Theory of Anisotropic Elasticity

recall the linear anisotropic constitutive law

$$\sigma_{ij} = C_{ijkl}\epsilon_{kl}$$

$$C_{ijkl} = C_{jikl} = C_{ijlk} = C_{klij}$$

$$\epsilon_{kl} = \frac{1}{2}(u_{k,l} + u_{l,k})$$

due to symmetry in elastic modulus, C_{ijkl}

$$\sigma_{ij} = C_{ijkl}u_{k,l}$$

consider 2D deformation where u_i $(i=1,2,3)$ depends only on x_1 and x_2

without loss of generality, let

$$u_i = a_i f(z)$$

$$z = x_1 + p x_2$$

$f(z)$ arbitrary function

a_i and p are unknowns dependent on material anisotropy and boundary conditions

SINCE displacements are defined, compatibility is automatically satisfied — equilibrium must be satisfied also $\longrightarrow$ $\sigma_{ij,j} = 0$

$$\cancel{C_{ijks}u_k}$$

$$\boxed{C_{ijks}u_{k,sj} = 0}$$

valid for homogeneous materials

(2) :

differentiate $u_k = a_k f(z)$

$$u_{k,s} = \left(\delta_{s1} + \rho\,\delta_{s2}\right) a_k f'(z)$$

δ_{si} - Kronecker delta

$$f'(z) = \frac{df}{dz}$$

now,

$$C_{ijks} u_{k,sj} = 0$$

$$C_{ijks}\left(\delta_{j1} + \rho\,\delta_{j2}\right)\left(\delta_{s1} + \rho\,\delta_{s2}\right) a_k = 0$$

$$\left(C_{i1ks} + \rho\, C_{i2ks}\right)\left(\delta_{s1} + \rho\,\delta_{s2}\right) a_k = 0$$

$$\left[C_{i1k1} + \rho\left(C_{i1k2} + C_{i2k1}\right) + \rho^2\left(C_{i2k2}\right)\right] a_k = 0$$

matrix notation

$$\left([Q] + \rho\left([R] + [R]^T\right) + \rho^2 [Q]\right)\{a\} = 0$$

$Q_{ik} = C_{i1k1}$, $R_{ik} = C_{i1k2}$, $T_{ik} = C_{i2k2}$

one solution to this problem is $\{a\} = 0$
This is trivial.
Nontrivial solution requires

$$\left|[Q] + \rho\left([R] + [R]^T\right) + \rho^2 [T]\right| = 0$$

gives six roots for ρ — eigenvalues
associated eigenvectors are a_k

determine stress and displacement solutions

from $\sigma_{ij} = C_{ijks} u_{k,s}$

$$\sigma_{ij} = C_{ijks}(\delta_{s1} + p\delta_{s2}) a_k f'(z)$$

$$\sigma_{ij} = (C_{ijk1} + pC_{ijk2}) a_k f'(z)$$

$$\cancel{\sigma_{i1} = (C_{1jk1} + pC_{1jk2}) a_k f'(z) = Q}$$

$$\sigma_{i1} = (C_{i1k1} + p\, C_{i1k2}) a_k f'(z) = (Q_{ik} + pR_{ik}) a_k f'(z)$$

$$\sigma_{i2} = (C_{i2k1} + pC_{i2k2}) a_k f'(z) = (R_{ki} + pQ_{ik}) a_k f'(z)$$

★ The eigenvalues must be imaginary $p_\alpha = p_{1\alpha} + ip_{2\alpha}$, $i = \sqrt{-1}$

★ this is required for the strain energy to be positive definite.

SINCE $\underline{u}$ is real, we take

$$\underline{u} = \sum_{\alpha=1}^{3} \left\{ a_\alpha f_\alpha(z_\alpha) + \bar{a}_\alpha f_{\alpha+3}(\bar{z}_\alpha) \right\}$$

$\operatorname{Im} p_\alpha > 0$

$p_{\alpha+3} = \bar{p}_\alpha$ ← complex conjugate

$a_{\alpha+3} = \bar{a}_\alpha$

$z_\alpha = x_1 + p_\alpha z_2$

similiarly stress can be defined using the previous equations for σ_{i1} and σ_{i2}

recall $t_i = \sigma_{ji} n_j$

$$t_1 = \sigma_{11} n_1 + \sigma_{21} n_2 + \sigma_{31} n_3$$

$$t_2 = \sigma_{12} n_1 + \sigma_{22} n_2 + \sigma_{32} n_3$$

the vector components of stress can be written as

$$(\underline{t}_1)_i = \sigma_{i1}$$

$$(\underline{t}_2)_i = \sigma_{i2}$$

such that now,

$$\underline{t}_1 = \sum_{\alpha=1}^{3} \left\{ (\underline{\underline{Q}} + p_\alpha \underline{\underline{R}}) \underline{a}_\alpha f_\alpha'(z_\alpha) + (\underline{\underline{Q}} + \bar{p}_\alpha \underline{\underline{R}}) \underline{\bar{a}}_\alpha f_{\alpha+3}'(\bar{z}_\alpha) \right\}$$

$$\underline{t}_2 = \sum_{\alpha=1}^{3} \left\{ (\underline{\underline{R}}^T + p_\alpha \underline{\underline{T}}) \underline{a}_\alpha f_\alpha'(z_\alpha) + (\underline{\underline{R}}^T + \bar{p}_\alpha \underline{\underline{T}}) \underline{\bar{a}}_\alpha f_{\alpha+3}'(\bar{z}_\alpha) \right\}$$

this can be simplified or rewritten in condensed notation as

$$\sigma_{i1} = -p b_i f'(z)$$

$$\sigma_{i2} = b_i f'(z)$$

where, $\underline{b} = (\underline{\underline{R}}^T + p \underline{\underline{T}}) \underline{a} = -\frac{1}{p} (\underline{\underline{Q}} + p \underline{\underline{R}}) \underline{a}$

introducing the stress function,

$$\varphi_i = b_i f(z)$$

$$\sigma_{i1} = -\varphi_{i,2}$$

$$\sigma_{i2} = \varphi_{i,1}$$

the general displacement and stress potential relations are now,

$$\underline{u} = \sum_{\alpha=1}^{3}\{\underline{a}_\alpha f_\alpha(z_\alpha) + \bar{\underline{a}}_\alpha f_{\alpha+3}(\bar{z}_\alpha)\}$$

$$\underline{\phi} = \sum_{\alpha=1}^{3}\{\underline{b}_\alpha f_\alpha(z_\alpha) + \bar{b}_\alpha f_{\alpha+3}(\bar{z}_\alpha)\}$$

and $b_{\alpha+3} = \bar{b}_\alpha$

This is the sextic formalism due to Stroh and $\underline{a}_\alpha$ and $\underline{b}_\alpha$ are the Stroh eigenvectors.

The only stress component missing is σ_{33}. It is determined in terms of other stress components for the plane strain condition, $\epsilon_{33}=0$.

How do you obtain the plane σ case?

Application to boundary value problems

Most applications (excluding bimaterials) f_α has the same functional form. We can then write,

$$f_\alpha(z_\alpha) = f(z_\alpha)\,q_\alpha$$

$$f_{\alpha+3}(\bar{z}_\alpha) = f(\bar{z}_\alpha)\,\bar{q}_\alpha$$

q_α are arbitrary complex constants that must satisfy boundary conditions

The equation above can now be written as

$$\underline{u} = 2\mathrm{Re}\{\underline{A}\langle f(z_*)\rangle \underline{q}\}$$

$$\underline{\phi} = 2\mathrm{Re}\{\underline{\underline{B}}\langle f(z_*)\rangle \underline{q}\}$$

$$\underline{\underline{A}} = [\underline{a}_1\ \ \underline{a}_2\ \ \underline{a}_3] \qquad \underline{\underline{B}} = [\underline{b}_1\ \ \underline{b}_2\ \ \underline{b}_3]$$

$$\langle f(z_*)\rangle = \mathrm{diag}[f(z_1), f(z_2), f(z_3)]$$

Example

Antiplane deformation

$$\sigma_{ij} = C_{ijks}\epsilon_{ks}$$

$$\sigma_{ij,j} = 0$$

$$\epsilon_{ij} = \tfrac{1}{2}(u_{i,j} + u_{j,i})$$

$$\sigma_{ij,j} = C_{ijks}u_{k,sj} = 0$$

must satisfy this equation subject to some set of boundary conditions

For the antiplane problem, consider special anisotropic materials which satisfy

$$C_{14} = C_{15} = C_{24} = C_{25} = C_{46} = C_{56} = 0$$

(Voigt ~~notion~~ notation)

1-11	4-23
2-22	5-31
3-33	6-12

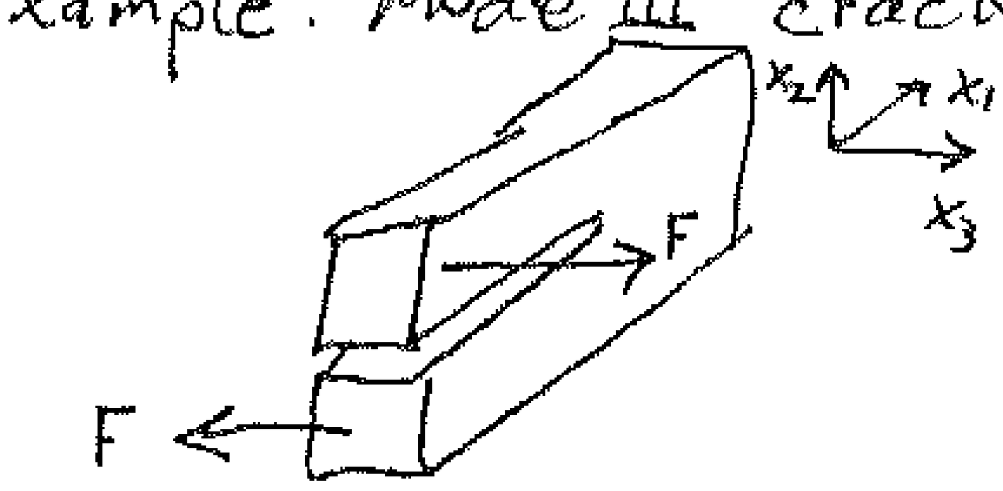
monoclinic materials with symmetry along x_3 satisfy this $\therefore$ fairly general

For anti-plane deformation

$$u_1 = u_2 = 0$$

$$u_3 = u(x_1, x_2)$$

Example: Mode III crack



$$\sigma_{ij} = C_{ij31}u_{,1} + C_{ij32}u_{,2}$$

$$\sigma_{31} = C_{55}u_{,1} + C_{45}u_{,2}$$

$$\sigma_{32} = C_{45}u_{,1} + C_{44}u_{,2}$$

$$\sigma_{33} = C_{35}u_{,1} + C_{34}u_{,2}$$

equilibrium requires, $\sigma_{31,1} + \sigma_{32,2} = 0$

$$\rightarrow C_{55}u_{,11} + 2C_{45}u_{,12} + C_{44}u_{,22} = 0$$

homogeneous second order differential equation for u

without loss of generality, let

$$u = \frac{-i}{2\mu} f(z)$$

$$z = x_1 + p x_2$$

$f(z)$ is an arbitrary function

$i = \sqrt{-1}$ and p must be determined

$\mu = \sqrt{C_{44} C_{55} - C_{45}^2} > 0$ to satisfy positive definite properties of C_{ijkl}

substitute u into 2nd order diff. eqn.

$$u_{,11} = \frac{-i}{2\mu} f''(z) \qquad \text{Note: } \frac{df}{dx_1} = \frac{df}{dz}\frac{dz}{dx_1} = \frac{df}{dz} = f'(z)$$

$$u_{,12} = \frac{-ip}{2\mu} f''(z) \qquad \frac{df}{dx_2} = \frac{df}{dz}\frac{dz}{dx_2} = p\frac{df}{dz} = p f'(z)$$

$$u_{,22} = \frac{-ip^2}{2\mu} f''(z)$$

$$\frac{-i}{2\mu}\left(C_{55} + 2p C_{45} + p^2 C_{44}\right) f''(z) = 0$$

$$C_{55} + 2p C_{45} + p^2 C_{44} = 0$$

2 roots for $p \Rightarrow p = \dfrac{-C_{45} + i\mu}{C_{44}}$

other is complex conjugate $\bar{p}$

☆ if material is isotropic, $C_{44} = C_{55} = \mu$; $C_{45} = 0 \Rightarrow p = i$ and μ is the shear modulus (This also holds for plane deformation)

now,

$$u = \frac{1}{2\mu}\left\{-i f(z) + i \bar{f}(\bar{z})\right\}$$

the stresses are

$$\sigma_{31} = \frac{1}{\mu} \operatorname{Im}\left\{(C_{55} + p C_{45}) \frac{df}{dz}\right\} = -\operatorname{Re}\left\{p \frac{df}{dz}\right\} = -\phi_{,2}$$

$$\sigma_{32} = \frac{1}{\mu} \operatorname{Im}\left\{(C_{45} + p C_{44}) \frac{df}{dz}\right\} = \operatorname{Re}\left\{\frac{df}{dz}\right\} = \phi_{,1}$$

$$\phi = \operatorname{Re} f(z)$$