Analysis in ME II
EML 4930/5061
Homework

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Do not print out all pages. Keep checking for changes. Complete assignment will normally be available the day after the last lecture whose material is included in the assignment (Monday, normally).

Contents

1 HW 1

Use vector analysis wherever possible.

1. 1st Ed: p13, q31a-f,h-j, 2nd Ed: p17, q31a-i. *if they can be vectors, count them as such.*

2. 1st Ed: p13, q32, 2nd Ed: p17, q32. Do it both graphically and analytically. Give length and angle.

3. 1st Ed: p14, q48, 2nd Ed: p19, q46. Use vector calculus only, no trig. No scalar equations at all. That includes cosine or sine rule.

4. 1st Ed: p32, q66, 2nd Ed: p38, q66. Use vector only, except when working out the final numbers.

5. 1st Ed: p32, q82, 2nd Ed: p40, q82a, where **B** should be corrected to $(1, -3, 4)$. Vector calculus only, no trig. Do it without finding the actual sides of the parallelogram. In particular, show that the area is half of $\vec{A} \times \vec{B}$. Also give a unit vector normal to the plane of the parallelogram.

6. 1st Ed: p33, q90, 2nd Ed: p41, q90a. Also give the area of the parallelogram with sides $\vec{B}$ and $\vec{C}$.

7. 1st Ed: p53, q32, 2nd Ed: p64, q32. Draw the curve neatly.

2 HW 2

1. Find the components of the acceleration in polar coordinates by differentiating the expression for the velocity in these coordinates. Identify the components a_r and a_θ.

2. 1st Ed: p78, q46, 2nd Ed: p91, q46. $r = \sqrt{x^2 + y^2 + z^2}$

3. 1st Ed: p78, q54, 2nd Ed: p92, q54. You may want to refresh your memory on total derivatives.

4. The height of the ground above sea level is $\sin(x)\sin(2y)$.

   (a) Draw the contour lines.

   (b) Consider the point $x = 0.5$ and $y = 1.5$. Find the gradient of height at that point and draw it in the graph.

   (c) If I want to climb at the fastest rate, for a given speed, in which direction should I move at that point? In particular, what is $\mathrm{d}y/\mathrm{d}x$?

   (d) If I am traveling along the line $y = 3x$ with speed 60, (ignoring the vertical component of velocity), how rapidly am I changing height?

5. (6 points). 1st Ed: p78, q60, 2nd Ed: p92, q60. Also find two scalar equations that describe the line through P that crosses the surface normally at P.

   Find the unit normal $\vec{n}$ to the surface at P. Now assume that the surface is reflective, satisfying Snell's law. An incoming light beam parallel to the x-axis hits the surface at P. Find a vector equation that describes the path of the *reflected* beam.

   Hint: let $\vec{v}$ be a vector along the light ray. The *component* of $\vec{v}$ in the direction of $\vec{n}$ is $\vec{n} \cdot \vec{v}$. The *component vector* in the direction of $\vec{n}$ is defined as $\vec{v}_1 = \vec{n}(\vec{n} \cdot \vec{v})$. Sketch this vector along with vector $\vec{n}$. In which direction is the remainder $\vec{v}_2 = \vec{v} - \vec{v}_1$? Now figure out what happens to $\vec{v}_1$ and $\vec{v}_2$ during the reflection. Take it from there.

3 HW 3

1. 1st Ed: p80, q102, 2nd Ed: p94, q102. Make sure that you find ϕ in a mathematically sound way, as discussed in class. No messing around and guessing a solution!

2. (9 points). Modified version of a question in the book. Maxwell's equations in vacuum are

$$\boxed{\begin{array}{llll} \nabla \times \vec{H} = \dfrac{1}{c}\dfrac{\partial \vec{E}}{\partial t} + \dfrac{4\pi}{c}\vec{\jmath} & \textit{(a)} & \nabla \times \vec{E} = -\dfrac{1}{c}\dfrac{\partial \vec{H}}{\partial t} & \textit{(b)} \\ \nabla \cdot \vec{H} = 0 & \textit{(c)} & \nabla \cdot \vec{E} = 4\pi\rho & \textit{(d)} \end{array}}$$

Here $\vec{E}$ is the electric field, $\vec{H}$ the magnetic field, ρ the charge density (the electric charge per unit volume), $\vec{\jmath}$ the current density (the current flowing per unit cross sectional area), and c the speed of light, a constant. Consider ρ and $\vec{\jmath}$ to be given functions of position and time. You need to show that any solution $\vec{E}, \vec{H}$ of the above equations is given by scalar and vector potentials $\phi, \vec{A}$ as described below.

Procedure to follow:

1. Explain why there must be a "vector potential" $\vec{A}_0$ so that
$$\vec{H} = \nabla \times \vec{A}_0$$

2. Next *define* a vector $\vec{E}_\phi$ by setting
$$\vec{E} = -\frac{1}{c}\frac{\partial \vec{A}_0}{\partial t} + \vec{E}_\phi$$

3. Prove that the $\vec{E}_\phi$ defined this way is minus the gradient of some "scalar potential" ϕ_0. Then the above equation becomes:
$$\vec{E} = -\frac{1}{c}\frac{\partial \vec{A}_0}{\partial t} - \nabla\phi_0$$

4. Unfortunately, $\vec{A}_0$ and ϕ_0 are not unique. We now want, given potentials $\vec{A}_0$ and ϕ_0, find modified potentials $\vec{A}$ and ϕ. These must still give
$$\boxed{\vec{H} = \nabla \times \vec{A} \quad \textit{(e)} \qquad \vec{E} = -\frac{1}{c}\frac{\partial \vec{A}}{\partial t} - \nabla\phi \quad \textit{(f)}}$$
However, in addition they must satisfy the famous "Lorenz condition"
$$\boxed{\nabla \cdot \vec{A} + \frac{1}{c}\frac{\partial \phi}{\partial t} = 0 \quad \textit{(1)}}$$
(No, there is no t in Lorenz. That is another Lorentz. The Lorenz condition is critical, because it is the only condition that all observers can agree on.) The potentials you need are of the form
$$\vec{A} = \vec{A}_0 + \nabla\psi \qquad \phi = \phi_0 - \frac{1}{c}\frac{\partial \psi}{\partial t}$$

Prove that in those terms, *(e)* and *(f)* above are true *regardless of what you take for* ψ. That is the famous "gauge property" of the electromagnetic field. It is central to quantum field theory. It *defines* the electromagnetic field in modern quantum theories, all the rest is derived.

5. Since you can take ψ whatever you like, you can choose it so that the Lorenz condition *(1)* is satisfied. Show that this leads to a partial differential equation for ψ. (This equation is called an inhomogeneous wave equation. The properties of this equation will be discussed in the second part of the class.)
6. Now substitute what you got so far into the four Maxwell equations and so find the requirements that $\vec{A}$ and ϕ must satisfy. (I.e. get rid of the electric and magnetic fields in favor of the vector and scalar potentials $\vec{A}$ and ϕ.)
7. How come only one vector equation and one scalar equation are left?
8. Clean up! You must obtain decoupled equations for the scalar and vector potentials.
9. Finally, combine *(a)* and *(d)* to get a relation between the charge and current densities. (This equation is similar to the continuity equation in incompressible flow and expresses that no charge can be created out of nothing.)

3. 1st Ed: p103, q44, 2nd Ed: p123, q44. (a) Use vector line integration. (b) Do it using Stokes.

4. CHANGE BELOW 1st Ed: p104, q62, 2nd Ed: p124, q62. Do the surface integrals both directly and using the divergence theorem. Make sure to include the flat circle of the cone. Note: in doing the surface integrals directly, you are required to write them down in Cartesian coordinates using the expression for $\vec{n}\,\mathrm{d}S$ given in class when $F(x, y, z) = 0$. After that, switch to polar coordinates to actually do the integration.

5. MODIFIED version of 1st Ed: p132, q50, 2nd Ed: p154, q50. Given

$$\vec{v} = \frac{(-y, x)}{x^2 + y^2}$$

   1. Evaluate $\nabla \times \vec{v}$.
   2. Also evaluate, presumably using polar coordinates,

$$\int_{\mathrm{I}} \vec{v} \cdot \mathrm{d}\vec{r} \qquad \int_{\mathrm{II}} \vec{v} \cdot \mathrm{d}\vec{r}$$

      where path I is the semi circle of radius r going clockwise from $(r, 0)$ to $(-r, 0)$, and path II is the semi circle of radius r going counter-clockwise from $(r, 0)$ to $(-r, 0)$.
   3. Explain why the integral over II minus the integral over I is the integral over the closed circle.

4. Explain why Stokes implies that the closed contour integral should be the integral of the z-component of $\nabla \times \vec{v}$ over the inside of the circle.
5. Then explain why you would then normally expect the contour integral to be zero. That means that the two integrals I and II should be equal, but they are not.
6. Explain what the problem is.
7. Do you expect integrals over closed circles of different radii to be equal? Why?
8. Are they actually equal?

Now assume that you allow singular functions to be OK, like Heaviside step functions and Dirac delta functions say. Then figure out in what part of the interior of the circle, $\int\int \nabla \times \vec{v} \cdot \hat{k}\, \mathrm{d}x\mathrm{d}y$ is not zero. So how would you describe $\nabla \times \vec{v}$ for this vector field in terms of singular functions?

4 HW 4

1. Derive $\vec{n}\,\mathrm{d}S$ in terms of $\mathrm{d}\theta$ and $\mathrm{d}\phi$, where (r,θ,ϕ) are spherical coordinates. Assume that the surface is described as $r = f(\theta,\phi)$ for some given function f. Use the formulae given earlier in class for $\vec{n}\,\mathrm{d}S$ in terms of two parameters $u = \theta$ and $v = \phi$. The formula requires you to differentiate $\vec{r}$ with respect to the parameters. Now in spherical,

$$\vec{r} = r\hat{\imath}_r$$

From class, the derivatives of $\hat{\imath}_r$ are

$$\frac{\partial\,\hat{\imath}_r}{\partial r} = 0 \qquad \frac{\partial\,\hat{\imath}_r}{\partial\theta} = \hat{\imath}_\theta \qquad \frac{\partial\,\hat{\imath}_r}{\partial\phi} = \sin\theta\,\hat{\imath}_\phi$$

So you can now write $\vec{n}\,\mathrm{d}S$ in terms of r and the derivatives f_θ and f_ϕ of function f.

Next assume that the surface is not given as $r = f(\theta,\phi)$, but as $F(r,\theta,\phi) =$ constant. Rewrite your expression for $\vec{n}\,\mathrm{d}S$ in terms of F instead of f. Hint: To get the derivatives of f in terms of those of F, look at the total differential of F at a point on the surface:

$$\mathrm{d}F \equiv \frac{\partial F}{\partial r}\mathrm{d}r + \frac{\partial F}{\partial\theta}\mathrm{d}\theta + \frac{\partial F}{\partial\phi}\mathrm{d}\phi$$

Now if you take $\mathrm{d}r = f_\theta\mathrm{d}\theta + f_\phi\mathrm{d}\phi$, you stay on the surface, so $\mathrm{d}F$ will then be zero:

$$0 = \frac{\partial F}{\partial r}(f_\theta\mathrm{d}\theta + f_\phi\mathrm{d}\phi) + \frac{\partial F}{\partial\theta}\mathrm{d}\theta + \frac{\partial F}{\partial\phi}\mathrm{d}\phi$$

From this you can find f_θ and f_ϕ in terms of the derivatives of F, by taking $\mathrm{d}\phi$, respectively $\mathrm{d}\theta$ zero. Plug that into the earlier expression for $\vec{n}\,\mathrm{d}S$ in terms of f and you have $\vec{n}\,\mathrm{d}S$ in terms of F. Write this expression in terms of the gradient of F in spherical coordinates, as given by the expression in your notes, or in any mathematical handbook. Compare with the Eulerian expression

$$\vec{n}\ \mathrm{d}S = \frac{\nabla F}{F_z}\,\mathrm{d}x\mathrm{d}y$$

as derived in class. Here $\mathrm{d}x\mathrm{d}y$ can be denoted symbolically as $\mathrm{d}S_z$: it is the area of a *surface of constant* z of dimensions $\mathrm{d}x \times \mathrm{d}y$. (In other words, it is the projection of surface element $\mathrm{d}S$ on a surface of constant z.) What is the equivalent to $\mathrm{d}S_z$ in your spherical coordinates expression?

2. 1st Ed: p133, q56, 2nd Ed: p155, q56.

3. 1st Ed: p160, q38, 2nd Ed: p183, q38. Simplify as much as possible. Sketch each surface, taking the z-axis upwards.

4. Finish finding the derivatives of the unit vectors of the spherical coordinate system using the class formulae. Then finish 1st Ed p160 q47, 2nd Ed p183 q47, as started in class, by finding the acceleration. As noted in class,

$$\frac{\partial\hat{\imath}_i}{\partial u_i} = \frac{1}{h_i}\frac{\partial h_i}{\partial u_i}\hat{\imath}_i - \sum_{k=1}^{3}\frac{1}{h_k}\frac{\partial h_i}{\partial u_k}\hat{\imath}_k \qquad \frac{\partial\hat{\imath}_i}{\partial u_j} = \frac{1}{h_i}\frac{\partial h_j}{\partial u_i}\hat{\imath}_j$$

5. Derive the heat equation for conduction in a bar physically. To do so, look at a small segment of length Δx of the bar. Fourier's law says that the heat going *into* the segment at its left hand surface at position x is given by

$$q_1 = -k u_x(x,t) A \Delta t$$

where k is the material's heat conduction coefficient in kW/m K, u the temperature, so $u_x(x,t)$ the partial derivative of the temperature with respect to x at the considered position and time (x,t), A is the cross sectional area of the bar, and Δt a small increase in time. Similarly, the heat going *out* the segment at its right hand surface at position $x + \Delta x$ is given by

$$q_2 = -k u_x(x + \Delta x, t) A \Delta t$$

Now apply that the *net* heat going into the segment must be the increase in temperature of the segment in time interval Δt, which can be written as $u_t(x,t)\Delta t$ times its volume, times the specific heat capacity of the segment, which can be written in terms of the specific heat per unit mass C_p of the material in kJ/kg K, its density ρ in kg/m^3, and the volume of the segment. From that, obtain the heat equation. In particular, identify the value and units of κ in terms of the material properties.

5 HW 5

1. Notes 18.2.1.3 and 18.2.1.4
2. Notes 18.2.1.6
3. Notes 18.3.1.1
4. Notes 18.3.1.2
5. Notes 18.2.3.5
6. Notes 18.3.3.5
7. Notes 18.3.4.2 Use the described procedure for $T = \pi$ to find an n so that $n\pi$ is within a distance of no more than $1/116$ from an integer.
8. 2.19 b, d, h. (DuChateau & Zachmann). In each case,
    - Show a picture of the different regions in the x, y-plane.
    - State in what regions would you have boundary value problems, and in what regions you would have initial value problems.
    - State in what regions singularities would be smoothed, and in what regions they would be propagated.

6 HW 6

1. 2.20.
2. Notes 18.6.2.1
3. 2.24. Show the complete transformed equation, fully converted to the new coordinates (ξ_1, ξ_2, ξ_3). (No remaining x, y, or z allowed.)
4. Notes 18.7.3.1

7 HW 7

1. 2.24 continued. Take the result of the previous homework question, as posted, and convert the principal part to the Laplacian. Then get rid of the first order derivatives to get a Helmholtz equation (with an imaginary k).

2. 2.22b,g. Draw the characteristics very neatly in the xy-plane,

3. 2.28d expanded. First find a particular solution. Try a quadratic

$$u_p = Ax^2 + Bxy + Cy^2 + Dx + Ey + F$$

   But obviously you do not need F, nor do you need B to create the x and y terms, so take these zero. Then find a particular solution. Next convert the remaining *homogeneous* problem for u_h to characteristic coordinates. Show that the homogeneous solution satisfies

$$2u_{h,\xi\eta} = u_{h,\eta}$$

   Put $u_{h,\eta} = v$. Solve the ODE to find $v = u_{h,\eta}$, noting that the integration constant is not really a constant. Next restore $v = u_{h,\eta}$ and integrate with respect to η to find u_h, watching again the integration constant. Finally write down the complete u, including particular solution, *in terms of x and y.*

4. 2.28f expanded. Solve this much like the previous problem, but in this case, leave the inhomogeneous term in there, don't try to find a particular solution for the original PDE. So transform the full problem to characteristic coordinates. (I think it is easiest to leave the logarithms in the coordinates you get, but you can try it either way.) Show that the solution satisfies

$$4u_{\xi\eta} - 2u_\xi \pm e^\eta = 0$$

   where $\pm$ indicates the sign of xy, or

$$4\xi\eta u_{\xi\eta} - 2\xi u_\xi + \eta = 0$$

   or

$$4\xi\eta u_{\xi\eta} + 2\xi u_\xi - \frac{1}{\eta} = 0$$

   or equivalent, depending on exactly how you define the characteristic coordinates. Solve this ODE for $v = u_\xi$, then integrate with respect to ξ to find u. Write the solution in terms of x and y.

5. 2.28c. Use the 2D procedure. Show that the equation may be simplified to

$$u_{\xi\xi} = 0$$

   Solve this equation and write the solution in terms of x and y. Watch any integration constants; they might not really be constants.

6. 2.28k. Reduce the PDE to the form

$$u_\eta = \left(e^{-\xi} + \frac{1}{\eta}\right) u_{\xi\xi}$$

Now discuss the properly posedness for the initial value problem, recalling from the class notes that the backward heat equation is not properly posed. In particular, assume that the "spatial coordinate" η is restricted to some finite interval $\xi_1 \leq \xi \leq \xi_2$, (like heat conduction in a finite bar). Assume that you have been given some suitable Dirichlet or Neumann boundary conditions at the ends ξ_1 and ξ_2. Finally assume that you have been given an initial condition at some value η_0 of the "time coordinate η. Under these conditions, determine whether the PDE be numerically solved to find u at large positive η. Can it be if η_0 is positive? If η_0 is a small negative number? If η_0 is a large negative number?

7. 2.28b. After transforming, describe a typical properly posed problem for the original equation. What sort of conditions could be prescribed on what sort of curves in the ξ, η-plane?

8 HW 8

1. Notes 19.1.1.1
2. Notes 19.1.1.2
3. Notes 19.2.2.1

9 HW 9

1. Notes 19.3.3.1 In doing the integral over the big circle, assume that u_{out} on it can be approximated as $C_0 + C_1/\rho + \ldots$, where C_0 is a constant and C_1 is independent of ρ.

2. 3.40. Use the Poisson integral formula as given in class.

10 HW 10

FOLLOW CLASS PROCEDURE IN ALL QUESTIONS.

1. 5.25. Also: (c) Assume that
$$f(x) = e^{-|x-2|}$$
   In a single very neat plot, draw $u(x,1)$, $u(x,2)$, and $u(x,3)$ versus x. Make sure you draw a complete covering of characteristics in the x, y-plane. And show the path of the singularity as a fattened characteristic in the x, y-plane.

2. 5.26b. IGNORE THE HINT. Include a very neat sketch of the complete set of characteristic lines. Fatten the asked characteristic in the x, y-plane. Simplify your answer as much as possible.

3. 5.27(a). Include a very neat sketch of the complete set of characteristic lines. Is the solution you get valid everywhere?

4. 5.27(b). Do not try to use an initial condition written in terms of two different, related, variables. Get rid of either x or y in the condition. Then call the argument of your undetermined function s and rewrite its expression in terms of s. Include a sketch of the complete set of characteristic lines and the initial condition line.

5. 5.29 Explain why there is no solution.

11 HW 11

1. In 7.27, acoustics in a pipe with closed ends, assume $\ell = 1$, $a = 1$, $f(x) = x$, and $g(x) = 1$. Graphically identify the extensions $F(x)$ and $G(x)$ of the given $f(x)$ and $g(x)$ to all x that allow the solution u to be written in terms of the infinite pipe D'Alembert solution.

2. Continuing the previous problem, in four separate graphs, draw $u(x,0)$, $u(x,0.25)$, $u(x,0.5)$, and $u(x,1.25)$. For all but the first graph, also include the separate terms $\frac{1}{2}F(x-at)$, $\frac{1}{2}F(x+at)$, and $\int_{x-at}^{x+at} G(\xi)\,\mathrm{d}\xi$. Use graph or raster paper or a plotting package. *Use the D'Alembert solution only to plot, do not use a separation of variables solution in your software package.* Comment on the boundary conditions. At which times are they satisfied? At which times are they not meaningful? Consider all times $0 \le t < \infty$ and do not approximate.

   Make sure to include your source code if any.

3. Using the D'Alembert solution of the previous problems, find $u(0.1, 3)$. Be sure to show the value of each term in the expression.

12 HW 12

1. Solve 7.26, by Laplace transforming the problem as given in time. This is a good way to practice back transform methods. Note that one factor in $\widehat{u}$ is a simpler function at a shifted value of coordinate s.

2. Solve 7.35 by Laplace transform in time. Clean up completely; only the given function may be in your answer, no Heaviside functions or other weird stuff. There is a minor error in the book's answer.

3. Write the *complete* (Sturm-Liouville) eigenvalue problem for the eigenfunctions of 7.27. Find the eigenfunctions of that problem. Make very sure you do not miss one. Write a symbolic expression for the eigenfunctions in terms of an index, and identify the values that that index takes. You may want, or not want, to write one eigenfunction explicitly instead of as a term in the sum.

4. Continuing the previous homework, write $f = x$ and $g = 1$ in terms of the eigenfunctions you found for the case $\ell = 1$. Be very careful with one particular eigenfunction. Note that sometimes you need to write a term in a sum or sequence out separately from the others.

5. Continuing the previous homework, substitute $u(x,t) = \sum_n u_n(t) X_n(x)$ (plus the additional term, if any) into the PDE to convert it into an ordinary differential for each separate coefficient $u_n(t)$. Solve the ODE. Be very careful with one particular case. By writing the initial conditions in terms of the eigenfunctions, identify the integration constants. Write out a complete summary of the solution. Make sure to identify the values of your numbering index in each expression.

13 HW 13

14 HW 14

1. (6 pts) Reconsider the separation of variables solution you derived. Using some programming language, evaluate the found solution at 101 equally spaced x-values from 0 to ℓ at times 0, 0.25, 0.5, and 1.25. Take $\ell = 1$ and $a = 1$. Include at least 50 nonzero terms in the summations. Plot the solution at these four times. Compare with the D'Alembert solution of the previous homework, which must be the same. (Check your D'Alembert solution first against the posted solution). Show also what happens if you only include 10 terms in the summations.

   To help you get started, a Matlab program that plots the solution to problem 7.28 is provided as an example. You need both p7_28.m[1] and p7_28u.m[2]. This program is valid for the PDE and BC solved in class, with the additional data

   $$a = \tfrac{1}{2}, \quad \ell = \tfrac{1}{2}\pi, \qquad f(x) = \tfrac{1}{2}\pi - x \Rightarrow f_n = \frac{1}{(2n-1)^2}, \qquad g(x) = 0 \Rightarrow g_n = 0.$$

   These may of course not apply for your problem.

   To run the program, enter matlab and type in `p7_28`. If you do not have matlab, a free replacement is octave. Or you can use some other programming and plotting facilities.

   Include your code.

2. Refer to problem 7.19. Find a function $u_0(x,t)$ that satisfies the inhomogeneous boundary conditions. Define $v = u - u_0$. Find the PDE, BC and IC satisfied by v.

3. Find suitable eigenfunctions in terms of which v may be written, and that satisfy the homogeneous boundary conditions. Write the relevant known functions in terms of these eigenfunctions and give the expressions for their Fourier coefficients. *Work out these integrals for arbitrary ℓ, f, g_0, and g_1 as far as possible.*

4. Continuing the previous question, solve for v using separation of variables in terms of integrals of the known functions $f(x)$, $g_0(t)$, and $g_1(t)$. Write the solution for u completely, worked out for arbitrary ℓ, f, g_0, and g_1 as far as possible.

5. Assume that $f = 0$, $k = \ell = 1$, and that $u_x = t$ at both $x = 0$ and $x = \ell$. Work out the solution for that case completely. Do not use the above values for earlier homeworks.

6. Plot the solution numerically at some relevant times. I suspect that for large times the solution is approximately

   $$u = (x - \tfrac{1}{2})t + \tfrac{1}{6}(x - \tfrac{1}{2})^3 - \tfrac{1}{8}(x - \tfrac{1}{2})$$

   Do your results agree?

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[1] `../p7_28.m`

[2] `../p7_28u.m`

7. Consider a simple problem of unsteady, axisymmetric, heat conduction in a ring (or unsteady axial flow between concentric pipes) of radii a and b:

$$u_t = \kappa\left(u_{rr} + \frac{1}{r}u_r\right) \qquad u(r,0) = f(r) \quad u(a,t) = 0 \quad u(b,t) = 0$$

Find the eigenvalue problem for the eigenfunctions $R(r)$. Do not try to solve it (or look under Bessell functions in our math handbook). Given an arbitrary function $g(r)$, figure out how to obtain the coefficients g_n in

$$g(r) = \sum_{\text{all } n} g_n R_n(r)$$