Kinematics

1 Notations

The book uses some weird notations for derivatives:

$$\boxed{\partial_i = \frac{\partial}{\partial x_i} \qquad \partial_0 = \frac{\partial}{\partial t}}$$

Also for tensors:

$$\boxed{T_{(ij)} = \tfrac{1}{2}\left(T_{ij} + T_{ji}\right) \qquad T_{[ij]} = \tfrac{1}{2}\left(T_{ij} - T_{ji}\right)}$$

Please do not do this. Follow the notations used in class.

2 Lagrangian

In a Lagrangian description, we want to find the motion of the fluid particles (little regions of fluid).

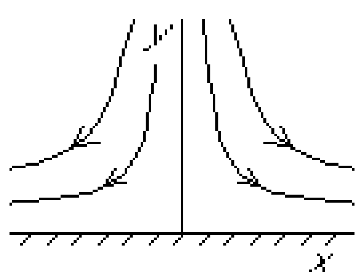
We must first label the fluid particles with tags. For example, let (ξ, η, θ) indicate the particle that was at $x = \xi$, $y = \eta$, $z = \theta$ at time $t = 0$.

We will want to find the position of every particle (ξ, η, θ) in time. In other words, we want to find $\vec{r}(t; \xi, \eta, \theta)$. Then

$$\boxed{\frac{D\vec{r}}{Dt} = \vec{v} \qquad \frac{D\vec{v}}{Dt} = \vec{a}}$$

Example:

2D ideal stagnation point flow



$$x = \xi e^{ct} \qquad y = \eta e^{-ct} \qquad z = \theta$$

$$u = c\xi e^{ct} \qquad v = -c\eta e^{-ct} \qquad w = 0$$

Shape of the particle paths: $xy = \xi\eta$ is constant.

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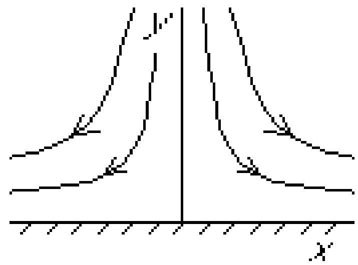
3 Eulerian

In a standard Eulerian description, we use space-time coordinates. We now want to find $\vec{v}\,(\vec{r},t)$, $p\,(\vec{r},t)$, ... In Cartesian coordinates, that would be finding $u(x,y,z,t)$, $v(x,y,z,t)$, $w(x,y,z,t)$, $p(x,y,z,t)$, ...

Example:

2D ideal stagnation point flow



$$u = cx \qquad v = -cy \qquad w = 0$$

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Particle paths: Lines traced by the fluid particles. Particle paths can be found from solving:

$$\boxed{D\vec{r} : Dt = \vec{v}} \qquad \text{Cartesian: } Dx : Dy : Dz : Dt = u : v : w : 1$$

Streamlines: Lines that are everywhere in the direction of the velocity field. Streamlines can be found from solving:

$$\boxed{d\vec{r}//\vec{v} \quad dt = 0} \qquad \text{Cartesian: } dx : dy : dz : dt = u : v : w : 0$$

Various numerical and experimental methods produce streamlines naturally.

Streaklines: Lines of smoke coming out of smoke generators. Streaklines can be found by integrating the path lines for all the smoke particles coming out the generator.

Steady Flow:

$$\boxed{\vec{v} = \vec{v}\,(\vec{r})}$$

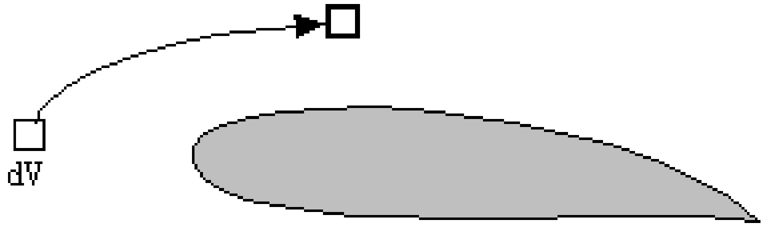
For steady flow, pathlines, streamlines, and streaklines are all the same.

4 Fluxions

To write physical laws, we must be able to find time derivatives *for a fixed point of fluid.* For example, the acceleration of a point of fluid is the time derivative of the velocity keeping that point constant.

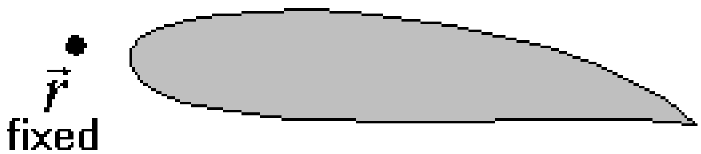
Derivatives depend on what you hold constant!

Lagrangian/material/substantial time derivatives keep the fluid constant (move along with the fluid):



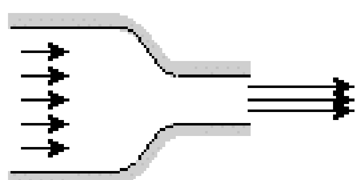
$$\frac{D}{Dt} \text{ keeps } \xi,\ \eta, \text{ and } \theta \text{ constant}$$

Eulerian time derivatives:



$$\frac{\partial}{\partial t} \text{ keeps } x,\ y, \text{ and } z \text{ constant}$$

Example:



$$\text{Lagrangian: } \frac{D\vec{v}}{Dt} \neq 0 \qquad \text{Eulerian: } \frac{\partial \vec{v}}{\partial t} = 0$$

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Exercise:

What can you say about the pressure field for the flow in the picture? Is the pressure increased on the curved walls to keep the incoming stream from going through the wall?

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Calculus:

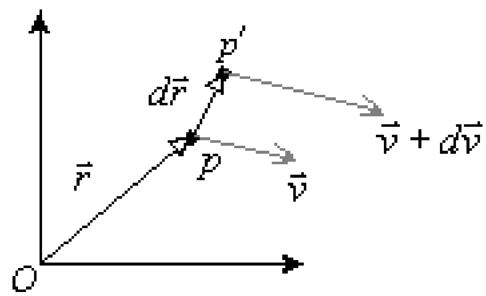
$$\frac{D}{Dt} = \frac{\partial}{\partial t} + \frac{Dx}{Dt}\frac{\partial}{\partial x} + \frac{Dy}{Dt}\frac{\partial}{\partial y} + \frac{Dz}{Dt}\frac{\partial}{\partial z}$$

Physics:

$$\boxed{\frac{D}{Dt} = \frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z}}$$

The additional terms are the *convective* terms.

5 Decomposition



While the motion of individual points of fluid tells us things like the force on the fluid at that point, in order to understand effects such as viscosity we must look at *deformation* of the fluid. The local deformation of the fluid near some point P depends on how arbitrary neigboring points P' move compared to P. If all neighboring points P' move with the same velocity as P, the fluid moves as a solid block, and there is no deformation.

The motion of point P' relative to neighboring point P is given by the difference in velocity $\mathrm{d}\vec{v}$.

$$\mathrm{d}\vec{v} = A\,\mathrm{d}\vec{r} \qquad A = \frac{\partial \vec{v}}{\partial \vec{r}} \qquad a_{ij} = \frac{\partial v_i}{\partial x_j}$$

So the velocity derivative tensor A determines the rate of local deformation of the fluid.

To understand this better, take A apart into symmetric and antisymmetric parts:

$$A = S + W \qquad S = \tfrac{1}{2}\left(A + A^T\right) \qquad W = \tfrac{1}{2}\left(A - A^T\right)$$

The symmetric part is called the strain-rate tensor.

$$S = \begin{pmatrix} \dfrac{\partial u}{\partial x} & \frac{1}{2}\left(\dfrac{\partial u}{\partial y} + \dfrac{\partial v}{\partial x}\right) & \frac{1}{2}\left(\dfrac{\partial u}{\partial z} + \dfrac{\partial w}{\partial x}\right) \\ \frac{1}{2}\left(\dfrac{\partial v}{\partial x} + \dfrac{\partial u}{\partial y}\right) & \dfrac{\partial v}{\partial y} & \frac{1}{2}\left(\dfrac{\partial v}{\partial z} + \dfrac{\partial w}{\partial y}\right) \\ \frac{1}{2}\left(\dfrac{\partial w}{\partial x} + \dfrac{\partial u}{\partial z}\right) & \frac{1}{2}\left(\dfrac{\partial w}{\partial y} + \dfrac{\partial v}{\partial z}\right) & \dfrac{\partial w}{\partial z} \end{pmatrix}$$

By rotating the coordinate system, to x', y', z', it becomes diagonal:

$$S' = \begin{pmatrix} \dfrac{\partial u'}{\partial x'} & 0 & 0 \\ 0 & \dfrac{\partial v'}{\partial y'} & 0 \\ 0 & 0 & \dfrac{\partial w'}{\partial z'} \end{pmatrix} \qquad \begin{array}{l} \rightarrow \mathrm{d}u' = \dfrac{\partial u'}{\partial x'}\,\mathrm{d}x' \\ \rightarrow \mathrm{d}v' = \dfrac{\partial v'}{\partial y'}\,\mathrm{d}y' \\ \rightarrow \mathrm{d}w' = \dfrac{\partial w'}{\partial z'}\,\mathrm{d}z' \end{array}$$

This are three simple straining motions.

The trace of the tensor S is the rate of (relative volume) expansion of the fluid.

The second, antisymmetric, part of A looks as:

$$W = \frac{1}{2}\begin{pmatrix} 0 & \frac{\partial u}{\partial y} - \frac{\partial v}{\partial x} & \frac{\partial u}{\partial z} - \frac{\partial w}{\partial x} \\ \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} & 0 & \frac{\partial v}{\partial z} - \frac{\partial w}{\partial y} \\ \frac{\partial w}{\partial x} - \frac{\partial u}{\partial z} & \frac{\partial w}{\partial y} - \frac{\partial v}{\partial z} & 0 \end{pmatrix} \equiv \frac{1}{2}\begin{pmatrix} 0 & -\omega_z & \omega_y \\ \omega_z & 0 & -\omega_x \\ -\omega_y & \omega_x & 0 \end{pmatrix}$$

where

$$\boxed{\vec{\omega} \equiv \nabla \times \vec{v}}$$

is the vorticity. By simply writing it out, you can see that the velocity due to this antisymmetric part is $\mathrm{d}\vec{v} = W\,\mathrm{d}\vec{r} = \frac{1}{2}\vec{\omega} \times \mathrm{d}\vec{r}$. That is the equation of a solid body rotation with an angular velocity vector equal to $\frac{1}{2}\vec{\omega}$.

Vorticity is important because it allows flows to behave in complicated, nonlinear ways.

The line integral of the velocity along a closed contour is called the circulation Γ of that contour:

$$\boxed{\Gamma \equiv \oint \vec{v} \cdot \mathrm{d}\vec{r}}$$

Exercise:

What can you say about Γ if $\vec{\omega} = 0$?

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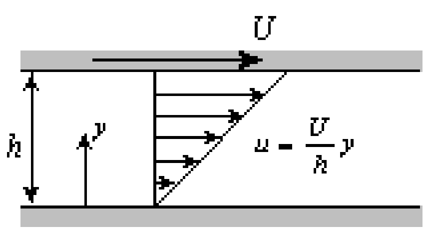
6 Shear

Linear shear flow:



Velocity:

$$\vec{v} = U\frac{y}{h}\hat{\imath}$$

Exercise:

Find the vorticity. How long until a small line segment of fluid completes one full rotation?

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Exercise:

Find the principal strain axes and corresponding strain rates. Is this an incompressible or compressible velocity field?

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