Navier Stokes Equations

1 Stress Tensor

$$T_{ij} = -p\delta_{ij} + \tau_{ij}$$

Inviscid fluid:

$$\boxed{\tau_{ij} = 0}$$

Newtonian fluid:

$$\boxed{\tau_{ij} = \lambda \frac{\partial v_k}{\partial x_k}\delta_{ij} + \mu\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right)}$$

$$\boxed{\tau_{ij} = \lambda\,\mathrm{div}\,(\vec{v})\,\delta_{ij} + 2\mu s_{ij}}$$

Stokes Hypothesis:

$$\lambda = -\tfrac{2}{3}\mu$$

You should now be able to do 6.1, 2, 3, 4, 5, 7.

2 Heat Flux

Fourier Law:

$$\boxed{q_i = -k\frac{\partial T}{\partial x_i} \qquad \vec{q} = -k\nabla T}$$

3 Navier-Stokes Equations

The governing equations for a Newtonian fluid satisfying Fourier's law are called the Navier-Stokes equations.
Compressible flow:
The students can write them out themselves.
Incompressible flow:
Assuming constant viscosity, the net stress force per unit volume simplifies a bit:

$$\frac{\partial T_{ji}}{\partial x_j} = \frac{\partial}{\partial x_j}\left[-p\delta_{ij} + \mu\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right)\right] = -\frac{\partial p}{\partial x_i} + \mu\frac{\partial^2 v_i}{\partial x_j \partial x_j}$$

Conservation form:

$$\boxed{\frac{\partial v_i}{\partial x_i} = 0 \qquad \frac{\partial \rho v_i}{\partial t} + \frac{\partial \rho v_j v_i}{\partial x_j} = -\frac{\partial p}{\partial x_i} + \mu\frac{\partial^2 v_i}{\partial x_j \partial x_j}}$$

Nonconservation form:

$$\boxed{\frac{\partial v_i}{\partial x_i} = 0 \qquad \rho\frac{\partial v_i}{\partial t} + \rho v_j\frac{\partial v_i}{\partial x_j} = -\frac{\partial p}{\partial x_i} + \mu\frac{\partial^2 v_i}{\partial x_j \partial x_j}}$$

Vector form:

$$\boxed{\mathrm{div}\,(\vec{v}) = 0 \qquad \rho\frac{D\vec{v}}{Dt} = -\nabla p + \mu\nabla^2\vec{v}}$$

Bottom line: for incompressible flow, the net pressure force is the gradient of the pressure (as always) and the net viscous force the Laplacian of the velocity.
Exercise:

> What happened to the energy equation?

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