmonic}}) = 0$

now $\nabla^2(p_1) = 0$ where $p_1 = \phi - xp - yq$

$\phi = p_1 + xp + yq$ ← general expression for Airy stress function

$\phi = 2xp + p_2$

$\phi = 2yq + p_3$

we next introduce $\chi(z) = p_1 + iq_1$

~~W~~ We showed that

$$\phi = Re[\bar{z}\psi(z) + \chi(z)]$$

$$\phi = Re\left[\bar{z}z\frac{\psi(z)}{z} + \chi(z)\right]$$

$r^2 = (x+iy)(x-iy)$

$$\phi = Re[r^2 p_4 + p_5]$$

Displacement in terms of ϕ

Plane stress soln.

$$E\frac{\partial u}{\partial x} = \sigma_x - \nu\sigma_y$$

$$E\frac{\partial v}{\partial y} = \sigma_y - \nu\sigma_x$$

$$G\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) = \tau_{xy}$$

$$\sigma_x = \phi_{,yy}$$

$$\sigma_y = \phi_{,xx}$$

$$\tau_{xy} = -\phi_{,xy}$$

$$E\frac{\partial u}{\partial x} = \phi_{,yy} - \nu\phi_{,xx} = P$$

$$\cancel{E\,\partial} \quad E\frac{\partial u}{\partial x} = P - \phi_{,xx} - \nu\phi_{,xx} = P - (1+\nu)\phi_{,xx} \qquad \Big| \qquad E\frac{\partial v}{\partial y} = P - (1+\nu)\phi_{,yy}$$

use $P = 4\frac{\partial p}{\partial x} = 4\frac{\partial q}{\partial y}$

$$E\frac{\partial u}{\partial x} = 4\frac{\partial p}{\partial x} - (1+\nu)\frac{\partial^2\phi}{\partial x^2} \rightarrow \frac{E}{1+\nu}\frac{\partial u}{\partial x} = \frac{4}{1+\nu}\frac{\partial p}{\partial x} - \cancel{(1+\nu)}\frac{\partial^2\phi}{\partial x^2}$$

$$\frac{E}{1+\nu}\frac{\partial u}{\partial x} = 2G\frac{\partial u}{\partial x} = \frac{4}{1+\nu}\frac{\partial p}{\partial x} - \frac{\partial^2\phi}{\partial x^2}$$

$$2G\frac{\partial v}{\partial y} = \frac{4}{1+\nu}\frac{\partial q}{\partial y} - \frac{\partial^2\phi}{\partial y^2}$$

$$Gu = \frac{2}{1+\nu}p - \frac{1}{2}\frac{\partial\phi}{\partial x} + f(y)$$

$$Gv = \frac{2}{1+\nu}q - \frac{1}{2}\frac{\partial\phi}{\partial y} + f_1(x)$$

$$G\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) = \frac{2}{1+\nu}\frac{\partial p}{\partial y} - \frac{1}{2}\frac{\partial^2\phi}{\partial x\partial y} + \frac{df}{dy} + \frac{2}{1+\nu}\frac{\partial q}{\partial x} - \frac{1}{2}\frac{\partial^2\phi}{\partial x\partial y} + \frac{df_1}{dx} = \tau_{xy}$$

$$\tau_{xy} = -\frac{\partial^2\phi}{\partial x\partial y} = -\frac{\partial^2\phi}{\partial x\partial y} + \frac{2}{1+\nu}\underbrace{\left(\frac{\partial p}{\partial y} + \frac{\partial q}{\partial x}\right)}_{} + \frac{df}{dy} + \frac{df_1}{dx}$$

$$0 = \frac{df_1}{\partial x} + \frac{df}{dy}$$

0 from Cauchy-Riemann $\frac{\partial p}{\partial y} = -\frac{\partial q}{\partial x}$

$f(y)$ and $f_1(x)$ give rigid body motion
let them equal zero

Once ϕ has been found

1) $P = \nabla^2\phi = \sigma_x + \sigma_y$
2) Find Q using $\frac{\partial P}{\partial x} = \frac{\partial Q}{\partial y}$, $\frac{\partial P}{\partial y} = -\frac{\partial Q}{\partial x}$
3) Integrate $f(z) = P + iQ$ to get $p + iq = \frac{1}{2}\int f(z)\,dz$
4) Determine u, v for plane stress

Stress and displacements in terms of complex potentials $\psi(z)$ and $\chi(z)$

$$\phi = Re\left[\bar{z}\psi(z) + \chi(z)\right]$$

complex conjugate $\rightarrow$ $f(z) = \alpha + i\beta$

$$\bar{f}(\bar{z}) = \alpha - i\beta$$

ex.: $f(z) = e^{inz} = e^{in(x+iy)} = e^{inx}e^{-ny}$

$$= (\cos(nx) + i\sin(nx))e^{-ny}$$

$$\bar{f}(\bar{z}) = e^{in\bar{z}} = e^{-in(x-iy)}$$

$$= (\cos(nx) - i\sin(nx))e^{-ny}$$

$$f(z) + \bar{f}(\bar{z}) = 2Re[f(z)]$$

$$f(z) - \bar{f}(\bar{z}) = 2iIm[f(z)]$$

this allows

$$2\phi = \bar{z}\psi(z) + \chi(z) + z\bar{\psi}(\bar{z}) + \bar{\chi}(\bar{z})$$

In the expressions for u, v we need $\phi_{,x}$ and $\phi_{,y}$

$z = x + iy$	$\bar{z} = x - iy$
$\frac{\partial z}{\partial x} = 1 \quad \frac{\partial z}{\partial y} = i$	$\frac{\partial \bar{z}}{\partial x} = 1 \quad \frac{\partial \bar{z}}{\partial y} = -i$
$\frac{\partial}{\partial x} = \frac{\partial}{\partial z}\left(\frac{\partial z}{\partial x}\right)$	$\frac{\partial}{\partial \bar{z}}\left(\frac{\partial \bar{z}}{\partial x}\right)$
$\frac{\partial}{\partial y} = \frac{\partial}{\partial z}\frac{\partial z}{\partial y}$	$\frac{\partial}{\partial \bar{z}}\left(\frac{\partial \bar{z}}{\partial y}\right)$

$$2\phi_{,x} = \bar{z}\psi'(z) + \psi(z) + \chi'(z) + z\bar{\psi}'(\bar{z}) + \bar{\psi}(\bar{z}) + \bar{\chi}'(\bar{z})$$

$$i\left[2\phi_{,y} = i\left[\bar{z}\psi'(z) - \psi(z) + \chi'(z) - z\bar{\psi}'(\bar{z}) + \bar{\psi}(\bar{z}) - \bar{\chi}'(\bar{z})\right]\right]$$

add above eqns.

$$\cancel{2}(\phi_{,x} + i\phi_{,y}) = \cancel{2}\left[\psi(z) + z\bar{\psi}'(\bar{z}) + \bar{\chi}'(\bar{z})\right]$$

from earlier,

$$2G(u+iv) = -\left(\frac{\partial\phi}{\partial x} + i\frac{\partial\phi}{\partial y}\right) + \frac{4}{1+\nu}(p+iq)$$

SINCE $\psi = p + iq$

$$2G(u+iv) = -\psi(z) - z\bar{\psi}'(\bar{z}) - \bar{\chi}'(\bar{z}) + \frac{4}{1+\nu}\psi(z)$$

$$= \frac{3-\nu}{1+\nu}\psi(z) - z\bar{\psi}'(\bar{z}) - \bar{\chi}'(\bar{z})$$

When ψ, χ are known; u, v can be found

For plane strain

substitute $\nu \to \frac{\nu}{1-\nu}$

To obtain stress in terms of ψ and χ start with

$$\phi_{,x} + i\phi_{,y} = \psi(z) + z\bar{\psi}'(\bar{z}) + \bar{\chi}'(\bar{z})$$

take $\frac{\partial}{\partial x}$: $\phi_{,xx} + i\phi_{,xy} = \psi'(z) + z\bar{\psi}''(\bar{z}) + \bar{\psi}'(\bar{z}) + \bar{\chi}''(\bar{z})$

$i\frac{\partial}{\partial y}$: $i\phi_{,xy} - \phi_{,yy} = -\psi'(z) + z\bar{\psi}''(\bar{z}) - \bar{\psi}'(\bar{z}) + \bar{\chi}''(\bar{z})$

subtract

$$\phi_{,xx} + \phi_{,yy} = 2\psi'(z) + 2\bar{\psi}'(\bar{z}) = 4\,\mathrm{Re}\left[\psi'(z)\right]$$

add

$$\sigma_{xx} - \sigma_{yy} - 2i\tau_{xy} = 2\left[z\bar{\psi}''(\bar{z}) + \bar{\chi}''(\bar{z})\right]$$

or take the conjugate

$$\sigma_{yy} - \sigma_{xx} + 2i\tau_{xy} = 2\left[\bar{z}\psi''(z) + \chi''(z)\right]$$