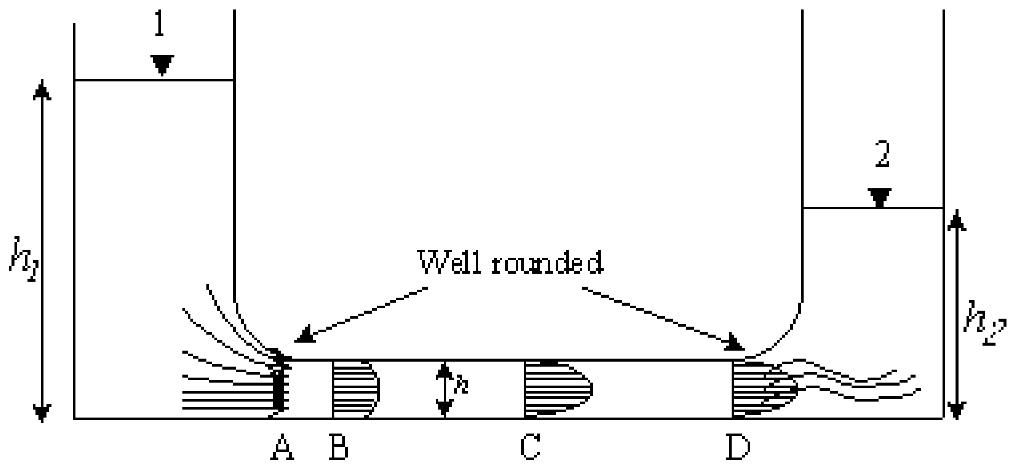
Incompressible Flows

1 Duct Flow



Bernoulli:

$$p_1 + \frac{1}{2}\rho V_1^2 + \rho g h_1 = p_2 + \frac{1}{2}\rho V_2^2 + \rho g h_2$$

Since $p_1 = p_2 = p_a$ and $V_1 = V_2$:

$$\rho g h_1 = \rho g h_2$$

hence $h_1 = h_2$.
Exercise:

What is wrong in this analysis?

1. Is $p_1 = p_2 = p_a$ correct?
2. Is $V_1 = V_2$ correct?
3. When does the Bernoulli law apply?
4. How about the energy balance?
5. Is the Bernoulli law the correct one?
6. Is the length of the connecting duct AD important?

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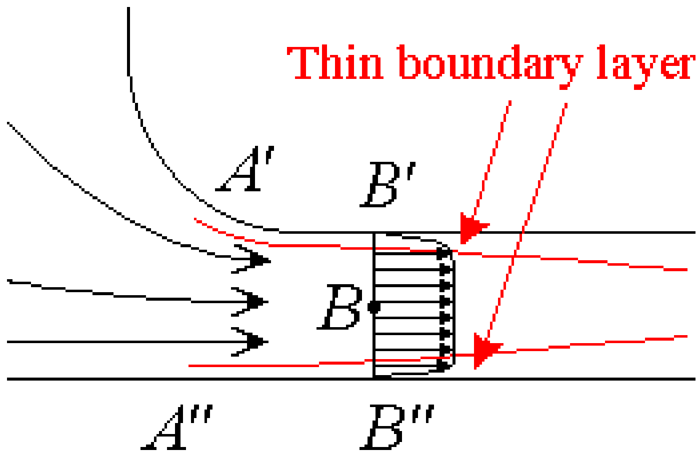
1.1 Entrance



Near the entrance we can learn a lot from Bernoulli:

$$p_1 + \frac{1}{2}\rho V_1^2 + \rho g h_1 = p_2 + \frac{1}{2}V_2^2 + \rho g h_2$$

Exercise:

Explain why $p_{B'} \approx p_B \approx p_{B''}$ although $V_B \neq 0$ while $V_{B'} = V_{B''} = 0$ and $h_{B'} \approx h_B \approx h_{B''}$

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Exercise:

Which one is larger:

- p_A or $p_{A'}$?
- V_A or $V_{A'}$?

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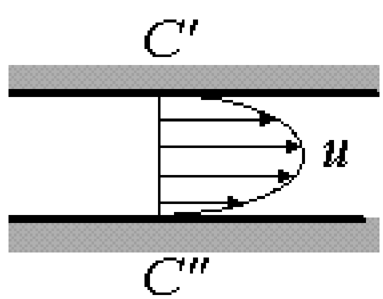
1.2 Developed



At and beyond station C the flow is called developed. We will assume that the flow is nonturbulent and that the streamlines have become parallel. These assumptions allow us to solve the incompressible Navier-Stokes equations exactly!
Note that for parallel streamlines, (unidirectional flow), $v = 0$.
Continuity:

$$\text{div}\,(\vec{v}) = 0 = \frac{\partial u}{\partial x} + \frac{\partial \not{v}}{\not{\partial} y} \quad \Longrightarrow \quad u = u(y)$$

y-momentum:

$$\rho\cancel{\frac{Dv}{Dt}} = -\rho g - \frac{\partial p}{\partial y} + \mu\nabla^2\cancel{v} \quad\Longrightarrow\quad p = -\rho g y + P(x)$$

x-momentum:

$$\rho\cancel{\frac{\partial u}{\partial t}} + \rho u\cancel{\frac{\partial u}{\partial x}} + \rho\cancel{v}\frac{\partial u}{\partial y} = -\frac{\partial p}{\partial x} + \mu\left(\cancel{\frac{\partial^2 u}{\partial x^2}} + \frac{\partial^2 u}{\partial y^2}\right)$$

hence:

$$\frac{\mathrm{d}^2 u}{\mathrm{d}y^2} = \frac{1}{\mu}\frac{\mathrm{d}P}{\mathrm{d}x} = \text{ constant}$$

Exercise:

Why is it constant? If the duct is long, how can you approximate the constant? What is the sign of $\mathrm{d}P/\mathrm{d}x$?

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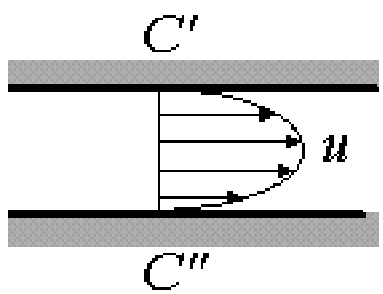
$$\frac{\mathrm{d}u}{\mathrm{d}y} = \frac{1}{\mu}\frac{\mathrm{d}P}{\mathrm{d}x}y + A$$

$$u = \frac{1}{2\mu}\frac{\mathrm{d}P}{\mathrm{d}x}y^2 + Ay + B$$



The boundary conditions $u(0) = u(h) = 0$ give the constants:

$$\boxed{\frac{\mathrm{d}P}{\mathrm{d}x} = \text{constant} \quad u = -\frac{1}{2\mu}\frac{\mathrm{d}P}{\mathrm{d}x}(h-y)y \quad v = 0 \quad p = -\rho g y + \frac{\mathrm{d}P}{\mathrm{d}x}x + \text{constant}}$$

Maximum velocity:

$$v_{\max} = -\frac{h^2}{8\mu}\frac{\mathrm{d}P}{\mathrm{d}x} \qquad u = \frac{4(h-y)y}{h^2}v_{\max}$$

Mass flux (per unit span):

$$\dot{m} = \int \rho\vec{v}\cdot\vec{n}\,\mathrm{d}S = \int_0^h \rho u\,\mathrm{d}y = \tfrac{2}{3}\rho v_{\max} h$$

The volumetric flow rate $Q = \dot{m}/\rho$
Average velocity:

$$\boxed{\dot{m} \equiv \rho Q \equiv \rho v_{\text{ave}} h \qquad v_{\text{ave}} = \tfrac{2}{3}v_{\max}}$$

Note that $Q = v_{\text{ave}}h$ and $\dot{m} = \rho v_{\text{ave}}h$.

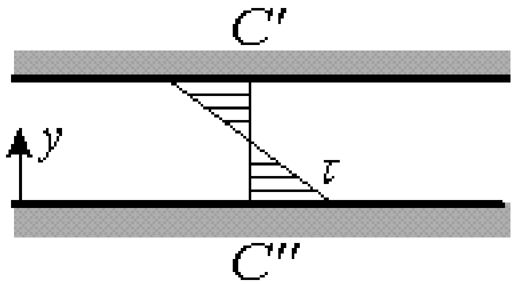
$$\boxed{\frac{\mathrm{d}P}{\mathrm{d}x} = -\frac{12\mu}{h^2}v_{\text{ave}}}$$

Vorticity:

$$\vec{\omega} = \hat{k}\left(\cancel{\frac{\partial v}{\partial x}} - \frac{\partial u}{\partial y}\right) = \hat{k}\frac{1}{2\mu}\frac{\mathrm{d}P}{\mathrm{d}x}(h-2y)$$

Shear:

$$\tau \equiv \tau_{xy} = \mu \left(\cancel{\frac{\partial v}{\partial x}} + \frac{\partial u}{\partial y} \right) = -\frac{1}{2} \frac{\mathrm{d}P}{\mathrm{d}x} (h - 2y)$$



Exercise:

Verify the integral momentum equation for any duct length L.

- What is the rate of change of linear momentum inside?
- What is the net outflow of momentum through the boundary?
- What are the forces on the control volume?

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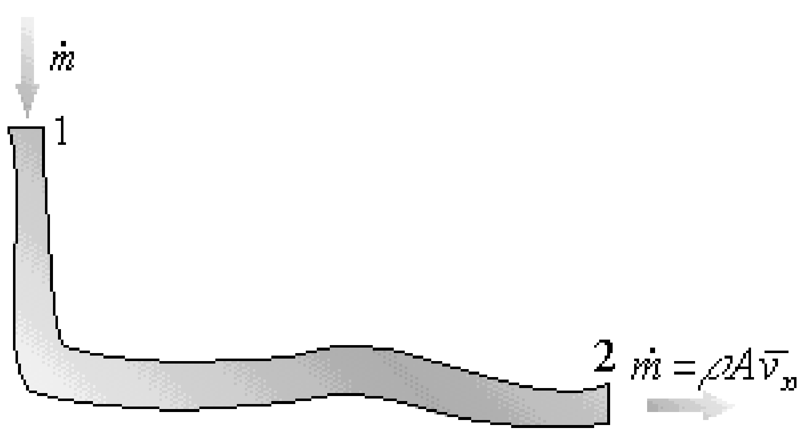
Note that this flow becomes turbulent at a Reynolds number of say 1,500 (in the range from 1,000 to 8,000). The above expressions do not apply to turbulent flow.
You should now be able to do 7.1, 2, and 5 to 11 Notes:

- In 7.5, write and solve the Navier-Stokes equations in cylindrical coordinates assuming that $\vec{v} = \hat{\imath}_\theta v_\theta(r)$
- In 7.9, write and solve the Navier-Stokes equations in cylindrical coordinates assuming that $\vec{v} = \hat{\imath}_z v_z(r)$
- In 7.11, write and solve the Navier-Stokes equations in 2D planar flow for each fluid separately assuming that $\vec{v}_1 = \hat{\imath} u_1(y)$ and $\vec{v}_2 = \hat{\imath} u_2(y)$. Carefully consider the boundary condition where the fluids meet.

2 Head Loss

Steady incompressible flows through pipes are very important for many applications. In the simplest case we will have a single duct with a mass flux $\dot{m} = \rho Q = \rho v S$ through it:



Note that according to continuity, $\dot{m}$ is constant, so that the average velocity v increases when S becomes smaller.

Ideally, the flow would be inviscid (no dissipation) and in each cross section the velocity, pressure and the height would be constant. In that case the Bernoulli law applies as:

$$\frac{p_2}{\rho} + \tfrac{1}{2}v_2^2 + gh_2 = \frac{p_1}{\rho} + \tfrac{1}{2}v_1^2 + gh_1 = \text{constant}$$

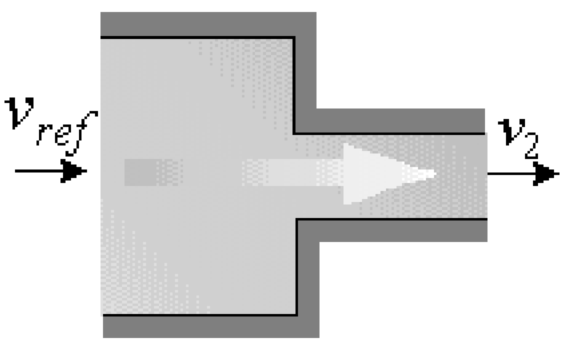
In real flows with dissipation and nonuniform velocity in the cross sections, we can write

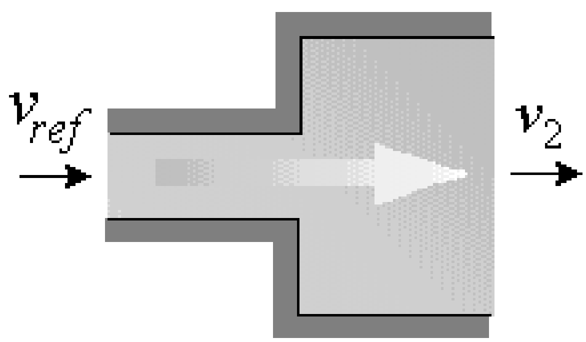
$$\frac{p_2}{\rho} + \tfrac{1}{2}\alpha_2 v_2^2 + gh_2 = \frac{p_1}{\rho} + \tfrac{1}{2}\alpha_1 v_1^2 + gh_1 - h_l$$

where h_l is the *head loss*, the effect of irreversible dissipation of energy. Also, $\frac{1}{2}\alpha v^2$ is the average kinetic energy per unit mass of the fluid at the cross section; $\alpha = 1$ as long as the flow is uniform at the cross section. For the developed duct with the parabolic profile, $\alpha = 54/35 \approx 1.5$. For laminar pipe flow, $\alpha = 2$. For turbulent flows, α is usually not very far from 1. Finally, p and h are the average pressure and height of the cross section.
Note: The above equation can be derived by integrating the mechanical energy equation over the duct. It may then be verified that all averages are weighted over the mass flux. The exception is v, which is still the plain average velocity.
Note: When treating air at low velocities as incompressible, use a single density: do not use p_1/ρ_1 and p_2/ρ_2, even if both densities are known precisely.

2.1 Values

Typical head loss values for important situations may be found in tables. For bends and area changes, they can be expressed as a head loss coefficient: $h_l = K\frac{1}{2}v_{\text{ref}}^2$.





Exercise:

Why express the headloss in terms of $\frac{1}{2}v_{\text{ref}}^2$? Why not, say, p_{ref}/ρ?

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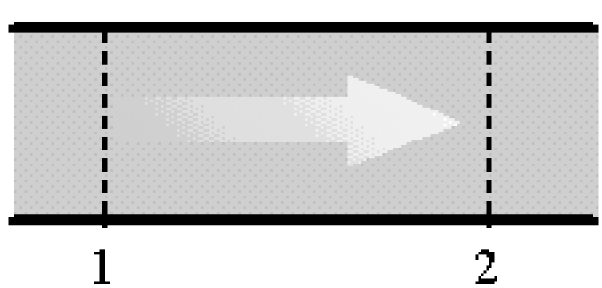
For the developed two-dimensional duct flow in the previous subsection, the head loss over a distance L of the duct is:

$$\frac{p_2}{\rho} + \tfrac{1}{2}\cancel{\alpha_2} v_2^2 + g\cancel{h_2} = \frac{p_1}{\rho} + \tfrac{1}{2}\cancel{\alpha_1} v_1^2 + g\cancel{h_1} - h_l$$



$$h_l = \frac{p_1 - p_2}{\rho} = -\frac{1}{\rho}\frac{\mathrm{d}P}{\mathrm{d}x}L = \frac{24\mu}{\rho v_{\text{ave}} h}\tfrac{1}{2}v_{\text{ave}}^2 \frac{L}{h}$$

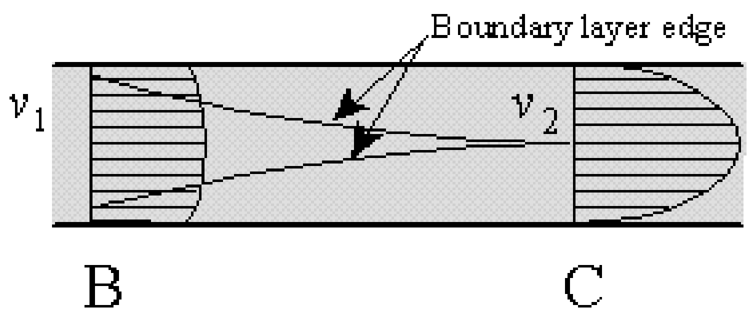
This head loss (called major head loss) can be given in terms of a friction factor:

$$\boxed{f_{\text{laminar duct}} = \frac{24\mu}{\rho v_{\text{ave}} h} = \frac{24}{Re_h} \qquad K = f\frac{L}{h}}$$

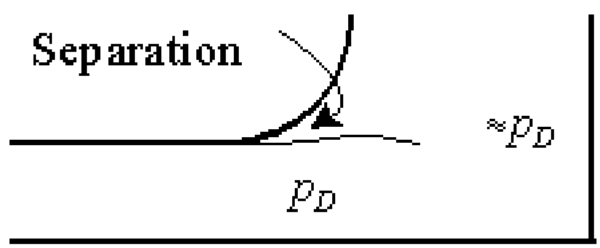
For laminar flow in a circular pipe,

$$\boxed{f_{\text{laminar pipe}} = \frac{64}{Re_D}}$$

There will be an additional head loss for the entrance effects (called minor head loss):

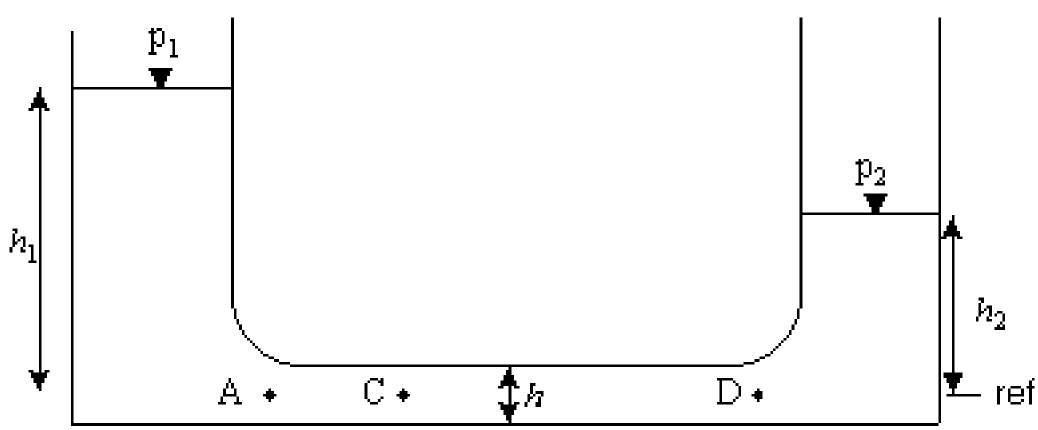


For the duct exit, the kinetic energy will probably be mostly lost:



You should now be able to do 7.3 and 4

2.2 Finish



$$0 = p_1 - p_A + p_A - p_D + p_D - p_2$$

Individual pressure differences:

$$\frac{p_1}{\rho} + gh_1 = \frac{p_A}{\rho} + \tfrac{1}{2}v^2$$

$$\frac{p_A}{\rho} + \tfrac{1}{2}v^2 = \frac{p_D}{\rho} + \alpha\tfrac{1}{2}v^2 + K_e\tfrac{1}{2}v^2 + f\frac{L}{h}\tfrac{1}{2}v^2$$

$$\frac{p_D}{\rho} + \alpha\tfrac{1}{2}v^2 = \frac{p_2}{\rho} + gh_2 + K_D\tfrac{1}{2}v^2 + gh_2$$

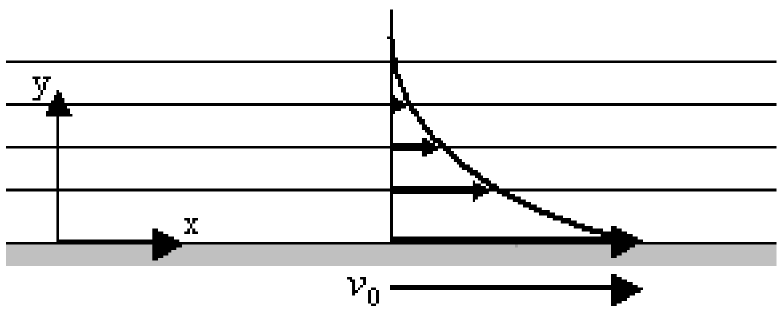
Total:

$$gh_1 - gh_2 = K_e\tfrac{1}{2}v^2 + f\frac{L}{h}\tfrac{1}{2}v^2 + K_D\tfrac{1}{2}v^2$$

where v is the average velocity in the duct.

3 Stokes' 2nd

Stokes' second problem, also erroneously called Rayleigh flow:



Continuity:

$$\operatorname{div}(\vec{v}) = 0 = \frac{\partial u}{\partial x} + \cancel{\frac{\partial v}{\partial y}} \quad \Longrightarrow \quad u = u(y,t)$$

y-momentum:

$$\rho\cancel{\frac{Dv}{Dt}} = -\rho g - \frac{\partial p}{\partial y} + \cancel{\mu\nabla^2 v} \quad \Longrightarrow \quad p = -\rho g y + P(\cancel{x},t)$$

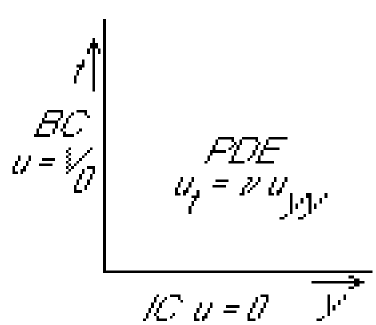
x-momentum:

$$\rho\frac{\partial u}{\partial t} + \cancel{\rho u\frac{\partial u}{\partial x}} + \cancel{\rho v}\frac{\partial u}{\partial y} = -\cancel{\frac{\partial p}{\partial x}} + \mu\left(\cancel{\frac{\partial^2 u}{\partial x^2}} + \frac{\partial^2 u}{\partial y^2}\right)$$

The x-momentum equation becomes:

$$u_t = \nu u_{yy}$$

where $\nu = \mu/\rho$ is the dynamic viscosity.



Exercise:

How would you normally find u?

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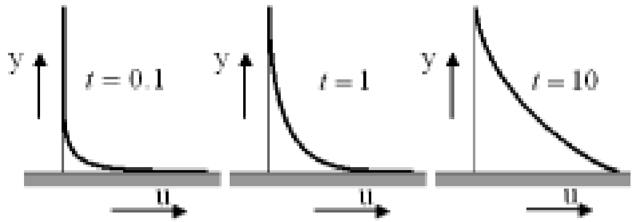
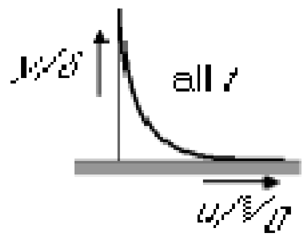
A simpler way to solve is to guess that the solution is *similar*: after rescaling u and y, all velocity profiles look the same.

Original profiles:



Supposed shape after scaling u with V_0, and y with a characteristic *boundary layer thickness* δ that increases with time:



Mathematical form of the similarity assumption:

$$\frac{u}{V_0} = f\left(\frac{y}{\delta(t)}, \cancel{t}\right)$$

The proof is in the pudding; if it satisfies the P.D.E., I.C., and B.C., it is OK.

$$u_t = \nu u_{yy} \quad \Longrightarrow \quad -V_0 f' \frac{y}{\delta^2}\delta_t = \nu V_0 f'' \frac{1}{\delta^2}$$

Put $\eta = y/\delta$:

$$-V_0 f' \eta \frac{\delta_t}{\delta} = \nu V_0 f'' \frac{1}{\delta^2}$$

Separate into terms depending only on η and terms depending only on t:

$$-\frac{f'\eta}{f''} = \frac{\nu}{\delta\delta_t} = \text{constant} = \frac{1}{2}$$

It does not make a difference what you take the constant; this merely changes the value of δ, not the physical solution.

Solving the O.D.E.s for δ and f, we solve the P.D.E. For the boundary layer thickness $\delta\delta_t = 2\nu$ so

$$\delta = \sqrt{4\nu t}$$

For the velocity profile $f'' = -2\eta f'$ hence

$$f = \text{erfc}(\eta)$$

where erfc is the *complementary error function* defined as

$$\boxed{\text{erfc}(x) \equiv \frac{2}{\sqrt{\pi}} \int_x^\infty e^{-\xi^2}\, \mathrm{d}\xi}$$

Exercise:

Derive the expressions for δ and f.

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Total:

$$\boxed{u = V_0 \ \text{erfc}\left(\frac{y}{\delta}\right) \qquad \delta = \sqrt{4\nu t}}$$

You should now be able to do 7.14, 16, 17