Green's function for infinite space and half space

$\underline{u} = 2\,\mathrm{Im}\{ \underline{\underline{A}} \langle f(z_\alpha) \rangle \underline{q} \}$

$\underline{\phi} = 2\,\mathrm{Im}\{ \underline{\underline{B}} \langle f(z_\alpha) \rangle \underline{q} \}$

$\sigma_{i1} = -\varphi_{i,2}$

$\sigma_{i2} = \varphi_{i,2}$

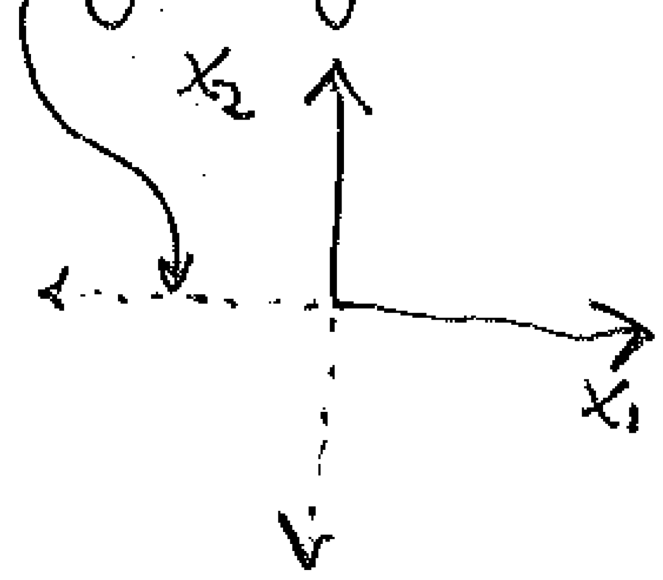
Define $\underline{f}^{(\frac{N}{m})}$ applied on x_3-axis with $\underline{b}$ (Burger's vector)

use the plane $\left.\begin{matrix} x_2 = 0 \\ x_1 < 0 \end{matrix}\right\}$ slip plane

Introduce a branch cut along negative x_1-axis

$r > 0$

$-\pi < \theta < \pi$



Boundary conditions,

$\sigma_{ij} = 0$ at $|\underline{x}| = \infty$

$u(r,\pi) - u(r,-\pi) = \underline{b}$

$\phi(r,\pi) - \phi(r,-\pi) = \underline{f}$

use the function,

$f(z_\alpha) = \ln(z_\alpha) = \begin{cases} \ln r & \theta = 0 \\ \ln r \pm i\pi & \theta = \pm\pi \end{cases}$

now,

$\underline{u} = \frac{1}{\pi}\mathrm{Im}\{ \underline{\underline{A}} \langle \ln z_* \rangle \underline{q}^\infty \}$

$\underline{\phi} = \frac{1}{\pi}\mathrm{Im}\{ \underline{\underline{B}} \langle \ln z_* \rangle \underline{q}^\infty \}$

Boundary conditions satisfied by subbing $\underline{\phi}$ into eqns. above for σ_{i1} and σ_{i2}

(2)

$$u(r,\pi)-u(r,-\pi)=\frac{1}{\pi}\left\{\mathrm{Im}\left\{\underline{\underline{A}}\langle \ln r + i\pi\rangle \underline{g}^{\infty}\right\}-\mathrm{Im}\left\{\underline{\underline{A}}\langle \ln r^{-i\pi}\rangle \underline{g}^{\infty}\right\}\right\}=\underline{b}$$

$$=\frac{1}{\pi}\left\{\mathrm{Im}\left\{\underline{\underline{A}}\langle 2i\pi\rangle g^{\infty}\right\}\right\}=\underline{b}$$

$$=2\mathrm{Re}\left\{\underline{\underline{A}}g^{\infty}\right\}=\underline{b}$$

similarly,

$$\phi(r,\pi)-\phi(r,-\pi)=2\mathrm{Re}\left\{\underline{\underline{B}}g^{\infty}\right\}=\underline{f}$$

this can be written as

$$\begin{bmatrix}\underline{\underline{A}} & \overline{\underline{\underline{A}}} \\ \underline{\underline{B}} & \overline{\underline{\underline{B}}}\end{bmatrix}\begin{bmatrix} g^{\infty} \\ \bar{g}^{\infty}\end{bmatrix}=\begin{bmatrix}\underline{b}\\ \underline{f}\end{bmatrix}$$

from orthogonality identities,

$$g^{\infty}=\underline{\underline{A}}^{T}\underline{f}+\underline{\underline{B}}^{T}\underline{b}$$

Now the solution can be cast in the form of the line force/length and Burgers vector $\underline{b}$

$$\begin{bmatrix}\underline{u}\\ \underline{\phi}\end{bmatrix}=\frac{1}{\pi}\mathrm{Im}\begin{bmatrix}\underline{\underline{A}}\langle \ln z_*\rangle \underline{\underline{B}}^{T} & \underline{\underline{A}}\langle \ln z_*\rangle \underline{\underline{A}}^{T}\\ \underline{\underline{B}}\langle \ln z_*\rangle B^{T} & \underline{\underline{B}}\langle \ln z_*\rangle \underline{\underline{A}}^{T}\end{bmatrix}\begin{bmatrix}\underline{b}\\ \underline{f}\end{bmatrix}$$

One component Green's function

recall
$$\underline{u}=\frac{1}{\pi}\mathrm{Im}\sum_{\alpha=1}^{3}(\ln z_\alpha)\, g_\alpha^{\infty}\underline{a}_\alpha$$
$$\underline{\phi}=\frac{1}{\pi}\mathrm{Im}\sum_{\alpha=1}^{3}(\ln z_\alpha)\, g_\alpha^{\infty}\underline{b}_\alpha$$

Each of the 3 solution ($\alpha=1,2,3$) is a one-component Green's function of the form

$$\underline{u}=\frac{1}{\pi}\mathrm{Im}\left\{\ln(z)\, g^{\infty}\underline{a}\right\}$$
$$\underline{\phi}=\frac{1}{\pi}\mathrm{Im}\left\{\ln(z)\, g^{\infty}\underline{\tilde{b}}\right\}$$

stroh eigenvector $\underline{b}_\alpha=\begin{bmatrix}\tilde{b}_1\\ \tilde{b}_2\\ \tilde{b}_3\end{bmatrix}^T$
not Burgers vector

③ the relation to line force and Burgers vector is

$$\underline{b} = 2\,\mathrm{Re}(g^{\infty}\underline{a}) \qquad \underline{f} = 2\,\mathrm{Re}(g^{\infty}\underline{\tilde{b}})$$

the equations for $\underline{u}$ and $\underline{\phi}$ illustrate that $\underline{u}$ is polarized on the eigenplane $\underline{a}$, while the stress function is polarized on the eigenplane $\underline{\hat{b}}$.

This means the line dislocation $\underline{b}$ and line force $\underline{f}$ are polarized on the $\underline{a}$ and $\underline{\tilde{b}}$ planes, respectively.

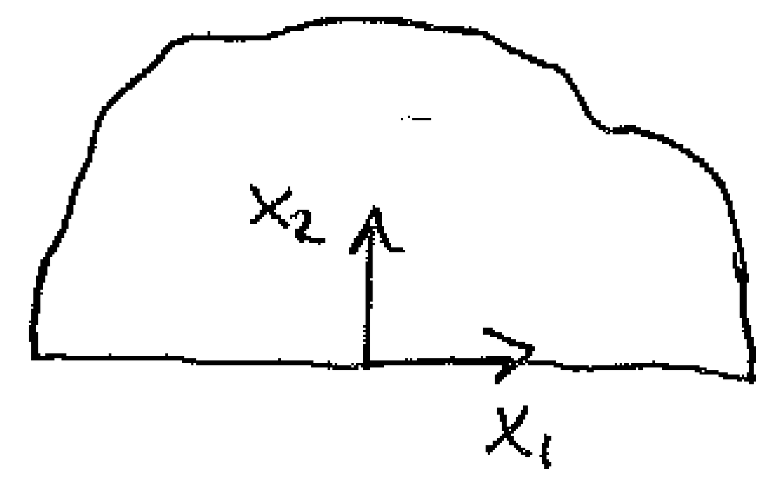
* This means only $\underline{b}$ or $\underline{f}$ can be prescribed, but not both!

If $\underline{b}$ and $\underline{f}$ are prescribed arbitrarily all 3 one-component Green's functions are required for the solution.

Green's Function for Half-Space

Let line force $\underline{f}$ and Burgers vector $\underline{b}$ be applied at

$(x_1, x_2) = (0, d) \quad d > 0$



let $f(z_\alpha - z_\alpha^\circ)$, $z_\alpha^\circ = x_1^\circ + p_\alpha x_2^\circ$, x_1°, x_2° are real constants (moves coordinate origin to x_1°, x_2°)

now let,

$$\underline{u} = \frac{1}{\pi}\,\mathrm{Im}\left\{\underline{\underline{A}}\langle \ln(z_* - p_* d)\rangle \underline{g}^{\infty}\right\} + \frac{1}{\pi}\,\mathrm{Im}\sum_{\beta=1}^{3}\left\{\underline{\underline{A}}\langle \ln(z_* - \bar{p}_\beta d)\rangle \underline{g}_\beta\right\}$$

$$\underline{\phi} = \frac{1}{\pi}\,\mathrm{Im}\left\{\underline{\underline{B}}\langle \ln(z_* - p_* d)\rangle \underline{g}^{\infty}\right\} + \underbrace{\frac{1}{\pi}\,\mathrm{Im}\sum_{\beta=1}^{3}\left\{\underline{\underline{B}}\langle \ln(z_* - \bar{p}_\beta d)\rangle \underline{g}_\beta\right\}}_{\text{this part is to satisfy B.C.'s at } x_2 = 0}$$

$$\underline{g}^{\infty} = \underline{\underline{A}}^T\underline{f} + \underline{\underline{B}}^T\underline{b}$$

④

First consider traction free surface

$$\underline{\phi} = 0 \quad \text{at } x_2 = 0$$

$$Im\{\underline{\underline{B}}\langle ln(x_1 - p_* d)\rangle \underline{q}^{\infty}\} + Im \sum_{\beta=1}^{3} \{ln(x_1 - \bar{p}_\beta d)\underline{\underline{B}}\,\underline{q}_\beta\} = 0$$

note: $Im\{\underline{\underline{B}}\langle ln(x_1 - p_* d)\rangle \underline{q}^{\infty}\} = -Im\{\underline{\underline{\bar{B}}}\langle ln(x_1 - \bar{p}_* d)\rangle \bar{\underline{q}}^{\infty}\}$

check: $B = B' + iB''$

$\underline{q}^{\infty} = \underline{q}^{\infty\prime} + i\underline{q}^{\infty\prime\prime}$

$Im\{(B' + iB'')(\underline{q}^{\infty\prime} + i\underline{q}^{\infty\prime\prime})\} = Im\{B'\underline{q}^{\infty\prime} - B''\underline{q}^{\infty\prime\prime} + i(B''\underline{q}^{\infty\prime} + B'\underline{q}^{\infty\prime\prime})$

$= B''\underline{q}^{\infty\prime} + B'\underline{q}^{\infty\prime\prime}$

$-Im\{\bar{B}\bar{\underline{q}}^{\infty}\} = B''\underline{q}^{\infty\prime} + B'\underline{q}^{\infty\prime\prime}$ ✓

also,

$$\langle ln(x_1 - \bar{p}_* d)\rangle = \sum_{\beta=}^{3} ln(x_1 - \bar{p}_\beta d)\,\underline{\underline{I}}_\beta$$

$I_1 = diag[1,0,0]$

$I_2 = diag[0,1,0]$

$I_3 = diag[0,0,1]$

now, from B.C.'s above we have

$$Im \sum_{\beta=1}^{3} ln(x_1 - \bar{p}_\beta d)\{-\underline{\underline{\bar{B}}} I_\beta \bar{\underline{q}}^{\infty} + \underline{\underline{B}}\,\underline{q}_\beta\} = 0$$

which gives

$$\underline{q}_\beta = \underline{\underline{B}}^{-1}\bar{B} I_\beta \bar{\underline{q}}^{\infty}$$

if $x_2 = 0$ is a rigidly clamped surface,

$$\underline{u} = 0 \quad \text{at } x_2 = 0$$

then the same procedure leads to

$$\underline{q}_\beta = \underline{\underline{A}}^{-1}\underline{\underline{\bar{A}}}\,\underline{\underline{I}}_\beta \bar{\underline{q}}^{\infty}$$

(5) if $d=0$

$$\underline{u} = \frac{1}{\pi}\mathrm{Im}\{\underline{\underline{A}}\langle \ln z_* \rangle \underset{\sim}{q}\}$$

$$\underline{\phi} = \frac{1}{\pi}\mathrm{Im}\{\underline{\underline{B}}\langle \ln z_* \rangle \underset{\sim}{q}\}$$

$$\underset{\sim}{q} = \underset{\sim}{q}^{(0)} + \sum_{\beta=1}^{3} \underset{\sim}{q}_\beta$$