**EML 4930/5930** **Analysis in M.E. II** **04/22/08**
*Closed book* *Van Dommelen* *3-5 pm*

Show all reasoning and intermediate results leading to your answer, or credit will be lost. One book of mathematical tables, such as Schaum's Mathematical Handbook, may be used, as well as a calculator and one handwritten letter-size single formula sheet.

1. (20%) Solve the PDE and initial condition

$$y^3 u_x - x^3 u_y = y^3 u \quad \text{for } y \geq 0; \qquad u(x,0) = 1 \quad \text{for } x \geq 0.$$

   Why is the initial condition only given for $x \geq 0$?

2. (20%) Use D'Alembert to find the pressure $u$ to the problem of acoustics in a pipe of length $\frac{1}{2}\pi$ with one end open and the other end closed,

$$u_{tt} = u_{xx} \qquad u(0,t) = 0 \quad u_x(\tfrac{1}{2}\pi, t) = 0$$

   if the initial conditions are

$$u(x,0) = x \quad u_t(x,0) = \sin x.$$

   In particular evaluate $u$ at $x = 1.5$ and $t = 2.5$. Note that the extension of $u_t(x,0)$ to all $x$ will take a simple analytical form.

3. (40%) Solve the following problem of heat conduction in a bar of length 1 using separation of variables:

$$u_t = u_{xx} \qquad u(x,0) = x \quad u(0,t) = 0 \quad u(1,t) = e^{-t}.$$

   Write out the fully worked-out solution precisely and completely.

4. (20%) Solve the following problem of vibrations in a semi-infinite string using the Laplace transform method:

$$u_{tt} = a^2 u_{xx} \qquad u(x,0) = u_t(x,0) = 0 \quad u(0,t) = f(t)$$

   where $f(t)$ is to be considered to be some given function and $x \geq 0, t \geq 0$. Show all steps and reasoning. Your answer should not have weird mathematics, but be in simple terms that anyone with a basic understanding of functions can understand. Does your solution agree with the general solution to the wave equation?

**Table 6.3: Properties of the Laplace Transform**

| $f(t)$ | $\hat{f}(s) = \int_0^\infty f(t)e^{-st}\mathrm{d}t$ |
|---|---|
| **1.** $C_1 f_1(t) + C_2 f_2(t)$ | $C_1\hat{f}_1(s) + C_2\hat{f}_2(s)$ |
| **2.** $f(at)$ | $a^{-1}\hat{f}(s/a) \qquad (a > 0)$ |
| **3.** $f^{(n)}(t)$ | $s^n\hat{f}(s) - s^{n-1}f(0) - \ldots - f^{(n-1)}(0) \qquad (n = 1, 2, \ldots)$ |
| **4.** $t^n f(t)$ | $(-1)^n\hat{f}^{(n)}(s) \qquad (n = 1, 2, \ldots)$ |
| **5.** $e^{ct}f(t)$ | $\hat{f}(s-c) \qquad (s = \text{const.})$ |
| **6.** $H(t-b)f(t-b)$, where $H(t) = \begin{cases} 0 & t < 0 \\ 1 & t > 0 \end{cases}$ | $e^{-bs}\hat{f}(s) \qquad (b > 0)$ |
| **7.** $f * g(t) \equiv \int_0^t f(t-\tau)g(\tau)\mathrm{d}\tau$ | $\hat{f}(s)\hat{g}(s)$ |

**Table 6.4: Laplace Transform Pairs**

| $f(t)$ | $\hat{f}(s) = \int_0^\infty f(t)e^{-st}\mathrm{d}t$ |
|---|---|
| **1.** $1$ | $\frac{1}{s}$ |
| **2.** $t^n$ | $\frac{n!}{s^{n+1}} \qquad (n = 1, 2, \ldots)$ |
| **3.** $e^{kt}$ | $\frac{1}{s-k}$ |
| **4.** $\sin(at)$ | $\frac{a}{s^2+a^2}$ |
| **5.** $\cos(at)$ | $\frac{s}{s^2+a^2}$ |
| **6.** $\frac{1}{\sqrt{\pi t}}$ | $\frac{1}{\sqrt{s}}$ |
| **7.** $\frac{1}{\sqrt{\pi t}}e^{-k^2/4t}$ | $\frac{1}{\sqrt{s}}e^{-k\sqrt{s}} \qquad (k > 0)$ |
| **8.** $\frac{k}{\sqrt{4\pi t^3}}e^{-k^2/4t}$ | $e^{-k\sqrt{s}} \qquad (k > 0)$ |
| **9.** $\operatorname{erfc}\left(k/2\sqrt{t}\right)$, where $\operatorname{erfc}(z) \equiv \frac{2}{\sqrt{\pi}}\int_z^\infty e^{-u^2}\mathrm{d}u$ | $\frac{1}{s}e^{-k\sqrt{s}} \qquad (k > 0)$ |