EML 4930/5061 | **Analysis in M.E. II** | **04/26/11**
Closed book | *Van Dommelen* | *3-5 pm*

Show all reasoning and intermediate results leading to your answer, or credit will be lost. One book of mathematical tables, such as Schaum's Mathematical Handbook, may be used, as well as a calculator and one handwritten letter-size single formula sheet.

1. (20%) Solve the following PDE and boundary condition

$$u_x + \cos(x)u_y = 0 \qquad u(x,0) = \cos(x)$$

   Clean up your answer!

   Draw the characteristics very neatly. Using your picture, answer the following questions:

   (a) Are there any points where the solution is not defined by the given boundary condition? If so, identify them. What happens to the solution that you found, if evaluated in this region?

   (b) A boundary condition cannot *in general* be specified at all $(x,0)$. Identify at what $(x,0)$ the boundary condition can be specified to ensure a properly posed problem. (Exclude the points mentioned in (a) from the domain. Otherwise it is automatically improperly posed.)

   (c) Is it OK to specify the given cosine initial condition at all $(x,0)$ anyway? Explain.

2. (20%) Use D'Alembert to find the deflection u of a vibrating string of length π satisfying

$$u_{tt} = \left(\tfrac{1}{2}\pi\right)^2 u_{xx} \qquad u(-\tfrac{1}{2}\pi,t) = u(\tfrac{1}{2}\pi,t) = 0$$

   if the initial conditions are

$$u(x,0) = \sin(x) \quad u_t(x,0) = \cos(x)$$

   Evaluate the solution at $x = \pi/6$ and $t = 13$. Make sure to explicitly list the value of each term in the expression for u. Exact value only.

3. (20%) Use the Laplace transform method to solve the following problem. At time zero, a plate is moving upwards with a given velocity V. Next to the plate is a viscous fluid that is at rest. For times greater than zero, the plate drags along more and more fluid next to it. The plate slows down accordingly. The motion of the viscous fluid is described by

$$v_t = \nu v_{xx}$$

   The initial condition is

$$v(x,0) = 0 \quad \text{for} \quad x \neq 0 \qquad v(0,0) = V$$

   (The single point where $v(x,0)$ is not zero can be ignored in transforming the PDE, but not necessarily in other manipulations.) The boundary condition is

$$v_t(0,t) = cv_y(0,t)$$

   where c is a constant, related to the fluid viscosity and plate mass per unit area, with units of velocity. Solve this problem, assuming that V, ν, and c are given constants. Note: the back transform is not in tables 6.3 or 6.4 and probably messy to derive. You can find it in Schaum's Mathematical Handbook, for one. Clean up.

   Use only the Laplace transform tables 6.3 and 6.4 except where stated otherwise.

4. (40%) Use separation of variables to solve the following problem for viscous flow in between two plates in which one plate exerts a quadratically increasing shear force on the fluid:

$$u_t = u_{xx} \qquad u(x,0) = 0 \quad u_x(0,t) = 0 \quad u_x(1,t) = 2t^2$$

   Show all reasoning. List the eigenfunctions and eigenvalues completely.

   At the end, write out the fully worked out solution completely, with all parameters in it clearly identified.

Table 6.3: Properties of the Laplace Transform

$f(t)$	$\hat{f}(s) = \int_0^\infty f(t)e^{-st}\mathrm{d}t$
1. $C_1 f_1(t) + C_2 f_2(t)$	$C_1 \hat{f}_1(s) + C_2 \hat{f}_2(s)$
2. $f(at)$	$a^{-1}\hat{f}(s/a) \qquad (a > 0)$
3. $f^{(n)}(t)$	$s^n \hat{f}(s) - s^{n-1} f(0) - \ldots - f^{(n-1)}(0) \qquad (n = 1, 2, \ldots)$
4. $t^n f(t)$	$(-1)^n \hat{f}^{(n)}(s) \qquad (n = 1, 2, \ldots)$
5. $e^{ct} f(t)$	$\hat{f}(s - c) \qquad (s = \text{const.})$
6. $H(t-b)f(t-b)$, where $H(t) = \begin{cases} 0 & t < 0 \\ 1 & t > 0 \end{cases}$	$e^{-bs}\hat{f}(s) \qquad (b > 0)$
7. $f * g(t) \equiv \int_0^t f(t-\tau)g(\tau)\mathrm{d}\tau$	$\hat{f}(s)\hat{g}(s)$

Table 6.4: Laplace Transform Pairs

$f(t)$	$\hat{f}(s) = \int_0^\infty f(t)e^{-st}\mathrm{d}t$
1. 1	$\frac{1}{s}$
2. t^n	$\frac{n!}{s^{n+1}} \qquad (n = 1, 2, \ldots)$
3. e^{kt}	$\frac{1}{s-k}$
4. $\sin(at)$	$\frac{a}{s^2 + a^2}$
5. $\cos(at)$	$\frac{s}{s^2 + a^2}$
6. $\frac{1}{\sqrt{\pi t}}$	$\frac{1}{\sqrt{s}}$
7. $\frac{1}{\sqrt{\pi t}} e^{-k^2/4t}$	$\frac{1}{\sqrt{s}} e^{-k\sqrt{s}} \qquad (k > 0)$
8. $\frac{k}{\sqrt{4\pi t^3}} e^{-k^2/4t}$	$e^{-k\sqrt{s}} \qquad (k > 0)$
9. $\operatorname{erfc}\left(k/2\sqrt{t}\right)$, where $\operatorname{erfc}(z) \equiv \frac{2}{\sqrt{\pi}} \int_z^\infty e^{-u^2}\mathrm{d}u$	$\frac{1}{s} e^{-k\sqrt{s}} \qquad (k > 0)$