Potential Flows

1 Introduction

(Book 18.1)

1.1 Potential flows

Based on Kelvin's theorem, large parts of common flow fields are irrotational. In those parts, it is possible to replace the *three* velocity components by a *single* scalar "velocity potential".

For irrotational flows

$$\boxed{\vec{\omega} = \nabla \times \vec{v} \equiv 0 \quad \Longleftrightarrow \quad \vec{v} = \nabla\phi}$$

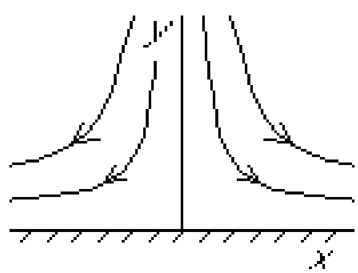
This will be outside the boundary layers and wakes. Function ϕ is called the *velocity potential*. The flow is called *potential flow* or *irrotational*.

Two-dimensional potential flow:

$$\boxed{u = \frac{\partial\phi}{\partial x} \qquad v = \frac{\partial\phi}{\partial y}}$$

Exercise:

For ideal stagnation point flow $(u, v) = a(x, -y)$,



- Does the velocity potential exist?
- If so, what is ϕ?

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1.2 2D Incompressible

Even if a flow is not irrotational, it is still possible to replace the velocity components by a single scalar unknown if the flow is two-dimensional and incompressible:

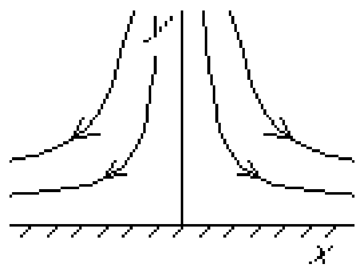
For two-dimensional incompressible flows:

$$\boxed{\mathrm{div}(\vec{v}) = \nabla \cdot \vec{v} \equiv 0 \quad \Longleftrightarrow \quad u = \frac{\partial \psi}{\partial y} \qquad v = -\frac{\partial \psi}{\partial x}}$$

Function ψ is called the *streamfunction*. Lines of constant ψ are the streamlines.

Exercise:

For ideal stagnation point flow $(u, v) = a(x, -y)$,



- Does the streamfunction exist?
- If so, what is ψ?
- How do the lines of constant ψ look?

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You should now be able to do 18.1

1.3 Both

We will now restrict ourselves to flows that are *both* irrotational and 2D incompressible. We will call those flows "ideal".

In terms of the velocity potential, by the simple existence of ϕ, irrotationality is automatic. But incompressibility requires:

$$\boxed{\nabla^2 \phi = 0}$$

Exercise:

Give a few examples of incompressible potential flows.

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In terms of ψ, incompressibility is satisfied automatically. But irrotationality,

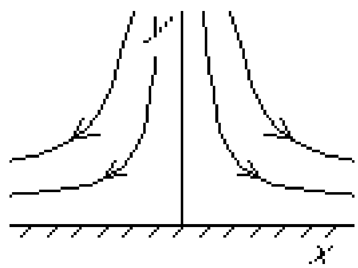
$$\omega_z = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = 0$$

requires:

$$\boxed{\nabla^2 \psi = 0}$$

Exercise:

For ideal stagnation point flow $(u, v) = a(x, -y)$,



- Is $\nabla^2 \phi = 0$?
- Is $\nabla^2 \psi = 0$?

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The boundary conditions for ϕ and ψ will be different. For flow around steady bodies, the body is a streamline, so that ψ is constant on the body (Dirichlet boundary condition). Also the velocity normal to the body surface will be zero, so that $\partial \phi / \partial n = 0$ (Neumann boundary condition.)

You should now be able to do 18.8

1.4 Bernoulli

For incompressible potential flows, the three momentum equations can be replaced by a single "potential flow Bernoulli equation."

The momentum equations for [in]viscid potential flow are:

$$\phi_{it} + \phi_j \phi_{ij} = -\frac{1}{\rho} p_i - g h_i + \mu \phi_{ijj}$$

Potential Flow Bernoulli equation:

$$\boxed{\frac{\partial \phi}{\partial t} + \tfrac{1}{2}v^2 + \frac{p}{\rho} + gh = C(t)}$$

Exercise:

Answer the following questions:

- How did I get this?
- Does this require points to be along the same streamline?
- Does this require the flow to be steady?
- Does this require the fluid to be inviscid?
- So how come there are viscous incompressible potential flows?
- Does this require the flow to be incompressible?
- How about compressible isentropic flows?
- Does the compressible isentropic flow have to be inviscid?

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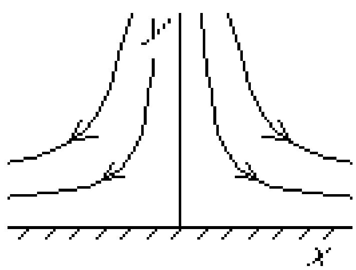
Exercise:

Draw the isobars for ideal stagnation point flow $(u, v) = a(x, -y)$.



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Bottom line: to find incompressible potential flow, solve

$$\phi_{ii} = 0$$

and find the pressure from the potential flow Bernoulli equation above.

1.5 Complex variables

Complex variables are very powerful in dealing with 2D irrotational incompressible flows.

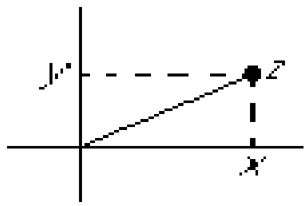
Define:

$$\boxed{\mathrm{i} = \sqrt{-1}}$$

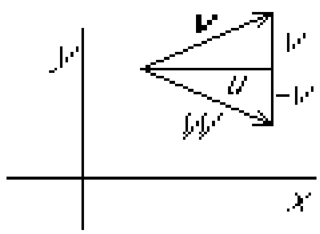
then i cannot be found anywhere on the real axis. This allows us to "pack" *two* real numbers into *one* complex number, eg:

$$\boxed{z = x + \mathrm{i}y \quad F = \phi + \mathrm{i}\psi \quad W = u - \mathrm{i}v}$$

Here z is the *complex position coordinate*,



F is the *complex velocity potential*, and W is the *complex conjugate* (because of the - sign) *velocity*.



What we have achieved is to replace the two dimensional *vectors* (x, y), (ϕ, ψ), and (u, v) by *scalar* (complex) numbers.

I can get the components of a given position z by writing z in the form $z = x + \mathrm{i}y$ where x and y are real. In that case, x is the x-component, and y is the y-component of the position z. We write $x = \Re(z)$ (the real part of z) and $y = \Im(z)$ (the imaginary part of z, i.e. the real number multiplying i.)

Exercise:

If $z = (1 + i) + i(2 + i)$, what are x and y?

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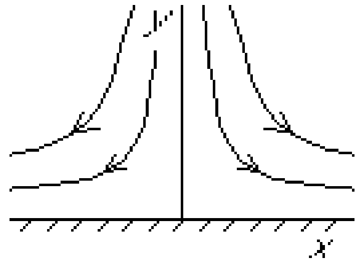
Complex numbers have the same general properties as ordinary numbers, except that they cannot be ordered (no $>$, $<$).

Exercise:

What are W and F for ideal stagnation point flow $(u, v) = a(x, -y)$? (Express in terms of z.)



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1.6 Differentiability

Any differentiable complex function $F(z)$ is the complex potential for a 2D incompressible potential flow. Further $dF/dz = W$ is the complex conjugate velocity.

Reason: differentiability requires that $\mathrm{d}F/\mathrm{d}z$ is the same whichever direction we take $\mathrm{d}z$. In particular if we take $\mathrm{d}z = \partial x$, a change in x only, we should get the same as when we take $\mathrm{d}z = \mathrm{i}\partial y$, a change in y only. That means, if $F = \phi + \mathrm{i}\psi$:

$$\frac{\partial \phi}{\partial x} + \mathrm{i}\frac{\partial \psi}{\partial x} = \frac{\partial \phi}{\mathrm{i}\partial y} + \mathrm{i}\frac{\partial \psi}{\mathrm{i}\partial y} \equiv u - \mathrm{i}v$$

Since $1/\mathrm{i} = \mathrm{i}/\mathrm{i}^2 = -\mathrm{i}$,

$$\frac{\partial \phi}{\partial x} + \mathrm{i}\frac{\partial \psi}{\partial x} = -\mathrm{i}\frac{\partial \phi}{\partial y} + \frac{\partial \psi}{\partial y} \equiv u - \mathrm{i}v$$

$$\frac{\partial \phi}{\partial x} = \frac{\partial \psi}{\partial y} = u \qquad \frac{\partial \psi}{\partial x} = -\frac{\partial \phi}{\partial y} = -v$$

from which it follows that u and v satisfy both the continuity equation and irrotationality:

$$u_x + v_y = \psi_{yx} - \psi_{xy} = 0 \qquad \omega_z = v_x - u_y = \phi_{yx} - \phi_{xy} = 0$$

Differentiable complex functions are easy to find:

- Constants are differentiable: $\mathrm{d}C/\,\mathrm{d}z = 0$.
- Function $F(z) = z$ is differentiable: $\mathrm{d}z/\,\mathrm{d}z = 1$.
- Sums of differentiable functions are differentiable (see your calculus book.)
- Products of differentiable functions are differentiable (see your calculus book.)
- Inverses of differentiable functions are differentiable (see your calculus book.)
- Functions with converging Taylor series are differentiable.

Not differentiable:

- Purely real functions such as x, $|z| \equiv \sqrt{x^2+y^2}$, ...
- Purely imaginary functions such as ix, i$|z|$, ...
- Complex conjugates such as $\bar{z} \equiv x - \mathrm{i}y$.

In other words, taking real or imaginary parts, absolute values, complex conjugates, ..., all make the expression nondifferentiable.

Exercise:

Give as many 2d incompressible potential flows as you can.

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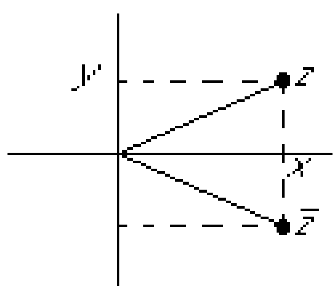
You should now be able to do 18.2, 3

1.7 Some Manipulations

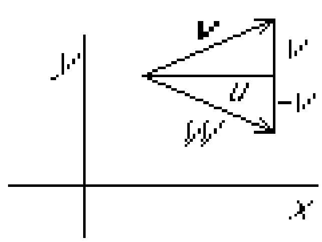
In order to deal with complex potentials, you need to know some important facts about complex numbers:

Parts If any complex number z is written in the form $z = x + \mathrm{i}y$ where x and y are both real, then the real part of z is $\Re(z) = x$. Similarly the imaginary part is $\Im(z) = y$. What are the real and imaginary parts of the complex streamfunction for ideal stagnation point flow, $F = \frac{1}{2}az^2$?

Complex conjugate To get the complex conjugate number, replace i everywhere by $-$i. Example: $\bar{z} = x - \mathrm{i}y$:



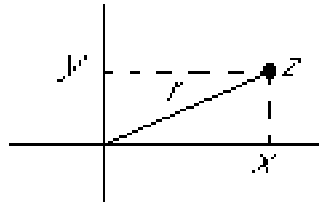
$\bar{V} = u - \mathrm{i}v = W$:



What is the complex conjugate of the complex streamfunction for ideal stagnation point flow, $F = \frac{1}{2}az^2$?

Magnitude To get the magnitude of a complex number, multiply by the conjugate and take the square root:

$$|z| = \sqrt{z\bar{z}} = \sqrt{(x + \mathrm{i}y)(x - \mathrm{i}y)} = \sqrt{x^2 + y^2} = r$$



$$|V| = \sqrt{\bar{W}W} = \sqrt{(u + \mathrm{i}v)(u - \mathrm{i}v)} = \sqrt{u^2 + v^2} = |\vec{v}|$$

What is the magnitude of the complex streamfunction for stagnation point flow, $F = \frac{1}{2}az^2$?

Note: z^2 is *not* a positive real number if z is complex. For complex numbers, only $z\bar{z}$ is always a positive real number.

Inverse To clean up the inverse of a complex number, try multiplying top and bottom with its complex conjugate. What is $(1 - \mathrm{i})/(1 + \mathrm{i})$?

Euler An exponential with a purely imaginary argument can be written:

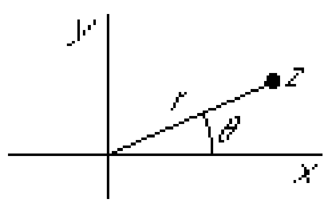
$$\boxed{e^{\mathrm{i}\theta} = \cos(\theta) + \mathrm{i}\sin(\theta)}$$

Verify this by writing out the Taylor series for the exponential.

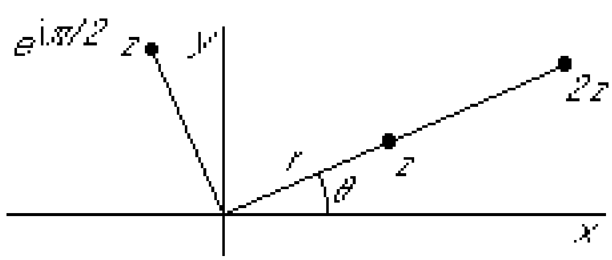
Polar form Any complex number z can be written in polar form as:

$$\boxed{z = re^{\mathrm{i}\theta}}$$

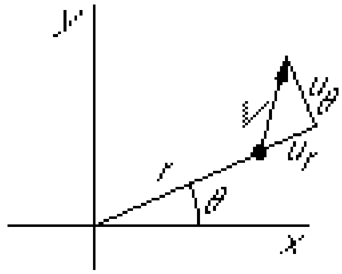
where real number r is the magnitude $|z| = \sqrt{z\bar{z}}$ of z and real number θ is the argument $\arg(z)$ of z.



Multiplication Multiplying a complex number with a real number a magnifies the number by a: $az = are^{\mathrm{i}\theta}$. Multiplying a complex number by $e^{\mathrm{i}\alpha}$ rotates the number counter-clockwise over an angle α: $e^{\mathrm{i}\alpha}z = re^{\mathrm{i}(\theta+\alpha)}$.



Application to finding the polar velocity components:



From the graph:

$$Ve^{-\mathrm{i}\theta} = u_r + \mathrm{i}u_\theta$$

Taking the complex conjugate:

$$\boxed{u_r - \mathrm{i}u_\theta = We^{\mathrm{i}\theta}}$$

What are the polar velocity components of ideal stagnation point flow, with complex streamfunction $F = \frac{1}{2}az^2$?

Logarithm If $z = re^{\mathrm{i}\theta}$ is any complex number,

$$\boxed{\ln(z) = \ln(r) + \mathrm{i}\theta}$$

where $\tan(\theta) = \Im(z)/\Re(z)$. What is the logarithm of the complex streamfunction for ideal stagnation point flow, $F = \frac{1}{2}az^2$?

You should now be able to do 18.4

2 Simplest Examples

(Book 18.2)

2.1 Constant F

Exercise:

If $F =$ constant, what is the velocity field?

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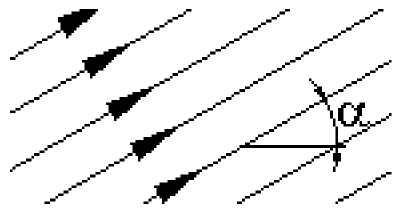
2.2 Uniform flow

$$\boxed{F = Ue^{-\mathrm{i}\alpha}z}$$

$$W = \frac{\mathrm{d}F}{\mathrm{d}z} = Ue^{-\mathrm{i}\alpha} \quad \Longrightarrow \quad V = Ue^{\mathrm{i}\alpha} \quad \Longrightarrow \quad u = U\cos(\alpha) \quad v = U\sin(\alpha)$$



2.3 Corners

$$\boxed{F = Az^n}$$

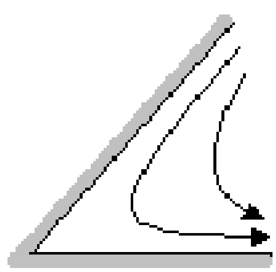
To identify how such a flow looks, the easiest is to use polar coordinates and find the streamfunction:

$$\phi + \mathrm{i}\psi = Ar^n e^{\mathrm{i}n\theta} = Ar^n\cos(n\theta) + \mathrm{i}Ar^n\sin(n\theta)$$

$$\boxed{\psi = Ar^n\sin(n\theta)}$$

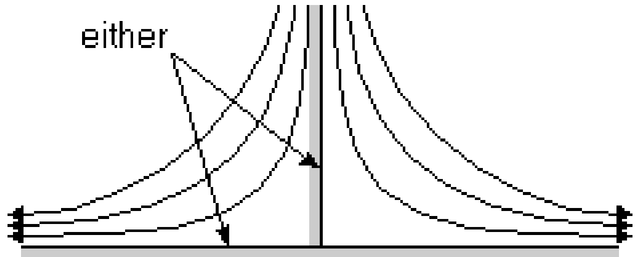
Case $n = 3$:

$$\psi = Ar^3\sin(3\theta) \quad \Longrightarrow \quad r = \sqrt[3]{\frac{\psi}{A\sin(3\theta)}}$$



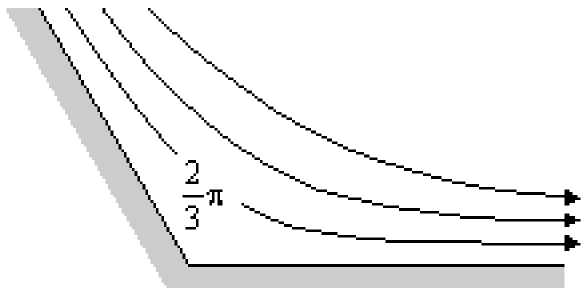
Case $n = 2$:

$$\psi = Ar^2\sin(2\theta) = 2Axy$$



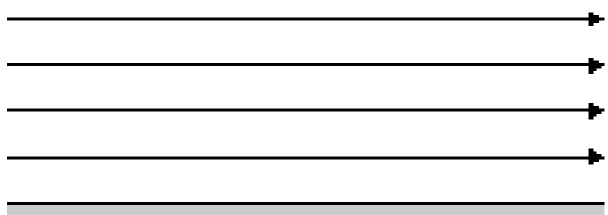
Case $n = 3/2$:

$$\psi = Ar^{3/2}\sin(\tfrac{3}{2}\theta)$$



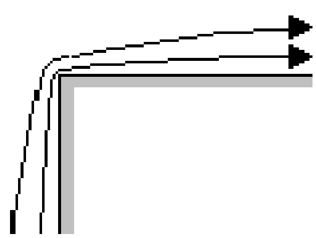
Case $n = 1$:

$$\psi = Ar\sin(\theta) = Ay$$



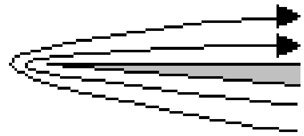
Case $n = 2/3$:

$$\psi = Ar^{2/3}\sin(\tfrac{2}{3}\theta)$$



Case $n = 1/2$:

$$\psi = Ar^{1/2}\sin(\tfrac{1}{2}\theta)$$



3 Logarithmic

(Book 18.3)

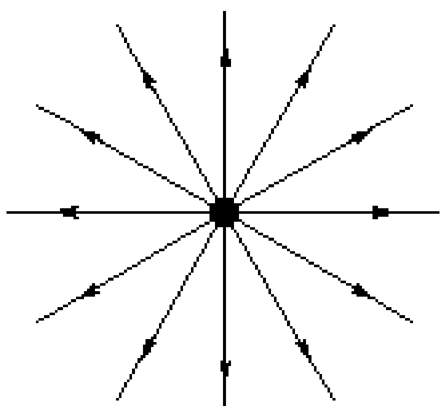
3.1 Source

Source flow:

$$\boxed{F = \frac{m}{2\pi}\ln z}$$

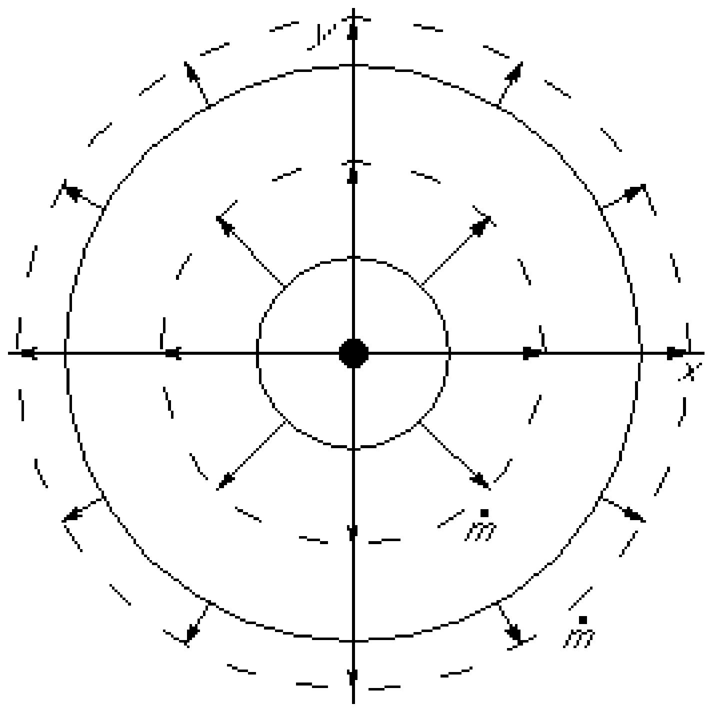
Polar coordinates $z = re^{\mathrm{i}\theta}$:

$$\phi + \mathrm{i}\psi = \frac{m}{2\pi}\left(\ln(r) + \mathrm{i}\theta\right)$$



Polar velocity components:

$$u_r - \mathrm{i}u_\theta = We^{\mathrm{i}\theta} = \frac{m}{2\pi z}e^{\mathrm{i}\theta} = \frac{m}{2\pi r} \quad\Longrightarrow\quad u_r = \frac{m}{2\pi r} \quad u_\theta = 0$$



Mass flux out of the source:

$$\dot{m} = \oint \rho \vec{v} \cdot \vec{n} \, \mathrm{d}s = \rho \frac{m}{2\pi r} 2\pi r = \rho m$$

and m is the volumetric flow rate of the source.

Exercise:

Since $m = \oint \vec{v} \cdot \vec{n} \, \mathrm{d}s$ and the divergence theorem states that

$$\oint \vec{v} \cdot \vec{n} \, \mathrm{d}s = \int \mathrm{div}(\vec{v}) \, \mathrm{d}x \, \mathrm{d}y,$$

what is the divergence of the velocity?

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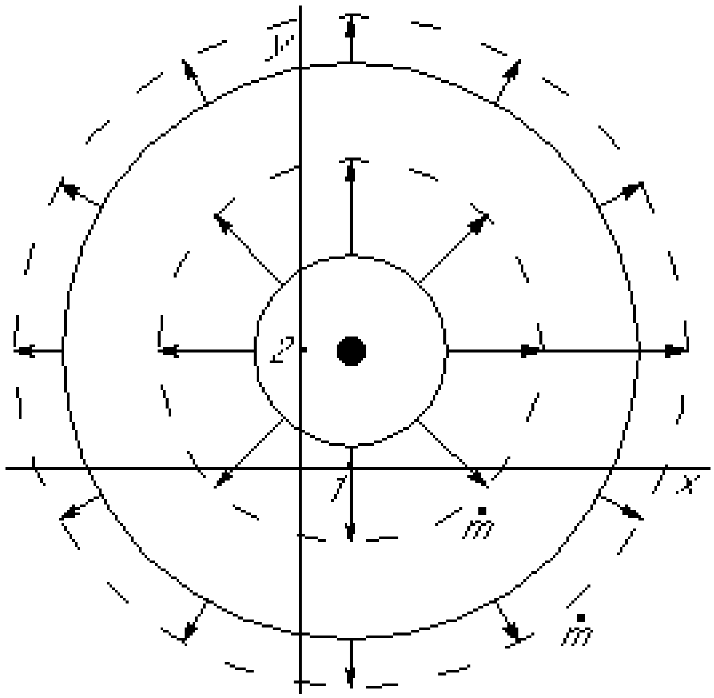
Source at a point $(x, y) = (1, 2)$:



$$F = \frac{m}{2\pi}\ln(z - z_0) \qquad z_0 = 1 + 2\mathrm{i}$$

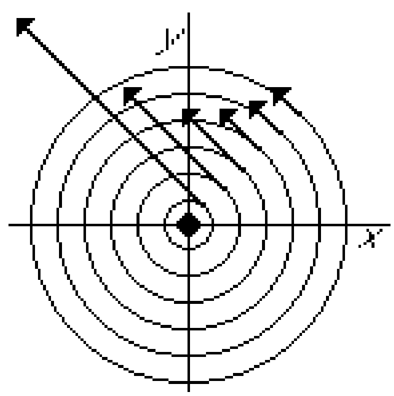
You should now be able to do 18.9

3.2 Vortex

$$\boxed{F = \frac{\tilde{\Gamma}}{2\pi\mathrm{i}}\ln z}$$

Polar coordinates $z = re^{\mathrm{i}\theta}$:

$$\phi + \mathrm{i}\psi = \frac{\tilde{\Gamma}}{2\pi}\left(-\mathrm{i}\ln(r) + \theta\right)$$



Polar velocity components:

$$u_r - \mathrm{i}u_\theta = We^{\mathrm{i}\theta} = \frac{\tilde{\Gamma}}{2\pi\mathrm{i}z}e^{\mathrm{i}\theta} = -\mathrm{i}\frac{\tilde{\Gamma}}{2\pi r} \quad \Longrightarrow \quad u_r = 0 \quad u_\theta = \frac{\Gamma}{2\pi r}$$

Circulation around the vortex:

$$\Gamma = \oint \vec{v} \cdot \mathrm{d}\vec{r} = \oint (\hat{\imath}_r v_r + \hat{\imath}_\theta v_\theta) \cdot (\hat{\imath}_r \,\mathrm{d}r + \hat{\imath}_\theta r \,\mathrm{d}\theta) = \oint \frac{\tilde{\Gamma}}{2\pi r} r \,\mathrm{d}\theta = \tilde{\Gamma}.$$

Exercise:

Since $\Gamma = \oint \vec{v} \cdot \mathrm{d}\vec{r}$ and the Stokes theorem states that

$$\oint \vec{v} \cdot \mathrm{d}\vec{r} = \int \omega_z \,\mathrm{d}x \,\mathrm{d}y,$$

what is the vorticity ω_z?

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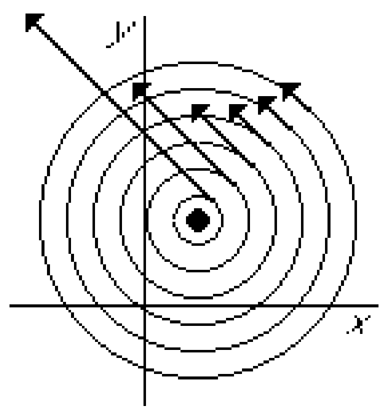
Vortex at a point z_0:



$$F = \frac{\tilde{\Gamma}}{2\pi\mathrm{i}} \ln(z - z_0)$$

You should now be able to do 18.6

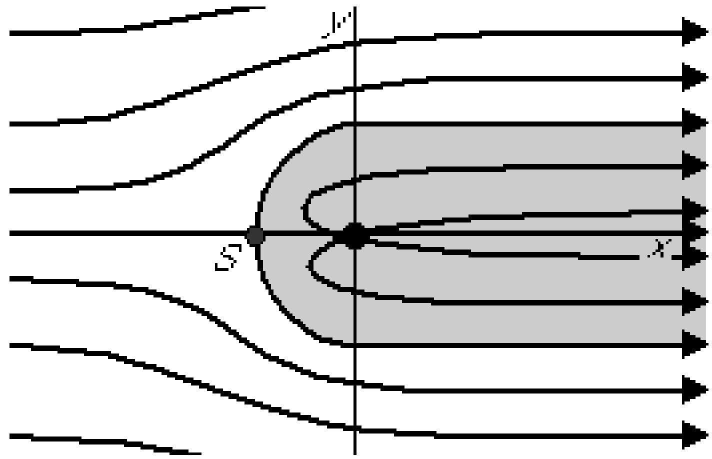
4 Nose

(Book 18.4)

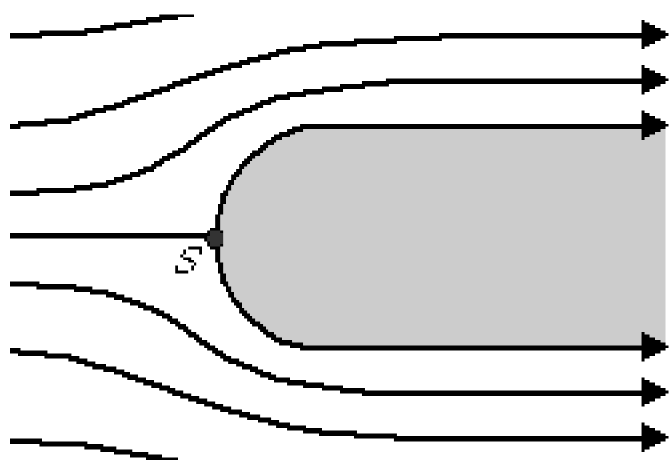
We can add simple flows and thus make more complicated ones. The reason we can do this is that the Laplace equation for the velocity potential is linear. Or, in terms of the complex potential, the sum of two differentiable functions is differentiable.

Add a source and a uniform flow:

$$F = Uz + \frac{m}{2\pi}\ln z \quad \begin{cases} \text{small } z: & F \approx \frac{m}{2\pi}\ln z \qquad \text{(isolated source)} \\ \text{large } z: & F \approx Uz \qquad \text{(uniform flow)} \end{cases}$$



We can replace the fluid coming out of the sink with a solid semi-infinite body:



Conversely, we can get the flow field around a body of this shape in a uniform stream by imagining a source inside the body.

Exercise:

Study the shape more closely:

- Find the location of the stagnation point for given U and m.
- Find the asymptotic thickness of the body.
- What is the mathematical equation for the contour of the body?

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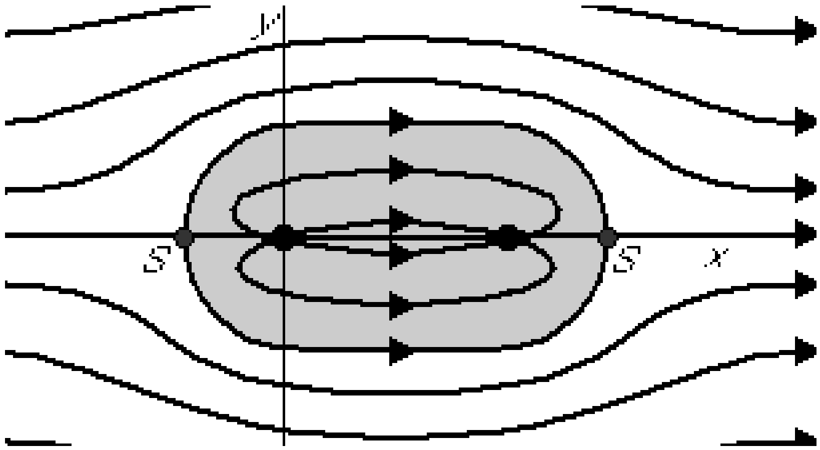
5 Cylinder

(Book 18.5, 6; 20.19)

We can create a finite body by placing a sink a distance e behind the source to absorb its fluid:



$$F = Uz + \frac{m}{2\pi}\Big(\ln z - \ln(z-e)\Big)$$

This is called a Rankine body.

If we let the sink and the source approach each other, $e = \epsilon \to 0$, while at the same time increasing their strength m, the body shape becomes circular:

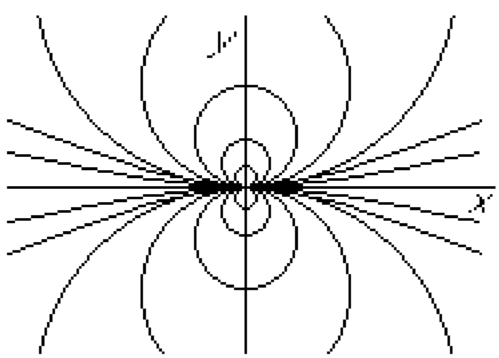
$$F = Uz + \frac{m}{2\pi}\Big(\ln z - \ln(z-\epsilon)\Big) = Uz + \frac{m}{2\pi}\ln\left(\frac{z}{z-\epsilon}\right)$$

$$= Uz + \frac{m}{2\pi}\ln\left(\frac{1}{1-(\epsilon/z)}\right) \approx Uz + \frac{m}{2\pi}\ln\left(1+\frac{\epsilon}{z}\right) \approx Uz + \frac{m}{2\pi}\frac{\epsilon}{z}$$

The second term is of the form

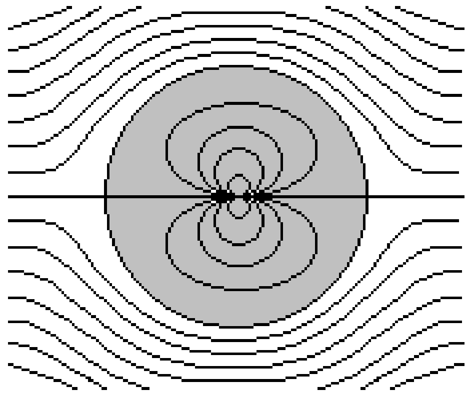
$$\boxed{F = \frac{\mu}{\pi z}}$$

which is called a *doublet*, with strength $\mu = \frac{1}{2}m\epsilon$. A doublet is the limit of an infinitely strong source and sink placed infinitely close together:



Changing the name of the constant once more, the potential for transverse flow past a circular cylinder becomes:

$$\boxed{F = U\left(z + \frac{r_0^2}{z}\right)}$$



Exercise:

Show that this is indeed flow past a circular cylinder and that the radius of the cylinder is r_0.

•

Exercise:

Assume that a cylinder of radius $r_0 = 1$ moves along the x-axis through an incompressible, irrotational fluid that is at rest at large distances from the cylinder. The position of the center is given as $z_0 = x_0(t)$. What is the complex velocity potential? (Hint: write the velocity potential $F_{\rm rel}$ in a coordinate system moving with the cylinder, then substract the coordinate system's translational velocity to find F.)

•

Exercise:

For the previous exercise, find the pressures along the surface of the cylinder, i.e. $x - x_0 = \cos\theta_{\rm rel}$, $y = \sin\theta_{\rm rel}$. Assume that the (gauge) pressure is zero at large distances from the cylinder. Ignore gravity.

•

Exercise:

To get a drag force, the pressures on the front half of the cylinder must be greater than those on the rear half. When is that the case? For motion at constant velocity, how much drag does the cylinder experience moving through this fluid? How much lift?

•

The derived solution only applies for short times after the flow is started by a gust of air or wind. After the air has moved a distance of about a diameter, the thin boundary layer around the cylinder breaks up:

http://www.eng.fsu.edu/~dommelen/research/ini2d/ini2dana.html

Pictures at the highest Reynolds number that can (hopefully) be computed accurately with current technology are at:

http://www.eng.fsu.edu/~dommelen/research/cylinder/cylinder.html

The boundary-layer break-up, or "separation" eventually leads to the formation of a large wake behind the cylinder. The streamlines outside the boundary layer and wake no longer follow the rear shape of the cylinder and the complex potential $U(z + r_0^2/z)$ is no longer valid.

The same separation process occurs on subsonic wings of aircraft if the aircraft flies at too large an angle of attack (in other words, too slow:)

http://www.eng.fsu.edu/~dommelen/research/airfoil/airfoil.html

Finally, no, the complex streamfunction solution *does not* apply to flow at very low Reynolds numbers, even though the streamlines look superficially the same:

Exercise:

Find the fluid velocity at the surface of the cylinder and thus establish that the complex potential $U(z + r_0^2/z)$ needs a thin boundary layer at the wall to satisfy no-slip.

•

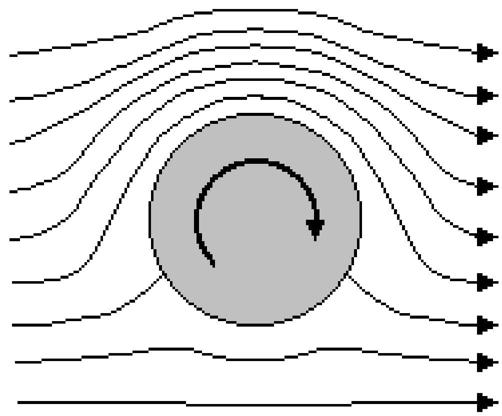
You should now be able to do 18.5

6 Circulation

(Book 18.7)

Add circulation to the flow past a cylinder to simulate cylinder rotation:

$$\boxed{F = U\left(z + \frac{r_0^2}{z}\right) + \frac{\mathrm{i}\Gamma}{2\pi}\ln z}$$



Exercise:

Show that this is still a flow past a cylinder.

•

Exercise:

Find the forces on the circular cylinder with transverse flow and circulation by integrating the wall pressure according to $F_i = -\oint p n_i \, \mathrm{d}s$

•

Exercise:

From the above, are the net forces on a steadily translating cylinder in an irrotational incompressible fluid still zero? How much work does the cylinder do on the fluid? Discuss the flow from an mechanical energy point of view.

•

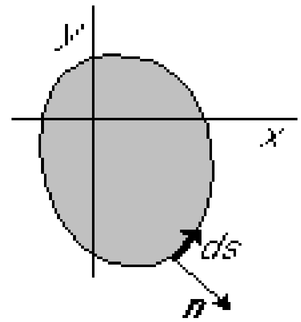
You should now be able to do The exercises above.

7 Forces

(Book 18.8)

7.1 Blasius

The Blasius theorem gives a convenient formula for the force on a two-dimensional body in an incompressible potential flow field.



The direct way to find the force on the body is to integrate the pressure forces over the surface:

$$F_x = \oint_{\text{body}} -pn_x \,\mathrm{d}s = -\oint_{\text{body}} pdy \qquad F_y = \oint_{\text{body}} -pn_y \,\mathrm{d}s = \oint_{\text{body}} pdx$$

(We ignore viscous stresses, if any.)

We can combine the force components in a complex conjugate force:

$$F_x - \mathrm{i}F_y = -\mathrm{i}\oint_{\text{body}} p\,\mathrm{d}\bar{z}$$

since $-\mathrm{i}d\bar{z} = -\mathrm{i}(\mathrm{d}x - \mathrm{i}\mathrm{d}y) = -\mathrm{i}\mathrm{d}x - \mathrm{d}y$

Assuming steady flow, the Bernoulli law gives:

$$F_x - \mathrm{i}F_y = \mathrm{i}\tfrac{1}{2}\rho\oint_{\text{body}} W\bar{W}\,\mathrm{d}\bar{z}$$

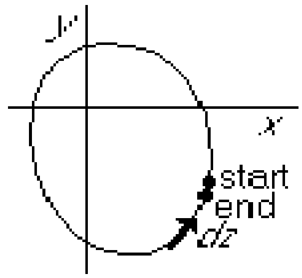
Now $\bar{W}\,\mathrm{d}\bar{z} = \mathrm{d}\bar{F} = \mathrm{d}\phi - \imath\,\mathrm{d}\psi = \mathrm{d}F$ since ψ is contant (the body is a streamline). So we get the Blasius formula:

$$\boxed{F_x - \mathrm{i}F_y = \mathrm{i}\tfrac{1}{2}\rho\oint_{\text{body}} W^2\,\mathrm{d}z}$$

In order to use this, we need to know how to do integrals of complex functions around closed contours.

7.2 Contour Integrals

We will look a bit at contour integrals of arbitrary complex functions f. Your first idea should be that a contour integral of a complex function, $\oint f\,\mathrm{d}z$ would be zero. After all, assuming the antiderivative of f exists, call it F, the integral would be the difference between F at start and end of integration. For a closed contour, start and end are the same.



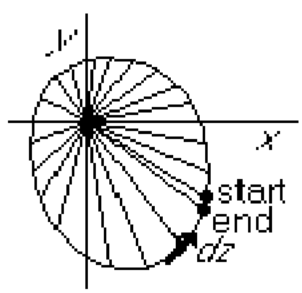
As an example,

$$\oint z^n \, \mathrm{d}z = \left. \frac{z^{n+1}}{n+1} \right|_{\text{start}}^{\text{end}} = 0 \qquad (n = 0, 1, 2, \ldots)$$

However, the integral is not always zero. F might be multiple-valued: in that case, we might need a different value for F at the end than at the beginning. For example, for any contour that goes once around the origin:

$$\oint \frac{1}{z} \, \mathrm{d}z = \ln(z)|_{\text{start}}^{\text{end}} = \ln(r) + \mathrm{i}\theta|_{\text{start}}^{\text{end}}$$

While r and hence $\ln(r)$ returns to the same value at the end, θ will have increased by an amount 2π, making the integral equal to $2\pi\mathrm{i}$.



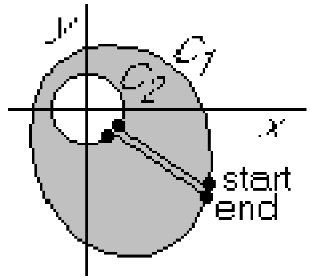
Note that function $f = 1/z$ is singular at $z = 0$. As this example shows, the value of a complex contour integral depends critically on singularities inside the contour:

$\boxed{\oint f \, \mathrm{d}z = 0 \text{ as long as } f \text{ is nonsingular inside and on the contour.}}$

As a consequence:

$\boxed{\text{The contour can be pushed around at will, as long as it does not cross a singularity.}}$

In the figure, the integral around contour C_1 is the same as around contour C_2 as long as there are no singularities inside the grey area:



Exercise:

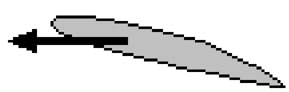
Integrate $\cos(z)/z$ along a circle of unit radius around the origin $z = 0$. Hint: contract the circle to a very small circle around the origin and then approximate.

•

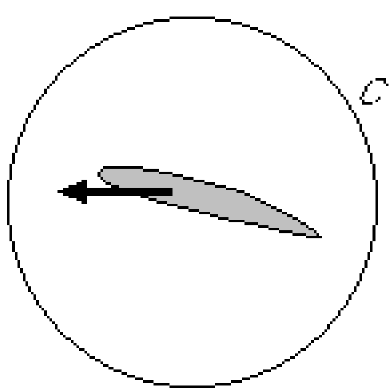
You should now be able to do The exercise above.

7.3 Far Field

Consider a two-dimensional body that moves through an irrotational, incompressible fluid. The fluid is at rest at large distances from the body.



Exercise:



Show that the circulation $\Gamma = \oint \vec{v} \cdot \mathrm{d}\vec{r}$ around the body is given by $\oint \mathrm{d}\phi = \Re\left(\oint \mathrm{d}F\right)$, where F is the complex velocity potential.

-

Since $\oint F\,\mathrm{d}z$ is a complex contour integral, we can push the contour of integration for Γ way out into the far field.

Exercise:

> For the same case, show that the volumetric flow rate $m = \oint \vec{v}\cdot\vec{n}\,\mathrm{d}s$ through a fixed contour around the body is given by $\oint \mathrm{d}\psi = \Im\left(\oint \mathrm{d}F\right)$, where F is the complex velocity potential. Why will this integral ordinarily be zero?

-

At large distances from the body, the complex conjugate velocity behaves as:

$$W \sim \frac{C}{z} + O\left(\frac{1}{z^2}\right)$$

After all, if we had logarithmic factors or broken powers, the velocity would not be unique. From W we find

$$F \sim C\ln(z) + O\left(\frac{1}{z}\right)$$

Note that the first term is in general a combination of a source (C real) and a vortex (C purely imaginary.)

Because the circulation and volumetric flow rate must still be the same at large distances

$$\boxed{F \sim \left(\frac{\Gamma}{2\pi\mathrm{i}} + \frac{m}{2\pi}\right)\ln(z)}$$

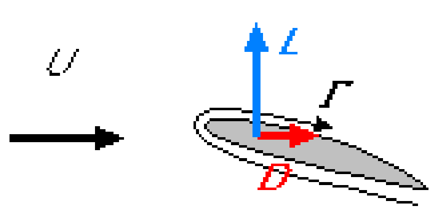
where m is the volumetric expansion rate of the body (zero for a solid body).

It follows that at large distances, a two-dimensional airfoil looks like a point vortex.

You should now be able to do The exercises above.

7.4 Laws

The aerodynamic forces on a body moving at constant velocity are the *drag* force D opposing the motion and the *lift* force L in the direction normal to the motion.



For unseparated incompressible potential flow past a two-dimensional body around which there is a circulation Γ:

D'Alembert's Paradox:

$$\boxed{D = 0}$$

According to D'Alembert, no energy is needed to overcome drag!

Kutta-Joukowski:

$$\boxed{L = \rho U \Gamma}$$

According to this, we can get nonzero lift without doing any work.

Proof of the laws: we saw before that at large distances, the velocity potential induced by the body approaches a vortex flow. So for large z:

$$F \sim Uz + \frac{\Gamma}{2\pi \mathrm{i}} \ln z + O\left(\frac{1}{z}\right) \qquad W \sim U + \frac{\Gamma}{2\pi \mathrm{i} z} + O\left(\frac{1}{z^2}\right)$$

Doing Blasius' integral along a very large circle around the body,

$$D - iL = \mathrm{i}\tfrac{1}{2}\rho \oint W^2 \, \mathrm{d}z = \mathrm{i}\tfrac{1}{2}\rho \oint U^2 + 2U\frac{\Gamma}{2\pi \mathrm{i} z} + O\left(\frac{1}{z^2}\right) \mathrm{d}z = \mathrm{i}\rho U \Gamma$$

Note that the circulation must be clockwise (increase the velocity above the airfoil) to produce upward lift.

Exercise:

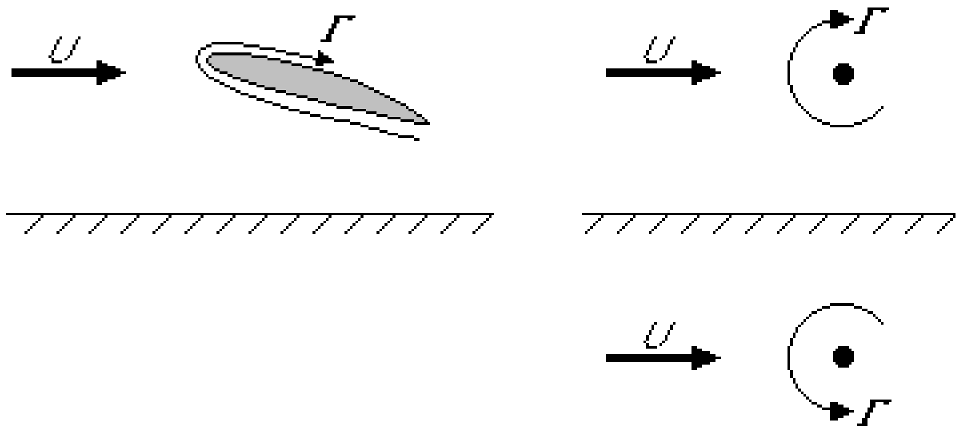
> Approximate the airfoil as a thin flat plate $0 < x < c$ where c is the cord length. Assume that the slip velocity above the plate is $U + u_u(x)$ and on the bottom $U + u_b(x)$ where U is the constant speed of flight. Use the Bernoulli law to find the pressures on the surface and then approximate for small u_u and u_b. Show that the Kutta-Joukowski value of the lift is obtained.

•

You should now be able to do The exercise above.

7.5 Mirror method

Given a two-dimensional airfoil . We want to approximate the effect of having a ground below the airfoil as compared to the airfoil in free space. To do so, we "mirror" the entire flow field into the ground. Then the ground is a streamline because of symmetry. It is really good enough to approximate the airfoil by a vortex while doing this:



By symmetry, the ground is now a streamline. The method of accounting for walls by mirroring the flow field into the wall is called the mirror method.

Exercise:

Write the combined potential of the two vortices and the uniform stream, and so verify that the ground is indeed a streamline.

•

It follows that the effect of the ground is the same as that of adding a mirror vortex below the ground. Near the airfoil, the effect of the second vortex (hence of the ground) is to decrease the effective wind velocity.

For three-dimensional wings, another effect of the mirror vortices is to reduce the induced drag caused by the trailing vortices.

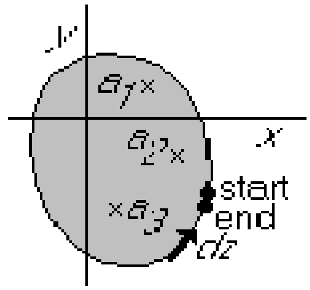
You should now be able to do 18.7, 19, 21

7.6 Residue theorem

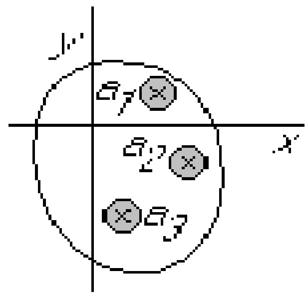
D'Alembert and Kutta-Joukowski only apply to bodies in an otherwise uniform stream. If there are other bodies or singularities (vortices, sources, ...) in the fluid, we need to actually integrate Blasius expression. This requires doing a complex contour integral.

We have already seen that the value of a complex contour integral $\oint f \, \mathrm{d}z$ depends on the singularities of f inside the contour: In fact, the quickest way to do this sort of integrals is usually to add the contributions of all the singularities together. These contributions are called the residues.

Let the function f have singularities at positions $z = a_1$, $z = a_2$. $z = a_3$, ... inside the contour:



The residues correspond to the integrals along small circles around the singular points:



The total integral can be written:

$$\oint f \, \mathrm{d}z = 2\pi \mathrm{i} \sum_{\text{all } i} \mathrm{Res}_f\,(a_i)$$

There are two ways to find the residue at a singular point $z = a_i$:

1. Use Taylor series expansions around the singular point $z = a_i$ to write f as a Laurent series:

$$\boxed{f \sim \frac{C_{-n}}{(z-a_i)^n} + \ldots + \frac{C_{-2}}{(z-a_i)^2} + \frac{C_{-1}}{z-a_i} + C_0 + C_1\,(z-a_i) + \ldots}$$

then

$$\boxed{\mathrm{Res}_f(a_i) = C_{-1}}$$

Note: To use the residue theorem, function f should only involve integer powers of $(z - a_i)$. Broken powers, logarithms, ... are not acceptable.

2. Use the following general formula:

$$\boxed{\mathrm{Res}_f\,(a_i) = \frac{1}{(n-1)!} \left\{ \frac{\mathrm{d}^{n-1}}{\mathrm{d}z^{n-1}} \left[(z-a_i)^n \, f \right] \right\}_{z=a_i}}$$

where n is chosen just big enough so that the term within the square brackets is *not* singular at $z = a_i$.

Exercise:

Evaluate $\oint \cot^3(z)\,\mathrm{d}z$ along circles of radius $r = 1, 2, 3, \ldots$ around the origin. Use Taylor series expansions for the sin and cos to find the Laurent series for the cot.

-

Exercise:

Evaluate $\oint \cot^3(z)\,\mathrm{d}z$ along circles of radius $r = 1, 2, 3, \ldots$ around the origin. Use the second method above.

-

As an example, the force on a Rankine body would be difficult to find by actual integration of the pressure. Using Blasius and the residue theorem, it is easy. The velocity potential is (uniform flow, source at $z = 0$, sink at $z = e$:)

$$F = Uz + \frac{m}{2\pi}\ln(z) - \frac{m}{2\pi}\ln(z-e),$$

so the complex conjugate velocity is:

$$W = U + \frac{m}{2\pi z} - \frac{m}{2\pi(z-e)}$$

There are two singular points; one at $z = 0$, the other at $z = e$. Blasius formula is:

$$F_x - \mathrm{i}F_y = \mathrm{i}\tfrac{1}{2}\rho \oint_{\text{body}} W^2\,\mathrm{d}z$$

Expanding W^2:

$$W^2 = U^2 + \frac{m^2}{4\pi^2 z^2} + \frac{m^2}{4\pi^2(z-e)^2} + 2U\frac{m}{2\pi z} - 2U\frac{m}{2\pi(z-e)} - \frac{m^2}{4\pi^2 z(z-e)}$$

To find the residue at $z = 0$, we can look at each term separately. The first, third, and fifth term do not have a residue since they are not singular at $z = 0$. The second term is a one-term Laurent series that does not have a $1/z$ term. The fourth term is a $1/z$ term with a coefficient (residue) $2Um/2\pi$. The residue of the sixth term can be found using the formula above with $n = 1$; it is $m^2/4\pi^2 e$:

$$\mathrm{Res}^2_W(0) = \frac{2Um}{2\pi} + \frac{m^2}{4\pi^2 e}$$

Similarly, at $z = e$ the first, second, and fourth terms are not singular. The third term is a one-term Laurent series with no $1/(z-e)$ term. The fifth term has a residue $-2Um/2\pi$, and the sixth has a residue $-m^2/4\pi^2 e$:

$$\mathrm{Res}^2_W(e) = -\frac{2Um}{2\pi} - \frac{m^2}{4\pi^2 e}$$

Since the sum of the residues is zero, there is no net force.

Exercise:

Use Blasius and the residue theorem to find the forces on a cylinder in a uniform stream U that has a circulation Γ. Compare with D'Alembert and Kutta-Joukowski.

•

You should now be able to do the exercises above.

8 Conformal Mappings

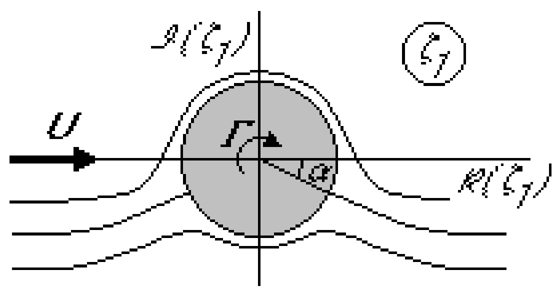
(Book 18.10)

Conformal transformations are a way to generate more complex flow fields from simple ones. They are based on distorting the *independent* variable: Suppose we are given a complex velocity potential $F(z)$ depending on the complex coordinate z. Now let ζ be another complex coordinate, then $F(\zeta)$ is also a complex velocity potential, provided only that ζ is a differentiable function of z.

Reason: according to the chain rule, the derivative of $F(\zeta)$ exists and is equal to:

$$\frac{\mathrm{d}F}{\mathrm{d}\zeta} = \frac{\mathrm{d}F}{\mathrm{d}z} \Big/ \frac{\mathrm{d}\zeta}{\mathrm{d}z}$$

As an example of a conformal transformation, we will find the complex potential around a *flat plate airfoil*, an airfoil of vanishing thickness. We start with a flow field in a ζ_1 plane that is simply a cylinder of unit radius with (clockwise) circulation Γ in a uniform stream U:

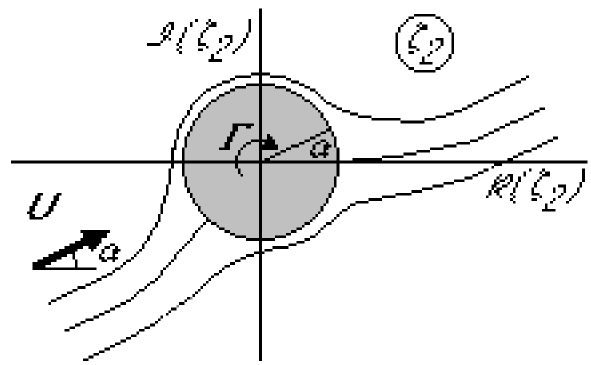


$$F = U\left(\zeta_1 + \frac{1}{\zeta_1}\right) - \frac{\Gamma}{2\pi \mathrm{i}} \ln \zeta_1$$

We are now going to deform this cylinder into a flat plate airfoil.

According to the *Kutta condition*, the rear stagnation point must be located at what will become the *trailing edge* of the airfoil. If the stagnation point is anywhere else, the flow must bend around the trailing edge:

Viscous effects do not allow that. So we must make sure that the rear stagnation point deforms into the trailing edge. To make this easier, we will first rotate the flow so that the rear stagnation point moves to the point 1:



We can rotate the independent coordinate counter-clockwise like this by multiplying with a complex exponential:

$$\boxed{\zeta_2 = \zeta_1 e^{\mathrm{i}\alpha}}$$

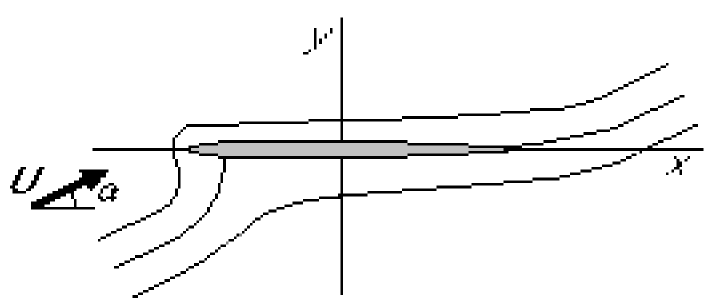
where α is the angular position of the rear stagnation point.

The potential is in terms of ζ_2:

$$\boxed{F = U\left(\zeta_2 e^{-\mathrm{i}\alpha} + \frac{e^{\mathrm{i}\alpha}}{\zeta_2}\right) - \frac{\Gamma}{2\pi\mathrm{i}} \ln \zeta_2}$$

(The constant arising from the logarithm is of no importance.)

Finally, we "squeeze the airfoil flat in the vertical direction" to a plate:



The transformation that does this is the Joukowski transformation:

$$\boxed{z = \zeta_2 + \frac{1}{\zeta_2}}$$

Exercise:

Show that the unit circle in the ζ_2-plane, corresponds to a flat plate on the x-axis in the z-plane.

•

Note that the rear stagnation point $\zeta_2 = 1$ becomes $z = 2$, while $\zeta_2 = -1$ becomes $z = -2$. As a result, the *chord* of the created airfoil is $c = 4$.

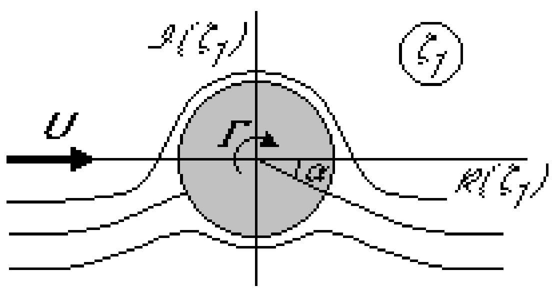
Since for large distances, $z \sim \zeta_2$, the potential at large distances is

$$F \sim U e^{-\mathrm{i}\alpha} z$$

which is a uniform flow at an angle alpha with the x-axis (the chord of the airfoil.) This angle between the chord and the direction of the incoming stream is called the *angle of attack* of the airfoil.

The lift generated by the airfoil, $L = \rho U \Gamma$ will be normal to the free stream U. This lift will depend on the angle of attack: the lift is zero by symmetry when $\alpha = 0$.

We now want to derive the relationship between the angle of attack α and the lift. Remember that in the ζ_1-plane, the angle α determines the location of the rear stagnation point:



$$\frac{\mathrm{d}F}{\mathrm{d}\zeta_1} = 0 = U\left(1 - \frac{1}{\zeta_1^2}\right) - \frac{\Gamma}{2\pi\mathrm{i}\zeta_1}$$

Writing ζ_1 in polar coordinates as $e^{-\mathrm{i}\alpha}$, we get

$$0 = U\left(1 - e^{2\mathrm{i}\alpha}\right) - \frac{\Gamma}{2\pi\mathrm{i}} e^{\mathrm{i}\alpha}$$

Dividing by $e^{\mathrm{i}\alpha}$ and solving for Γ:

$$\Gamma = 4\pi U \sin\alpha \quad \Longrightarrow \quad L = \rho U \Gamma = 4\pi\rho U^2 \sin\alpha$$

The lift of an airfoil is expressed in terms of a *lift coefficient:*

$$\boxed{c_l \equiv \frac{L}{\frac{1}{2}\rho U^2 c} = 2\pi\sin\alpha \approx 2\pi\alpha}$$

This formula is good approximation for the lift of thin airfoils in general. However, for airfoils that are not symmetric top/bottom, the angle α should be measured from the direction of zero lift, instead of from the chord.

Exercise:

Find the velocity on top and bottom of the flat plate airfoil as a function of x.

- **Exercise:**

  Integrate this velocity to find the circulation around the airfoil. Is it equal to Γ?

- **Exercise:**

  Integrate the pressure difference over the airfoil to find the net force on it.

- **Exercise:**

  Is the result of the previous exercise as you expected? If not, what do you think is the difficulty?

- **Exercise:**

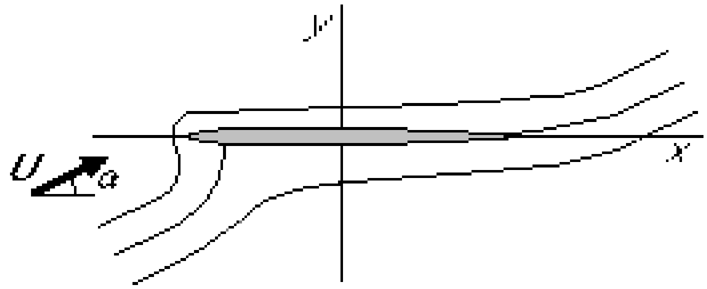
  Integrate the pressure difference over the the airfoil to find the moment around the center. From it , show that the resultant force acts at the point one-quarter chord behind the leading edge.

- You should now be able to do The exercises above, 18.10, 11

9 Joukowski Airfoils

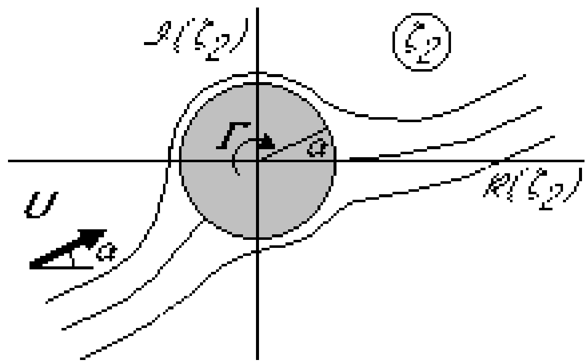
(Book 18.11)

The flat plate airfoil does not work well in practice, since the flow does not really want to bend about the sharp leading edge, like it does not want to bend around the sharp trailing edge:

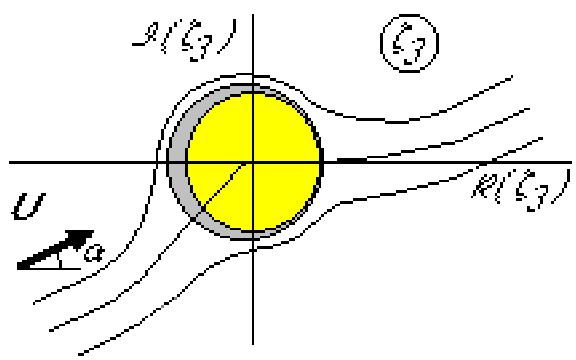


To get a decent airfoil, we need to round the leading edge.

Joukowski airfoils achieve a rounded leading edge as follows: after the mapping ζ_2:



they add an additional mapping ζ_3. This additional mapping leaves the trailing edge alone, but makes the cylinder slightly larger so that the leading edge is no longer on the circle that maps onto the plate:

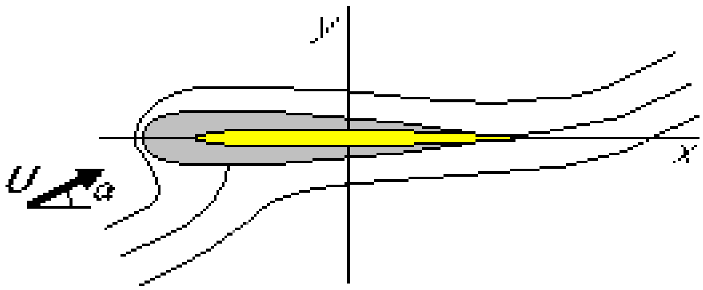


(The smaller yellow circle is the unit circle, the same size as the circle in the ζ_2 plane.) The mapping is

$$\zeta_3 - 1 = C\,(\zeta_2 - 1) \qquad C = 1 + \epsilon$$

Now we map the (yellow) unit circle again onto a plate using the Joukowki transormation:

$$z = \zeta_3 + \frac{1}{\zeta_3}$$



By giving C also a small imaginary part, we can also move the circle vertically, giving asymmetric airfoils.

Exercise:

Set $C = 1 + \epsilon - \mathrm{i}\delta$ and use your plotting software to plot the shape of the Joukowski airfoil for various small values of ϵ and δ. Use lots of points to plot.

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Joukowski airfoils have a trailing edge that ends in a cusp.

You should now be able to do the exercise above.