Governing Eqns.

- Equilibrium
- Compatibility
- $\underline{\underline{\epsilon}}-\underline{u}$ relations
- $\underline{\underline{\sigma}}-\underline{\underline{\epsilon}}$ relations
- B.C.'s

2D problems

plane stress ($\sigma_{zz}=0$)

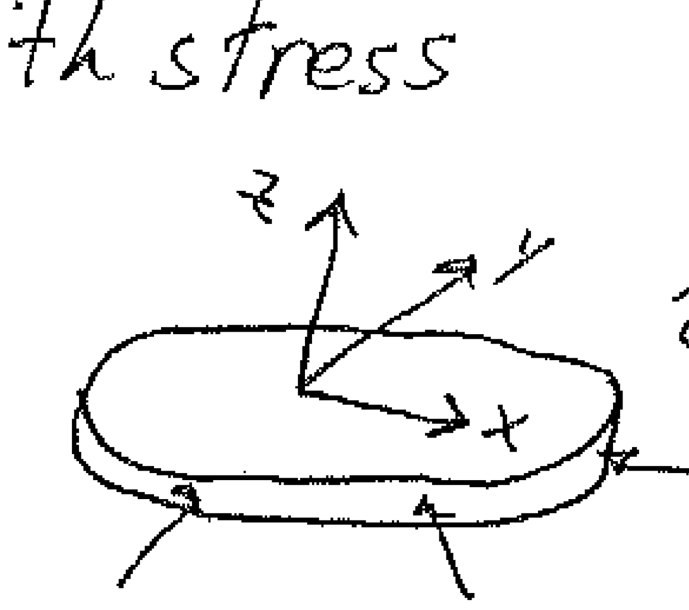
plane strain ($\epsilon_{zz}=0$)

Compatibility

$$\epsilon_{xx,yy}+\epsilon_{yy,xx}=2\epsilon_{xy,xy}$$

use Hooke's law to replace with stress

$$\epsilon_{ij}=\frac{1+\nu}{E}\sigma_{ij}-\frac{\nu}{E}\sigma_{kk}\delta_{ij}$$



$\tau_{xz}=\tau_{yz}=0$

for plane stress,

$$\epsilon_{xx}=\frac{1+\nu}{E}\sigma_{xx}-\frac{\nu}{E}(\sigma_{xx}+\sigma_{yy}+\cancel{\sigma_{zz}}^{\,0})$$

$$\left.\begin{aligned}\epsilon_{xx}&=\frac{\sigma_{xx}}{E}-\frac{\nu}{E}\sigma_{yy}\\ \epsilon_{yy}&=\frac{\sigma_{yy}}{E}-\frac{\nu}{E}\sigma_{xx}\end{aligned}\right\}\text{ sub into compatibility}$$

$$\epsilon_{xy}=\frac{1+\nu}{E}\sigma_{xy}$$

$$\frac{\sigma_{xx,yy}}{\cancel{E}}-\frac{\nu}{\cancel{E}}\sigma_{yy,yy}+\frac{\sigma_{yy,xx}}{\cancel{E}}-\frac{\nu}{\cancel{E}}\sigma_{xx,xx}=2\frac{1+\nu}{\cancel{E}}\sigma_{xy,xy}$$

$$\bigstar\ \sigma_{xx,yy}+\sigma_{yy,xx}-\nu(\sigma_{yy,yy}+\sigma_{xx,xx})=2(1+\nu)\sigma_{xy,xy}$$

recall equilibrium: $\sigma_{ji,j}=0$

$$\sigma_{xx,x}+\sigma_{yx,y}+\cancel{\sigma_{zx,x}}^{\,0}=0$$

$$-\sigma_{xx,xx}=\sigma_{yx,yx}$$

from 2nd equil. eqn.

$$\sigma_{xy,x}+\sigma_{yy,y}=0$$

$$-\sigma_{xy,xy}=\sigma_{yy,yy}$$

$$\underbrace{-\sigma_{xx,xx}-\sigma_{yy,yy}=2\sigma_{xy,xy}}_{\text{sub into}}$$

this gives,

$$\sigma_{xx,xx}+\sigma_{yy,yy}+\sigma_{xx,yy}+\sigma_{yy,xx}=0$$

$$\boxed{\nabla^2(\sigma_{xx}+\sigma_{yy})=0}\ \text{(same for plane strain)}$$

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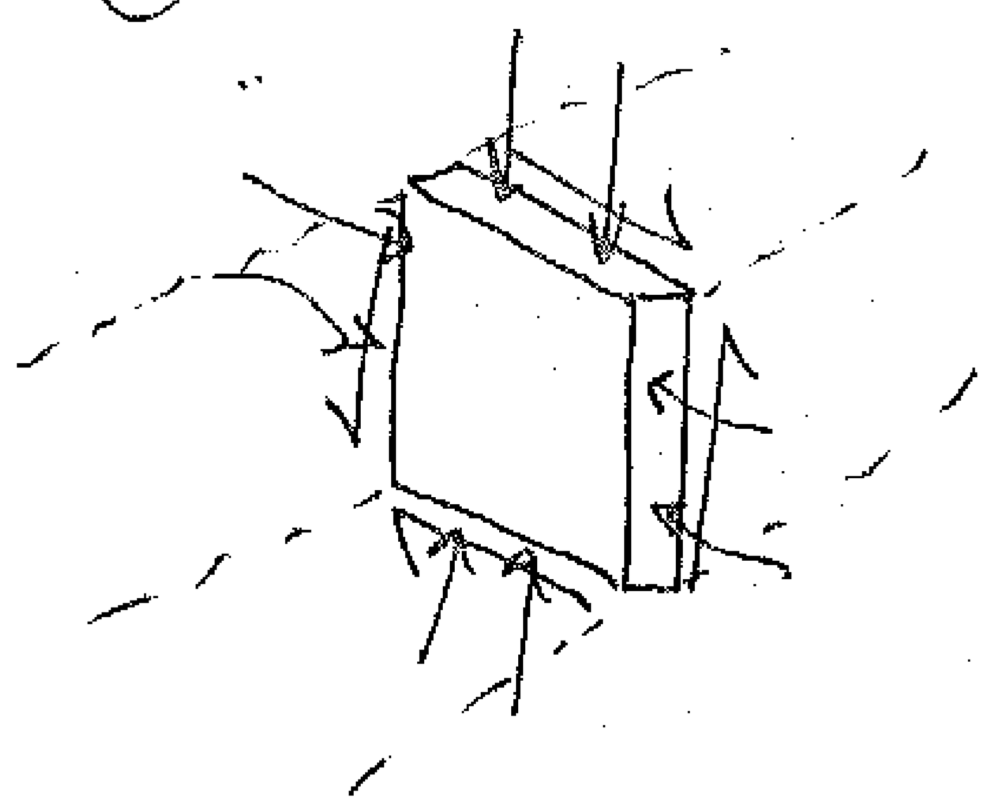


$\sigma_{xx}(x,y)$
$\sigma_{yy}(x,y)$
$\sigma_{xy}(x,y)$

plane stress analysis doesn't depend on whether problem is plane stress or plane strain (in-plane stress) but σ_{zz} is different

ϵ_{ij} are different for 2 cases

u_i are also different

1) find $\underline{\underline{\sigma}}(x,y)$ that matches B.C. (harmonic functions)
2) Find the $\underline{\underline{\epsilon}}(x,y)$ field by putting $\underline{\underline{\sigma}}$ into Hooke's Law
3) Integrate $\underline{\underline{\epsilon}}$ to get $\underline{u}$

Airy introduced a scalar potential function ϕ

$$\left.\begin{aligned}\sigma_{xx} &= \phi_{,yy}\\ \sigma_{yy} &= \phi_{,xx}\\ \sigma_{xy} &= -\phi_{,xy}\end{aligned}\right\}$$

similar to compatibility eqn.
a more general 3D form will be given later

$$\nabla^2(\sigma_{xx}+\sigma_{yy}) = 0$$

$$\vec{\nabla} = \frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j}$$

$$\nabla^2(\sigma_{xx}+\sigma_{yy}) = \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j}\right)\cdot\left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j}\right)(\sigma_{xx}+\sigma_{yy})$$

$$= \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)(\sigma_{xx}+\sigma_{yy})$$

$$\sigma_{xx}+\sigma_{yy} = \underbrace{\phi_{,yy}+\phi_{,xx}} \rightarrow \nabla^2(\sigma_{xx}+\sigma_{yy}) = \nabla^2(\nabla^2\phi)$$

$$\rightarrow \nabla^2\phi = \sigma_{xx}+\sigma_{yy}$$

$$\boxed{\nabla^4\phi = 0}$$ Biharmonic equation

"What can we do with $\nabla^4\phi = 0$?

Semi-inverse method

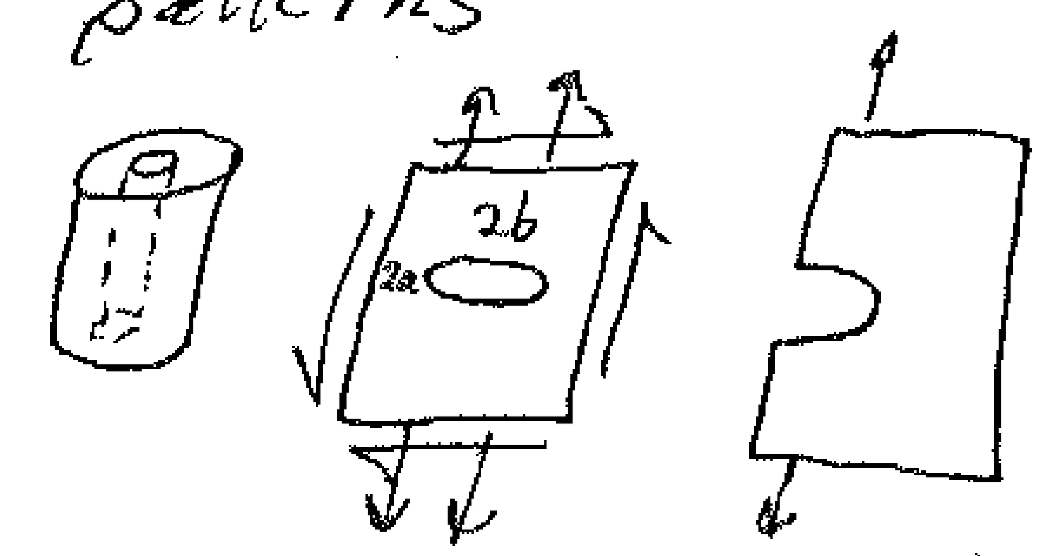
Guess what ϕ is.

$$(\vec{\nabla}\cdot\vec{\nabla})(\vec{\nabla}\cdot\vec{\nabla})\phi = 0$$

__Cartesian 2D__

$$\vec{\nabla} = \frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j}$$

$\left.\begin{array}{l}\text{cylindrical}\\ \text{spherical}\\ \text{elliptical}\\ \text{hyperbolic}\end{array}\right\}$ other coordinate systems ← used to model shapes and loading patterns



hyperbolic shrink to get edge crack

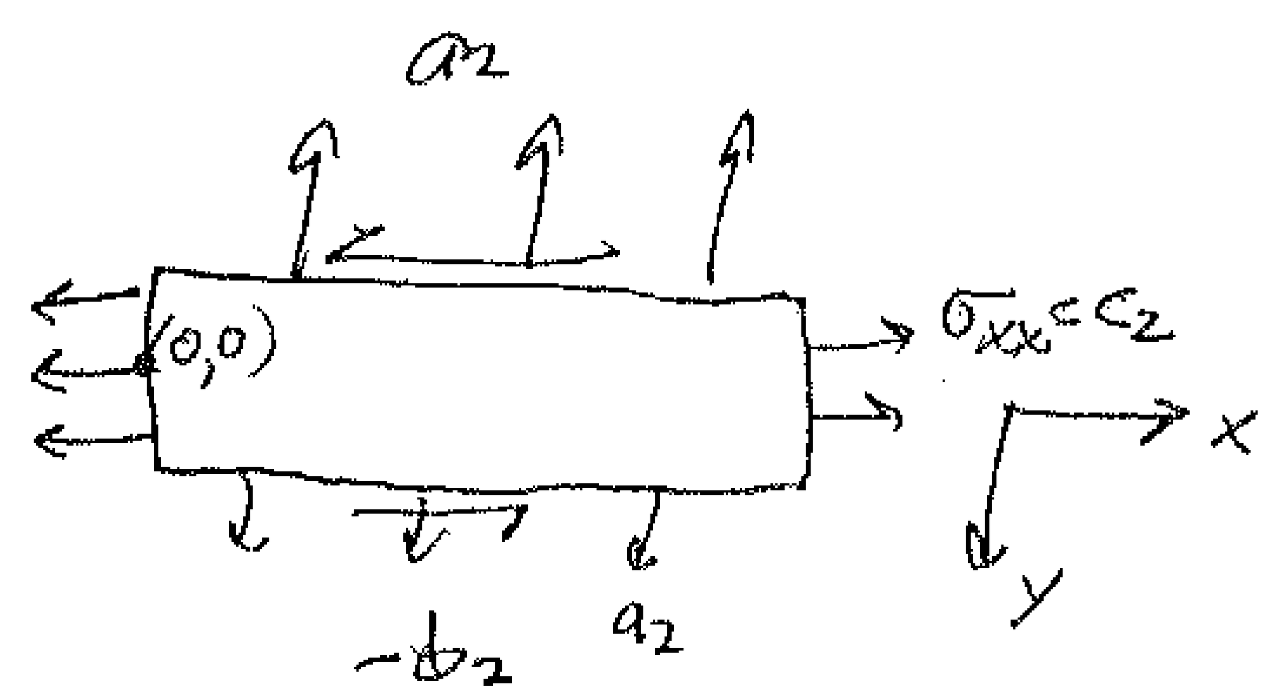
$$\phi^{(2)} = \frac{a_2}{2}x^2 + b_2 xy + \frac{c_2}{2}y^2$$

$$\nabla^4\phi = \phi_{,xxxx} + 2\phi_{,xxyy} + \phi_{,yyyy} = 0$$

$$\sigma_{xx} = \phi^{(2)}_{,yy} = c_2$$

$$\sigma_{yy} = \phi^{(2)}_{,xx} = a_2$$

$$\sigma_{xy} = -\phi^{(2)}_{,xy} = -b_2$$



if plate is thin (pl. σ ok)
if thick variation in stress through thickness (no plane stress)

take $c_2, a_2 \neq 0$

use Hookes Law (pl. σ or pl. ε)
to get $\underline{\underline{\epsilon}}$
integrata $\underline{\underline{\epsilon}}$ to get $\underline{u}$
set rigid body rotation + displacement
to zero

3rd order polynomial

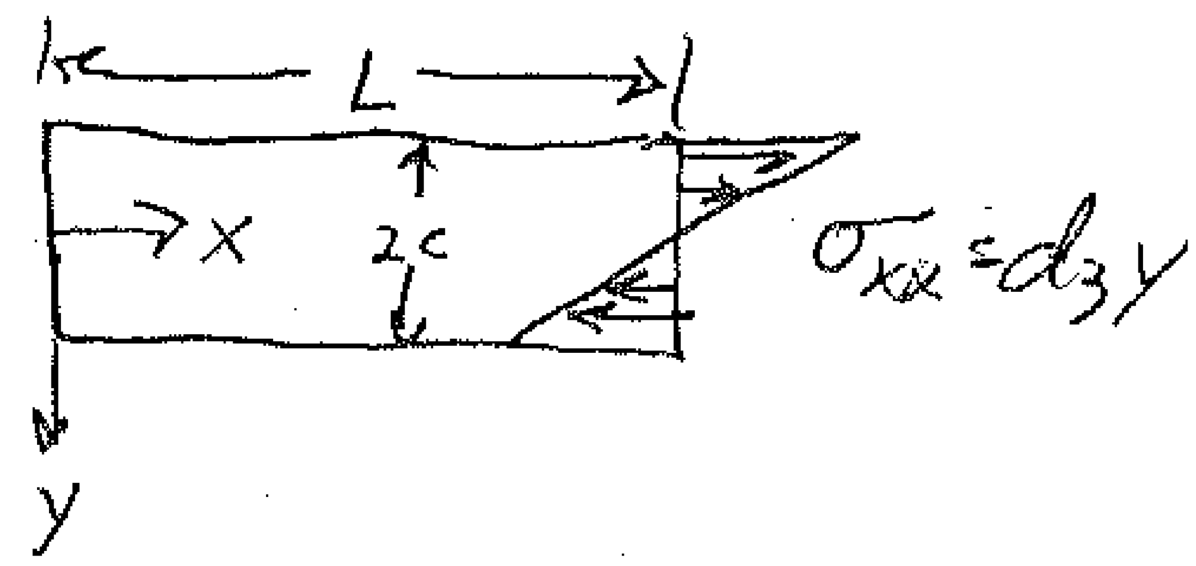
$$\phi^{(3)} = \frac{a_3 x^3}{3(2)} + \frac{b_3 x^2 y}{2} + \frac{c_3 x y^2}{2} + \frac{d_3 y^3}{3(2)}$$

$$\sigma_{xx} = \phi^{(3)}_{,yy} = c_3 x + d_3 y$$

$$\sigma_{yy} = \phi^{(3)}_{,xx} = a_3 x + b_3 y$$

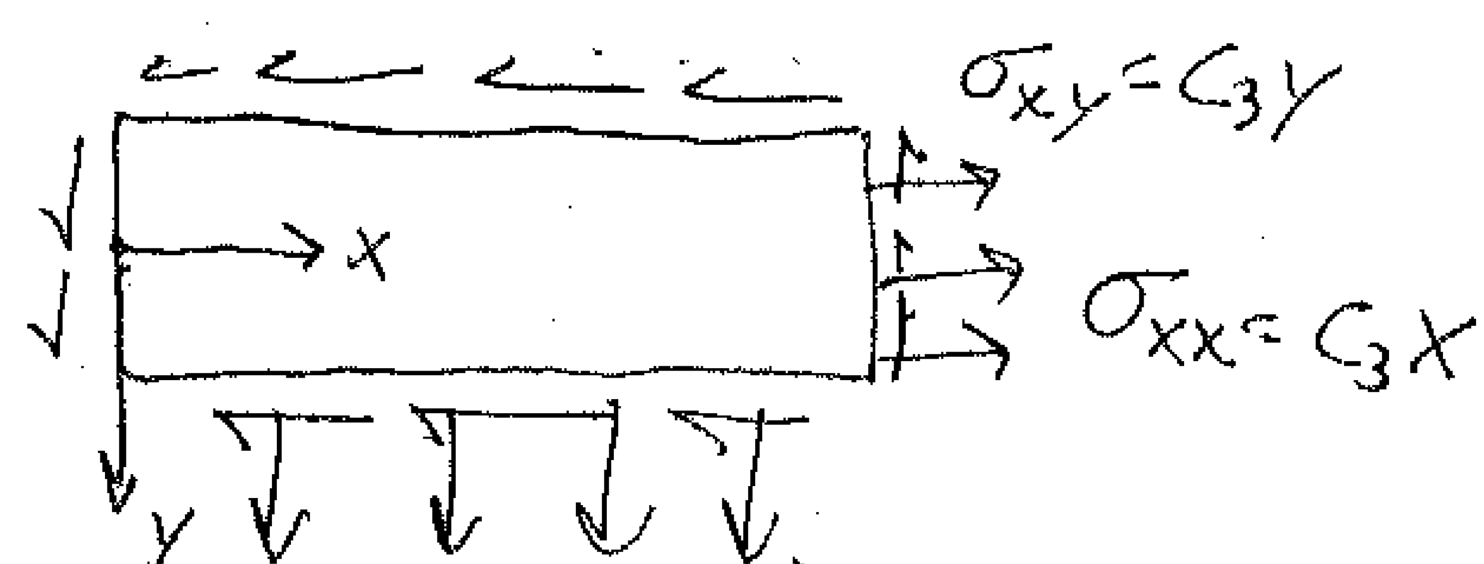
$$\sigma_{xy} = -\phi^{(3)}_{,xy} = -(b_3 x + c_3 y)$$

$d_3 \neq 0$, others $= 0$



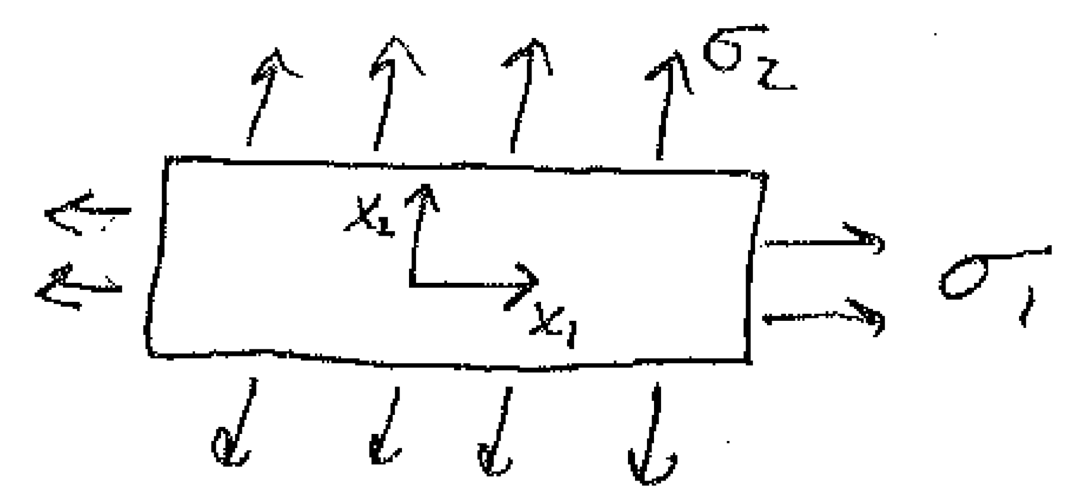
beam example

let $c_3 \neq 0$



can be added from different Airy stress function

Superposition example



$$\phi^{(2)} = \frac{a_2}{2}x^2 + b_2 xy + \frac{c_2}{2}y^2$$

$$\sigma_{11} = \phi^{(2)}_{,22} = a_2$$

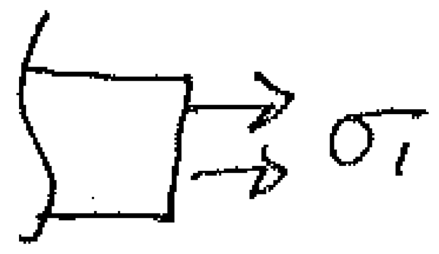
$$\sigma_{22} = \phi^{(2)}_{,11} = c_2$$

$$\sigma_{12} = -\phi^{(2)}_{,12} = -b_2$$

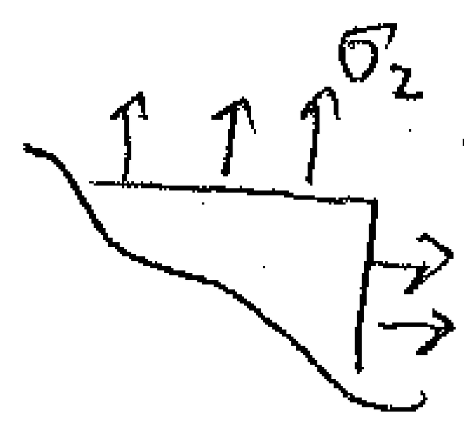
apply boundary conditions

$$t_1 = \sigma_{11} n_1 + \sigma_{21} n_2$$

$$t_2 = \sigma_{12} n_1 + \sigma_{22} n_2$$

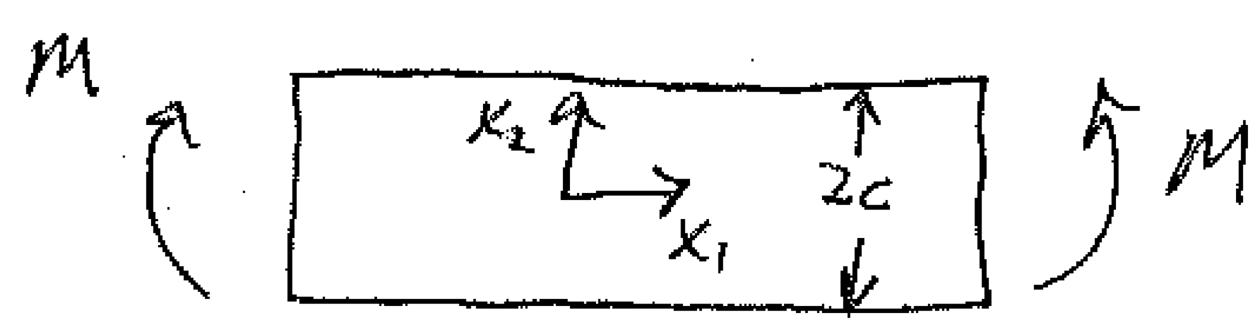


$$t_1 = \sigma_{11} n_1 + \sigma_{21} \cancel{n_2}^{\,0} \rightarrow \sigma_{11} = a_2$$



$$t_2 = \sigma_{12} \cancel{n_1}^{\,0} + \sigma_{22} n_2 \rightarrow \sigma_{22} = c_2$$

$$-b_2 = 0$$



$$\phi^{(3)} = \frac{a_3 x^3}{6} + \frac{b_3}{2}x^2 y + \frac{c_3}{2}xy^2 + \frac{d_3}{6}y^3$$

$$\sigma_{22} = a_3 x + b_3 y$$

$$\sigma_{11} = c_3 x + d_3 y$$

$$\sigma_{12} = -(b_3 x + c_3 y)$$

B.C.

$$\sigma_{12} = 0 = b_3 x + c_3 y \qquad y = \pm c$$

$$M = \int \sigma_{11} x_2 \, dA$$

$$= \int \sigma_{11} x_2 \, dx_2 \, \cancel{dx_3}^{\,1} = \int_{-c}^{c} (c_3 x_1 + d_3 x_2) x_2 \, dx_2$$

$$= \int_{-c}^{+c} (c_3 x_2 x_1 + d_3 x_2^2) dx_2 = \frac{c_3 x_1 x_2^2}{2} + \frac{d_3}{3} x_2^3 \Big|_{-c}^{c}$$

$$= \left(\frac{c_3}{2} x_1 c^2 + \frac{d_3}{3} c^3\right) - \left(\frac{c_3 x_1 c^2}{2} - \frac{d_3}{3} c^3\right)$$

$$M = \frac{2 d_3}{3} c^3$$

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$$\boxed{\frac{3M}{2c^3} = d_3}$$

$$\Sigma F = 0$$

$$\int_A \sigma_{11}\, dA = 0 = \int_{-c}^{+c} (c_3 x_1 + d_3 x_2)\, dx_2 = c_3 x_1 x_2 + \frac{d_3}{2} x_2^2 \Big|_{-c}^{c}$$

$$= c_3 x_1 c + \frac{d_3}{2} c^2 - \left(-c_3 c x_1 + \frac{d_3}{2} c^2\right)$$

$$2 c_3 c x_1 = 0$$

$$c_3 = 0$$

$$\sigma_{12} = 0 = b_3 x_1 + \cancel{c_3}^{\,0} x_2 \quad @ \; y = \pm c$$

$$b_3 x = 0$$

$$b_3 = 0$$

$$\boxed{\begin{array}{l} \sigma_{11} = \sigma_{22} = 0 \\ \sigma_{11} = \frac{3M}{2c^3} x_2 \\ \sigma_{22} = a_3 x_1 = 0 \rightarrow a_3 = 0 \end{array}}$$

Superposition gives:

$$\phi = \phi^{(2)} + \phi^{(3)} = \left(\frac{a_2}{2} x_1^2 + b_2 x_1 x_2 + \frac{c_2}{2} x_2^2\right) + \left(\frac{a_3}{6} x_1^3 + \frac{b_3}{2} x_1^2 x_2 + \frac{c_3}{2} x_1 x_2^2 + \frac{d_3}{6} x_2^3\right)$$

$$\sigma_{11} = \sigma_{xx} = \phi_{,22} = c_2 + \cancel{c_3}^{\,0} x_1 + d_3 x_2$$

$$\sigma_{22} = \sigma_{yy} = \phi_{,11} = a_2 + \cancel{a_3}^{\,0} x_1 + \cancel{b_3}^{\,0} x_1$$

$$\sigma_{12} = \sigma_{xy} = -\phi_{,xy} = \cancel{b_2}^{\,0} + \cancel{b_3}^{\,0} x_1 + \cancel{c_3}^{\,0} x_2$$

$$a_3 = b_2 = c_3 = b_3 = 0$$

$$\boxed{\begin{array}{l} \sigma_{yy} = a_2 \\ \sigma_{xx} = c_2 + d_3 x_2 = c_2 + \frac{3M}{2c^3} x_2 \\ \sigma_{xy} = 0 \end{array}}$$

Summary of Governing Eqns.

(w/ body forces and thermal expansion)

① Equilibrium

$$\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} + F_x = 0$$

$$\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \tau_{yz}}{\partial z} + F_y = 0$$

$$\frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \sigma_{zz}}{\partial z} + F_z = 0$$

General form

$\sigma_{ji,j} + \rho b_i = \rho \ddot{u}_i = \rho a_i$

② compatibility

$$\frac{\partial^2 \epsilon_{xx}}{\partial y^2} + \frac{\partial^2 \epsilon_{yy}}{\partial x^2} = \frac{\partial^2 \gamma_{xy}}{\partial x \partial y}$$

③ Hooke's Law

$$\epsilon_{xx} = \frac{1}{E}\left[\sigma_x - \nu(\sigma_{yy} + \sigma_{zz})\right] + kT$$

$\vdots$

sub ③ into ②

$$\frac{\partial^2}{\partial y^2}\left(\sigma_{xx} - \nu\sigma_{yy} + EkT\right) + \frac{\partial^2}{\partial x^2}\left(\sigma_{yy} - \nu\sigma_{xx} + EkT\right) = 2(1+\nu)\frac{\partial^2 \tau_{xy}}{\partial x \partial y}$$

assume plane σ

$\sigma_{zz} = \tau_{zx} = \tau_{zy} = 0$

$$\left.\begin{aligned} \frac{\partial^2 \sigma_{xx}}{\partial x^2} + \frac{\partial^2 \sigma_{yx}}{\partial x \partial y} + \frac{\partial F_x}{\partial x} &= 0 \\ \frac{\partial^2 \sigma_{yy}}{\partial y^2} + \frac{\partial^2 \sigma_{yx}}{\partial y \partial x} + \frac{\partial F_y}{\partial y} &= 0 \end{aligned}\right\}$$

add and sub. into solving for $\frac{\partial^2 \tau_{xy}}{\partial x \partial y}$

Plane stress

$$-(1+\nu)\left[\frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y}\right] = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\left(\sigma_{xx} + \sigma_{yy} + EkT\right) = \nabla^2\left(\sigma_{xx} + \sigma_{yy} + EkT\right)$$

Plane strain

$\sigma_z = \nu(\sigma_{xx} + \sigma_{yy}) \neq 0$

$\epsilon_z = 0$

$$\nabla^2\left(\sigma_{xx} + \sigma_{yy} + EkT\right) = \frac{-1}{1-\nu}\left(\frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y}\right)$$

p.281
Beltrami-Mitchell compatibility

Generalized Plane Stress p.340 (Boresi + Chong)

Assumptions: 1) Upper + lower surfaces free of external forces, $\sigma_{zz} = \sigma_{zx} = \sigma_{zy} = 0$

2) displacement in z small (thin plate)

3) variations in u, v small through thickness

Average displacements applied:

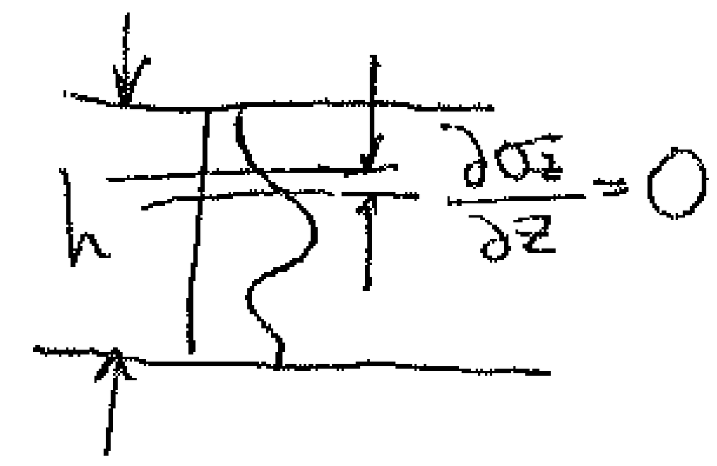
$$\bar{u}(x,y) = \frac{1}{h}\int_{-\frac{h}{2}}^{\frac{h}{2}} u(x,y,z)\,dz$$

$\bar{\underline{\underline{\sigma}}}$ and $\bar{\underline{\underline{\epsilon}}}$ also treated as mean value

From equilibrium eqns. in z-direction

SINCE $\tau_{xz} = \tau_{yz} = 0$ where $z = \pm \frac{h}{2}$

then $\dfrac{\partial \sigma_{zz}}{\partial z} = 0$



Equilibrium is then,

$$\frac{\partial \bar{\sigma}_x}{\partial x} + \frac{\partial \bar{\tau}_{xy}}{\partial y} + \bar{X} = 0$$

$$\frac{\partial \bar{\tau}_{xy}}{\partial x} + \frac{\partial \bar{\sigma}_y}{\partial y} + \bar{Y} = 0$$

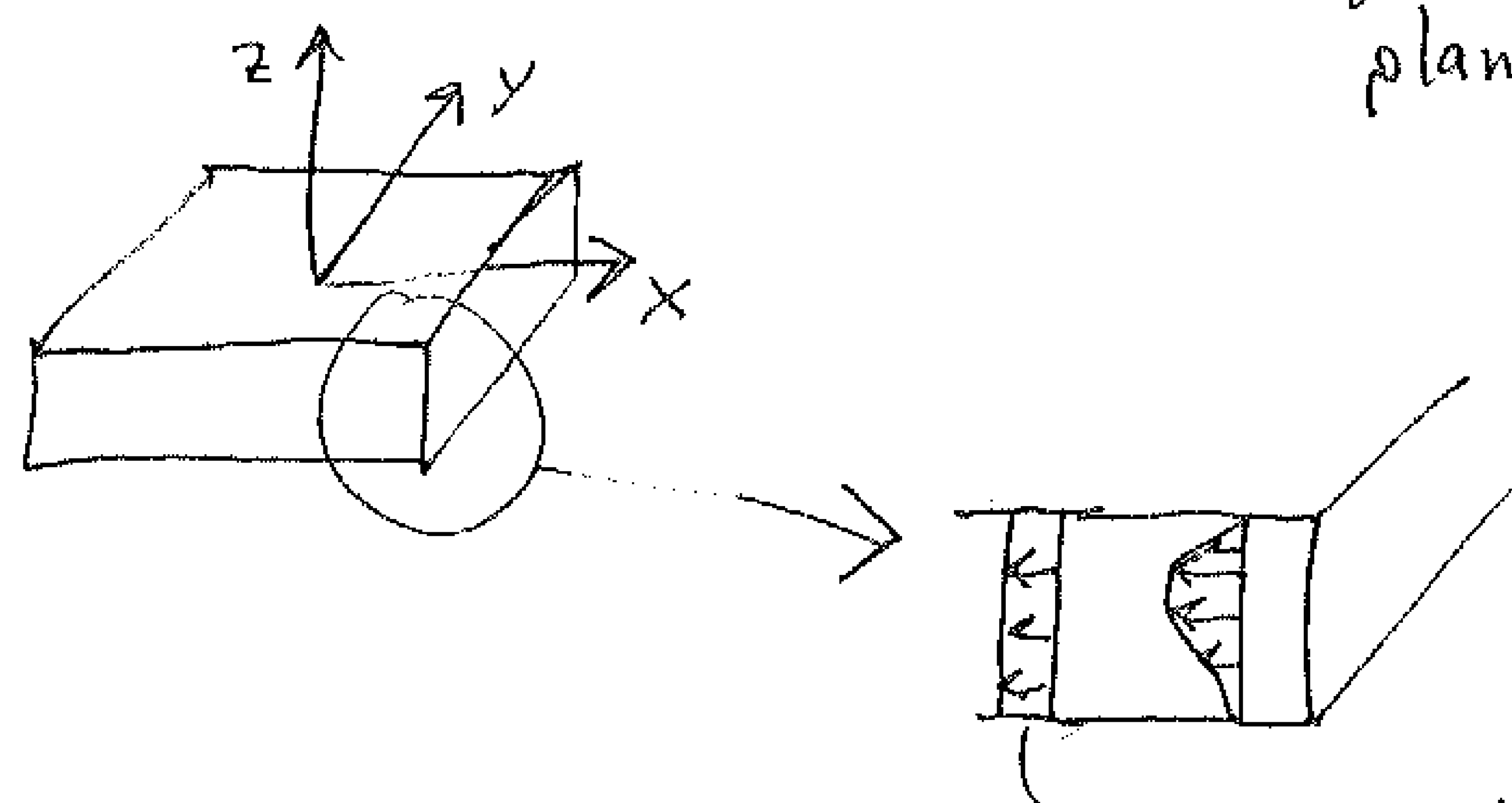
Hooke's Law

$$\bar{\sigma}_x = \bar{\lambda}(\bar{\epsilon}_x + \bar{\epsilon}_y) + 2G\,\bar{\epsilon}_x$$

$$\bar{\sigma}_y = \ldots$$

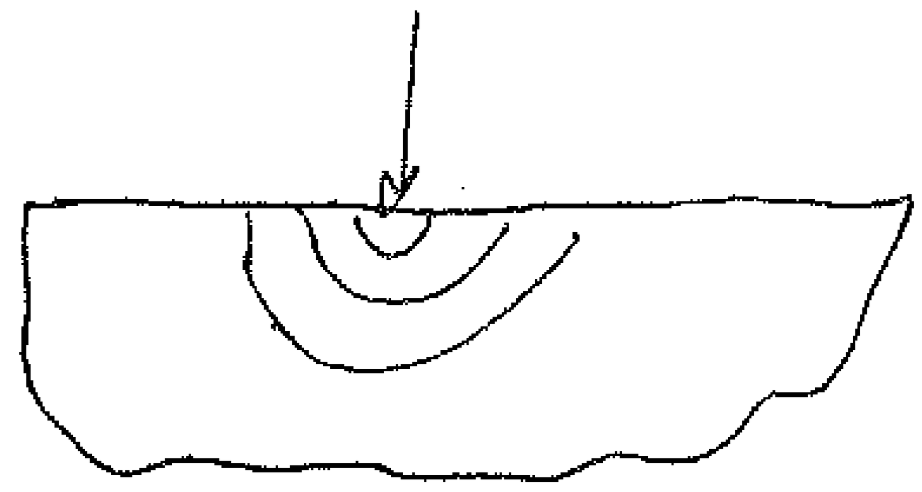
$$\bar{\tau}_{xy} = G\left(\frac{\partial \bar{u}}{\partial y} + \frac{\partial \bar{v}}{\partial x}\right)$$

$\bar{\lambda} = \lambda$ for generalized plane stress and plane strain

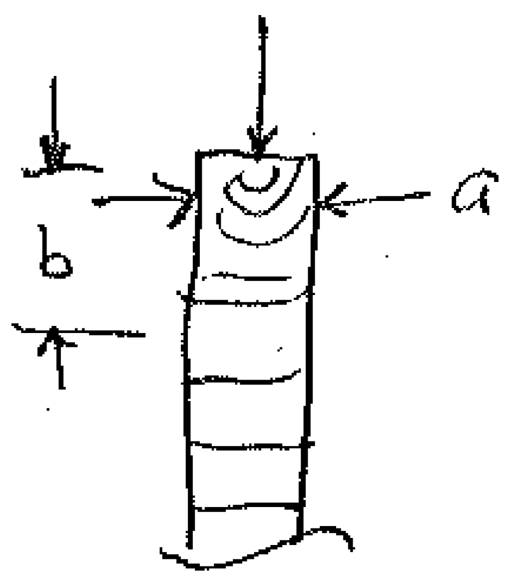


average, independent of z

End effects and St. Venant's Principle



Semi-infinite



$\frac{b}{a} \geq 5$ for uniform stress over cross section

For the same boundary loads, the stress distribution is the same for plane σ and plane ϵ, but displacements are different

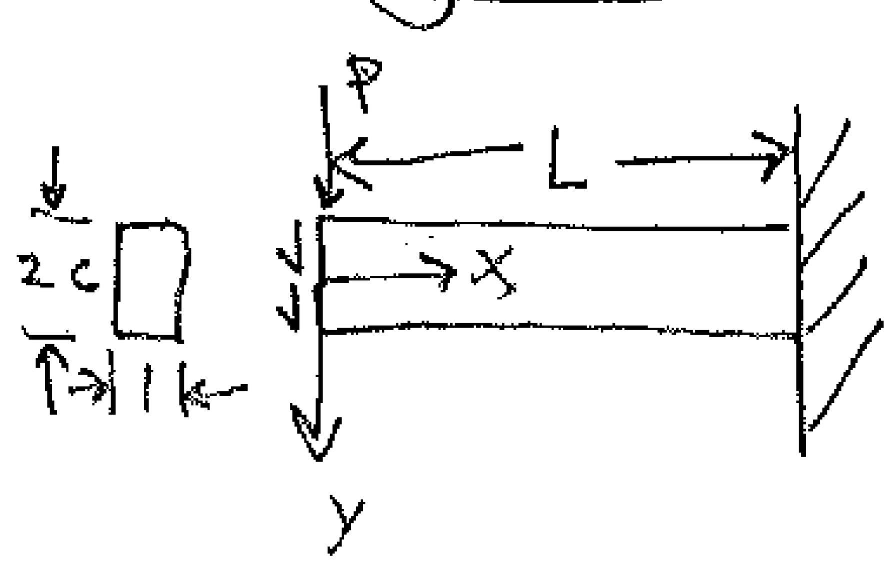
Plane stress ($\sigma_{zz}=0$)

$$\epsilon_{xx} = \frac{1}{E}(\sigma_{xx} - \nu\sigma_{yy})$$
$$\epsilon_{yy} = \frac{1}{E}(\sigma_{yy} - \nu\sigma_{xx})$$
$$\gamma_{xy} = 2\epsilon_{xy} = \frac{1}{G}\tau_{xy}$$

Plane strain ($\epsilon_{zz}=0$)

$$E \rightarrow \frac{E}{1-\nu^2}$$
$$\nu \rightarrow \frac{\nu}{1-\nu}$$

Bending of a beam



superposition

from $\phi^{(2)}$, $\tau_{xy} = -b_2$

stress from $\phi^{(4)}$

Given by $\sigma_x = d_4 xy$

$\sigma_y = 0$

$\tau_{xy} = -\frac{d_4}{2}y^2 - b_2 \rightarrow 0 \text{ @ } \pm c$

relate P to constants

@ $x=0$

b/c neg. face, +dir.

$$-\int_{-c}^{c}\int_{0}^{1} \tau_{xy}\, dy\, dz = P$$

$$-\int_{-c}^{c}\left(-\frac{d_4}{2}y^2-b_2\right)dy=P$$

$$-\left(-\frac{d_4}{6}y^3-b_2y\right)\Big|_{-c}^{c}=P$$

$$-\left[\left(\frac{d_4}{6}c^3-b_2c\right)-\left(-\frac{d_4}{6}c^3+b_2c\right)\right]=P \qquad (1)$$

also from B.C.

$$\frac{d_4}{2}c^2-b_2=0$$

$$\frac{d_4}{2}c^2=b_2$$

$$d_4=\frac{2b_2}{c^2}$$

sub back into (1)

$$\boxed{\begin{aligned} b_2 &= \frac{3P}{4c} \\ d_4 &= \frac{3}{2}\frac{P}{c^3} \end{aligned}}$$

$$\sigma_x=\frac{3}{2}\frac{P}{c^3}xy$$

$$\sigma_{yy}=0$$

$$\sigma_{xy}=\frac{3P}{4c^3}y^2-\frac{3P}{4c}=-\frac{3P}{4c}\left(1-\frac{y^2}{c^2}\right)$$

$$I=\frac{1}{12}bh^3=\frac{1}{12}\cdot 1(2c)^3=\frac{2c^3}{3}$$

$$\boxed{\begin{aligned} \sigma_{xy} &= -\frac{P}{2I}(c^2-y^2) \\ \sigma_{xx} &= -\frac{P}{I}xy \end{aligned}}$$

Find u, v for plane σ

$$\epsilon_{xx} = \frac{\sigma_{xx}}{E} = \frac{1}{E}\left(\frac{-P}{I}xy\right) = \frac{\partial u}{\partial x}$$

$$\epsilon_{yy} = -\frac{\nu \sigma_{xx}}{E} = \frac{\nu}{E}\left(\frac{P}{I}xy\right) = \frac{\partial v}{\partial y}$$

$$\gamma_{xy} = \frac{\tau_{xy}}{G} = -\frac{1}{G}\left(\frac{P}{2I}(c^2-y^2)\right) = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}$$

$$u = -\frac{P}{EI}\frac{x^2}{2}y + f_1(y)$$

$$v = \frac{\nu}{E}\frac{P}{I}\frac{xy^2}{2} + f_2(x)$$

$$\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = -\frac{Px^2}{2EI} + \frac{\partial f_1}{\partial y} + \frac{\nu P y^2}{2EI} + \frac{\partial f_2}{\partial x} = -\frac{P}{2GI}(c^2-y^2)$$

$$F(x) + G(y) = K$$

F and G are constant

$$-\frac{Px^2}{2EI} + \frac{\partial f_2}{\partial x} = F$$

$$-\frac{Py^2}{2GI} + \frac{\nu P y^2}{2EI} + \frac{\partial f_1}{\partial y} = G$$

$$f_2 = Fx + \frac{Px^3}{6EI} + C_1$$

$$f_1 = Gy - \frac{\nu P y^3}{6EI} + \frac{Py^3}{6GI} + C_2$$

$$u = -\frac{Px^2y}{2EI} - \frac{\nu P y^3}{6EI} + \frac{Py^3}{6GI} + Gy + C_2$$

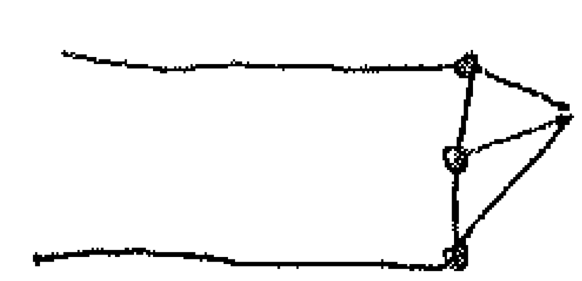
$$v = \frac{\nu P}{2EI}xy^2 + \frac{Px^3}{6EI} + Fx + C_1$$

apply B.C.'s

@ $x = L$, $y = 0$ $\quad u, v = 0$

$$C_2 = 0$$

$$c_1 = -\frac{Px^3}{6EI} - FL$$



B.C.'s at these points give different results

also, $\frac{dv}{dx} = 0$ @ $x = L$, $y = 0$ gives $F = \frac{-PL^2}{2EI}$

$$G = \frac{-Pc^2}{2GI} + \frac{PL^2}{2EI}$$