Analysis in ME II
EML 4930/5061
Homework

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Do not print out all pages. Keep checking for changes. Complete assignment will normally be available the day after the last lecture whose material is included in the assignment (Monday, normally).

Contents

1 01/16 F

Use vector analysis wherever possible.

1. 1st Ed: p13, q31a-f,h-j, 2nd Ed: p17, q31a-i. *if they can be vectors, count them as such.*

2. 1st Ed: p13, q32, 2nd Ed: p17, q32. Do it both graphically and analytically. Give length and angle.

3. 1st Ed: p14, q48, 2nd Ed: p19, q46. Use vector calculus only, no trig. No scalars at all.

4. 1st Ed: p32, q66, 2nd Ed: p38, q66. Use vector only, except when working out the final numbers.

5. 1st Ed: p32, q82, 2nd Ed: p40, q82a, where **B** should be corrected to $(1, -3, 4)$. Vector calculus only, no trig. Do it without finding the actual sides of the parallelogram. In particular, show that the area is half of $\vec{A} \times \vec{B}$. Also give a unit vector normal to the plane of the parallelogram.

6. 1st Ed: p33, q90, 2nd Ed: p41, q90a. Also give the area of the parallelogram with sides $\vec{B}$ and $\vec{C}$.

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1. 1st Ed: p53, q32, 2nd Ed: p64, q32. Draw the curve neatly.

2. 1st Ed: p54, q47, 2nd Ed: p65, q47. (9 points)

3. 1st Ed: p78, q46, 2nd Ed: p91, q46. $r = \sqrt{x^2 + y^2 + z^2}$

4. 1st Ed: p78, q54, 2nd Ed: p92, q54. You may want to refresh your memory on total derivatives.

5. The height of the ground above sea level is $\sin(x)\sin(2y)$.

   (a) Draw the contour lines.

   (b) Consider the point $x = 0.5$ and $y = 1.5$. Find the gradient of height at that point and draw it in the graph.

   (c) If I want to climb to the nearest peak in the shortest possible distance, in which direction should I move at that point? In particular, what is $\mathrm{d}y/\mathrm{d}x$?

   (d) If I am traveling along the line $y = 3x$ with speed 60, how rapidly am I changing height?

3 01/30 F

1. (6 points). 1st Ed: p78, q60, 2nd Ed: p92, q60. Also find two scalar equations that describe the line through P that crosses the surface normally at P.

   Find the unit normal $\vec{n}$ to the surface at P. Now assume that the surface is reflective, satisfying Snell's law. An incoming light beam parallel to the x-axis hits the surface at P. Find a vector equation that describes the path of the *reflected* beam.

   Hint: let $\vec{v}$ be a vector along the light ray. The *component* of $\vec{v}$ in the direction of $\vec{n}$ is $\vec{n} \cdot \vec{v}$. The *component vector* in the direction of $\vec{n}$ is defined as $\vec{v}_1 = \vec{n}(\vec{n} \cdot \vec{v})$. Sketch this vector along with vector $\vec{n}$. In which direction is the remainder $\vec{v}_2 = \vec{v} - \vec{v}_1$? Now figure out what happens to $\vec{v}_1$ and $\vec{v}_2$ during the reflection. Take it from there.

2. 1st Ed: p78, q62, 2nd Ed: p92, q62.

3. 1st Ed: p79, q64, 2nd Ed: p92, q64.

4. 1st Ed: p79, q70, 2nd Ed: p92, q70.

5. 1st Ed: p80, q84, 2nd Ed: p93, q84.

6. 1st Ed: p80, q87, 2nd Ed: p93, q87. (6 points) Compare with a point sink in which

   $$\vec{v} = -\frac{x\hat{\imath} + y\hat{\imath}}{x^2 + y^2}$$

   Assume these are incompressible flows, in which the fluid density is constant. For each flow, compute the divergence, draw streamlines, and figure out how much fluid passes through a circle of arbitrary radius r. (Since the velocity is radial, the fluid flow through a circle is the magnitude of the velocity times the circumference of the circle.) Now look at a ring between two slightly different radii, and compare the fluid that goes in at one radius with the fluid that goes out at the other radius. Based on the results, argue that the divergence of the velocity is a measure of the "source strength," the amount of fluid created out of nothing. (A sink being a negative source, where fluid disappears into nothing.) So, what do you think of the value of the divergence of the point sink at the origin (assuming that you smooth out the singularity a bit)? Note: if the fluid is not incompressible, it is really *volume* flows you are comparing, not mass flows, and the divergence is a measure of the relative rate of specific volume expansion. Additional volume is created out of nothing, not mass.

7. 1st Ed: p80, q102, 2nd Ed: p94, q102. Make sure that you find ϕ in a mathematically sound way, as discussed in class. No messing around and guessing a solution!

8. (9 points). Modified version of a question in the book. Maxwell's equations in vacuum are

   $$\nabla \times \vec{H} = \frac{1}{c}\frac{\partial \vec{E}}{\partial t} + \frac{4\pi}{c}\vec{\jmath} \qquad (a) \qquad\qquad \nabla \times \vec{E} = -\frac{1}{c}\frac{\partial \vec{H}}{\partial t} \qquad (b)$$

   $$\nabla \cdot \vec{H} = 0 \qquad (c) \qquad\qquad \nabla \cdot \vec{E} = 4\pi\rho \qquad (d)$$

Here $\vec{E}$ is the electric field, $\vec{H}$ the magnetic field, ρ the charge density (the electric charge per unit volume), $\vec{\jmath}$ the current density (the current flowing per unit cross sectional area), and c the speed of light, a constant. Consider ρ and $\vec{\jmath}$ to be given functions of position and time. You need to show that any solution $\vec{E}, \vec{H}$ of the above equations is given by scalar and vector potentials $\phi, \vec{A}$ as described below.

Procedure to follow:

1. Explain why there must be a "vector potential" $\vec{A}_0$ so that

$$\vec{H} = \nabla \times \vec{A}_0$$

2. Next *define* a vector $\vec{E}_\phi$ by setting

$$\vec{E} = -\frac{1}{c}\frac{\partial \vec{A}_0}{\partial t} + \vec{E}_\phi$$

3. Prove that the $\vec{E}_\phi$ defined this way is minus the gradient of some "scalar potential" ϕ_0. Then the above equation becomes:

$$\vec{E} = -\frac{1}{c}\frac{\partial \vec{A}_0}{\partial t} - \nabla\phi_0$$

4. Unfortunately, $\vec{A}_0$ and ϕ_0 are not unique. We now want, given potentials $\vec{A}_0$ and ϕ_0, find modified potentials $\vec{A}$ and ϕ. These must still give

$$\boxed{\vec{H} = \nabla \times \vec{A} \quad (e) \qquad \vec{E} = -\frac{1}{c}\frac{\partial \vec{A}}{\partial t} - \nabla\phi \quad (f)}$$

However, in addition they must satisfy the famous "Lorenz condition"

$$\boxed{\nabla \cdot \vec{A} + \frac{1}{c}\frac{\partial \phi}{\partial t} = 0 \quad (1)}$$

(No, there is no t in Lorenz. That is another Lorentz. The Lorenz condition is critical, because it is the only condition that all observers can agree on.) The potentials you need are of the form

$$\vec{A} = \vec{A}_0 + \nabla\psi \qquad \phi = \phi_0 - \frac{1}{c}\frac{\partial \psi}{\partial t}$$

Prove that in those terms, *(e)* and *(f)* above are true *regardless of what you take for* ψ. That is the famous "gauge property" of the electromagnetic field. It is central to quantum field theory. It *defines* the electromagnetic field in modern quantum theories, all the rest is derived.

5. Since you can take ψ whatever you like, you can choose it so that the Lorenz condition *(1)* is satisfied. Show that this leads to a partial differential equation for ψ. (This equation is called an inhomogeneous wave equation. The properties of this equation will be discussed in the second part of the class.)

6. Now substitute what you got so far into the four Maxwell equations and so find the requirements that $\vec{A}$ and ϕ must satisfy. (I.e. get rid of the electric and magnetic fields in favor of the vector and scalar potentials $\vec{A}$ and ϕ.)
7. How come only one vector equation and one scalar equation are left?
8. Clean up! You must obtain decoupled equations for the scalar and vector potentials.
9. Finally, combine *(a)* and *(d)* to get a relation between the charge and current densities. (This equation is similar to the continuity equation in incompressible flow and expresses that no charge can be created out of nothing.)

9. 1st Ed: p102, q32, 2nd Ed: p122, q32. Use vector integration only.

10. 1st Ed: p103, q44, 2nd Ed: p123, q44. Use vector line integrations only.

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1. 1st Ed: p103, q44, 2nd Ed: p123, q44. Do it using Stokes.

2. 1st Ed: p104, q62, 2nd Ed: p124, q62. Do the surface integrals both directly and using the divergence theorem. Make sure to include the flat circle of the cone. Note: in doing the surface integrals directly, you are required to write them down in Cartesian coordinates using the expression for $\vec{n}\,\mathrm{d}S$ given in class when $F(x,y,z)=0$. After that, switch to polar coordinates to actually do the integration.

3. MODIFIED version of 1st Ed: p132, q50, 2nd Ed: p154, q50. Given

$$\vec{v} = \frac{(-y,x)}{x^2+y^2}$$

   1. Evaluate $\nabla \times \vec{v}$.
   2. Also evaluate, presumably using polar coordinates,

   $$\int_{\mathrm{I}} \vec{v}\cdot \mathrm{d}\vec{r} \qquad \int_{\mathrm{II}} \vec{v}\cdot \mathrm{d}\vec{r}$$

   where path I is the semi circle of radius r going clockwise from $(r,0)$ to $(-r,0)$, and path II is the semi circle of radius r going counter-clockwise from $(r,0)$ to $(-r,0)$.
   3. Explain why the integral over II minus the integral over I is the integral over the closed circle.
   4. Explain why Stokes implies that the closed contour integral should be the integral of the z-component of $\nabla \times \vec{v}$ over the inside of the circle.

5. Then explain why you would then normally expect the contour integral to be zero. That means that the two integrals I and II should be equal, but they are not.
6. Explain what the problem is.
7. Do you expect integrals over closed circles of different radii to be equal? Why?
8. Are they actually equal?

Now assume that you allow singular functions to be OK, like Heaviside step functions and Dirac delta functions say. Then figure out in what part of the interior of the circle, $\int\int \nabla \times \vec{v} \cdot \hat{k}\, \mathrm{d}x\mathrm{d}y$ is not zero. So how would you describe $\nabla \times \vec{v}$ for this vector field in terms of singular functions?

4. 1st Ed: p133, q56, 2nd Ed: p155, q56.

5. Derive $\vec{n}\,\mathrm{d}S$ in terms of $\mathrm{d}\theta$ and $\mathrm{d}\phi$, where (r, θ, ϕ) are spherical coordinates. Assume that the surface is described as $r = f(\theta, \phi)$ for some given function f. Use the formulae given earlier in class for $\vec{n}\,\mathrm{d}S$ in terms of two parameters u and v. The formula requires you to differentiate $\vec{r}$ with respect to the parameters. Now in spherical,

$$\vec{r} = r\hat{\imath}_r$$

From class, the derivatives of $\hat{\imath}_r$ are

$$\frac{\partial\,\hat{\imath}_r}{\partial r} = 0 \qquad \frac{\partial\,\hat{\imath}_r}{\partial \theta} = \hat{\imath}_\theta \qquad \frac{\partial\,\hat{\imath}_r}{\partial \phi} = \sin\theta\,\hat{\imath}_\phi$$

So you can now write $\vec{n}\,\mathrm{d}S$ in terms of r and the derivatives f_θ and f_ϕ of function f.

Next assume that the surface is not given as $r = f(\theta, \phi)$, but as $F(r, \theta, \phi) = \text{constant}$. Rewrite your expression for $\vec{n}\,\mathrm{d}S$ in terms of F instead of f. Hint: To get the derivatives of f in terms of those of F, look at the total differential of F at a point on the surface:

$$\mathrm{d}F \equiv \frac{\partial F}{\partial r}\mathrm{d}r + \frac{\partial F}{\partial \theta}\mathrm{d}\theta + \frac{\partial F}{\partial \phi}\mathrm{d}\phi$$

Now if you take $\mathrm{d}r = f_\theta \mathrm{d}\theta + f_\phi \mathrm{d}\phi$, you stay on the surface, so $\mathrm{d}F$ will then be zero:

$$0 = \frac{\partial F}{\partial r}(f_\theta \mathrm{d}\theta + f_\phi \mathrm{d}\phi) + \frac{\partial F}{\partial \theta}\mathrm{d}\theta + \frac{\partial F}{\partial \phi}\mathrm{d}\phi$$

From this you can find f_θ and f_ϕ in terms of the derivatives of F, by taking $\mathrm{d}\phi$, respectively $\mathrm{d}\theta$ zero. Plug that into the earlier expression for $\vec{n}\,\mathrm{d}S$ in terms of f and you have $\vec{n}\,\mathrm{d}S$ in terms of F. Write this expression in terms of the gradient of F in spherical coordinates, as given by the expression in your notes, or in any mathematical handbook. Compare with the Eulerian expression

$$\vec{n}\ \mathrm{d}S = \frac{\nabla F}{F_z}\,\mathrm{d}x\mathrm{d}y$$

as derived in class. Here $\mathrm{d}x\mathrm{d}y$ can be denoted symbolically as $\mathrm{d}S_z$: it is the area of a *surface of constant* z of dimensions $\mathrm{d}x \times \mathrm{d}y$. (In other words, it is the projection of surface element $\mathrm{d}S$ on a surface of constant z.) What is the equivalent to $\mathrm{d}S_z$ in your spherical coordinates expression?

6. 1st Ed: p160, q38, 2nd Ed: p183, q38. Simplify as much as possible. Sketch each surface, taking the z-axis upwards.

7. Finish finding the derivatives of the unit vectors of the spherical coordinate system using the class formulae. Then finish 1st Ed p160 q47, 2nd Ed p183 q47, as started in class, by finding the acceleration. As noted in class,

$$\frac{\partial \hat\imath_i}{\partial u_i} = \frac{1}{h_i}\frac{\partial h_i}{\partial u_i}\hat\imath_i - \sum_{k=1}^{3}\frac{1}{h_k}\frac{\partial h_i}{\partial u_k}\hat\imath_k \qquad \frac{\partial \hat\imath_i}{\partial u_j} = \frac{1}{h_i}\frac{\partial h_j}{\partial u_i}\hat\imath_j$$

8. Express the acceleration in terms of the spherical velocity components v_r, v_θ, v_ϕ and their first time derivatives, instead of time derivatives of position coordinates. Like $a_r = \dot v_r + \ldots$, etc. This is how you do it in fluid mechanics, where time-derivatives of particle position coordinates are normally not used. (So, get rid of all position coordinates with dots on them in favor of the velocity components and position coordinates without dots.) Hint: you may want to differentiate the expressions for the velocity components with respect to time to get expressions for the second order derivatives of the position coordinates. Then get rid of the second order derivatives first.

5 02/13 F

1. Notes 18.2.1.1

2. Notes 18.2.1.2

3. Notes 18.2.1.3 and 18.2.1.4

4. Notes 18.2.1.6

5. Notes 18.2.1.7

6. Notes 18.2.1.8

7. Notes 18.3.1.1

8. Notes 18.3.1.2

6 02/20 F

1. Notes 18.2.3.3

2. Notes 18.2.3.5

3. Notes 18.3.3.3
4. Notes 18.3.3.5
5. 2.19 b, h. (DuChateau & Zachmann) Show a picture of the different regions.
6. 2.20.
7. Notes 18.6.2.1

7 02/27 F

1. 2.26. Also show the transformation formulae from and to the new coordinates. DONE IN CLASS!!
2. Notes 18.7.3.1
3. Notes 18.7.4.1
4. 2.22b,g. Draw the characteristics very neatly in the xy-plane,
5. 2.28d. (20 pt) First find a particular solution. Try a quadratic

   $$u_p = Ax^2 + Bxy + Cy^2 + Dx + Ey + F$$

   But obviously you do not need F, nor do you need B to create the x and y terms, so take these zero. Next convert the remaining *homogeneous* problem for u_h to characteristic coordinates. Show that the homogeneous solution satisfies

   $$2u_{h,\xi\eta} = u_{h,\eta}$$

   Put, say, $v = u_{h,\eta}$. Solve this ODE to find $v = u_{h,\eta}$, then integrate $u_{h,\eta}$ with respect to η to find u_h. Finally find the complete u, in terms of x and y. Watch any integration constants; they might not be constants.

6. 2.28f. (20 pt) In this case, leave the inhomogeneous term in there, don't try to find a particular solution for the original PDE. Transform the full problem to characteristic coordinates. (I think it is easiest to leave the logarithms in the coordinates, but you can try it either way.) Show that the solution satisfies

   $$4u_{\xi\eta} - 2u_\xi \pm e^\eta = 0$$

   where $\pm$ indicates the sign of xy, or

   $$4\xi\eta u_{\xi\eta} - 2\xi u_\xi + \eta = 0$$

   or

   $$4\xi\eta u_{\xi\eta} + 2\xi u_\xi - \frac{1}{\eta} = 0$$

   or equivalent, depending on exactly how you define the characteristic coordinates. Solve this ODE for $v = u_\xi$, then integrate with respect to ξ to find u. Write the solution in terms of x and y. Watch any integration constants; they might not be constants.

7. 2.28c. (20 pt) Use the 2D procedure. Show that the equation may be simplified to

$$u_{\xi\xi} = 0$$

Solve this equation and write the solution in terms of x and y. Watch any integration constants; they might not be constants.

8. 2.28k. Reduce the PDE to the form

$$u_\eta = \left(e^{-\xi} + \frac{1}{\eta}\right) u_{\xi\xi}$$

Now discuss the properly posedness for the initial value problem, recalling from the class notes that the backward heat equation is not properly posed. In particular, given an interval $\xi_1 \le \xi \le \xi_2$, with an initial condition at some value of η_0 and boundary conditions at ξ_1 and ξ_2, can the PDE be numerically solved to find u at large η? If η_0 is positive? If η_0 is a small negative number? If η_0 is a large negative number?

9. 2.28b. Describe a typical properly posed problem for the original equation.

8 03/06 F

1. 3.44. This is mostly the uniqueness proof given in class, which can also be found in the notes and more generally in solved problems 3.14-3.16. However, here you will want to write out the two parts of the surface integral separately since the boundary conditions are a mixture of the two cases 3.14 and 3.15 (with $c = 0$).
2. Notes 18.4.1.1
3. Notes 18.4.1.2
4. Notes 19.1.1.1
5. Notes 19.1.1.2

9 03/20 F

1. Notes 19.2.2.1
2. Notes 19.3.3.1 In doing the integral over the big circle, assume that u_{out} on it can be approximated as $C_0 + C_1/\rho + \ldots$, where C_0 and C_1 are constants.

10 03/27 F

1. Notes 2.3.6.1
2. Notes 2.3.6.2
3. 5.25. Also: (c) Assume that

   $$f(x) = e^{-|x-2|}$$

   In a single very neat plot, draw $u(x,1)$, $u(x,2)$, and $u(x,3)$ versus x. Make sure you draw a complete covering of characteristics in the x,y-plane. And show the path of the singularity as a fattened characteristic in the x,y-plane.
4. 5.27(a). Include a very neat sketch of the complete set of characteristic lines. Is the solution you get valid everywhere?
5. 5.27(b). Do not try to use an initial condition written in terms of two different, related, variables. Get rid of either x or y in the condition. Then call the argument of your undetermined function s and rewrite its expression in terms of s. Include a sketch of the complete set of characteristic lines and the initial condition line.

11 04/03 F

1. The viscous Burger's equation is:

   $$u_t + uu_x = \nu u_{xx}$$

   where ν is a positive constant. (This equation can be solved analytically, by setting $u = 2\nu U_x$ where e^U satisfies the heat equation, showing that it has smooth solutions.) Derive the conservation law for fixed intervals satisfied by the solutions of the viscous Burger's equation.

   When the viscosity is ignored as negligibly small, you get the inviscid Burger's equation:

   $$u_t + uu_x = 0$$

   Deduce from it, using the simplified conservation law, the speed, call it v, at which the conserved quantity is inviscidly propagated.

   Now the inviscid Burger's equation is a nonlinear first order equation that can have discontinuities, "shocks". Derive the propagation speed, call it a_s, of these shocks, in terms of the quantities immediately before and behind the shock. Make use of the fact that shocks are not really discontinuities, but small regions, if small but nonzero ν is considered. And therefore the viscous Burger's equation ensures that no conserved quantity is created out of nothing nor destroyed inside the shock.
2. Continuing the previous question, in regions in the x,t plane where the solution of the inviscid Burger's equation is smooth, it can be solved with the method of characteristics. Do that and write the general solution in two alternative forms, depending on which integration constant you consider to be a function of which other. Also give the propagation velocity, call it a, of the characteristics. Compare a and a_s.

3. Continuing the previous question, now consider the solution due to an initial unit pulse:

$$u(x,0) = \begin{cases} 0 \text{ for } x < 0 \\ 1 \text{ for } 0 < x < 1 \\ 0 \text{ for } 1 < x \end{cases}$$

   Neatly draw this initial condition in a u, x-plane.

   There are two proposed solutions for $u(x,t)$ with this initial condition:

$$u_1(x,t) = \begin{cases} 0 \text{ for } x < \frac{1}{2}t \\ 1 \text{ for } \frac{1}{2}t < x < 1 + \frac{1}{2}t \\ 0 \text{ for } 1 + \frac{1}{2}t < x \end{cases} \qquad u_2(x,t) = \begin{cases} 0 \text{ for } x < 0 \\ x/t \text{ for } 0 < x < t \\ 1 \text{ for } t < x < 1 + \frac{1}{2}t \\ 0 \text{ for } 1 + \frac{1}{2}t < x \end{cases}$$

   (a) Draw both solutions in u, x planes at time $t = 1$. (b) Draw the characteristics of both solutions in x, t planes, for t up to 2. (Graph or raster paper recommended.) (c) Check that both solutions satisfy the given initial condition. (d) Check that both solutions satisfy at least one of your characteristic solutions in the regions between the singularities. (e) Which one of the two solutions above is correct, if any? Why?

4. Continuing the previous question, describe what happens to solution u_2 when t becomes larger than 2. What happens to the shock strength and velocity? Extend your earlier x, t plane to include times $t > 2$.

5. In 7.27, acoustics in a pipe with closed ends, assume $\ell = 1$, $a = 1$, $f(x) = x$, and $g(x) = 1$. Graphically identify the extensions $F(x)$ and $G(x)$ of the given $f(x)$ and $g(x)$ to all x that allow the solution u to be written in terms of the infinite pipe D'Alembert solution.

6. Continuing the previous problem, in four separate graphs, draw $u(x,0)$, $u(x,0.25)$, $u(x,0.5)$, and $u(x,1.25)$. For all but the first graph, also include the separate terms $\frac{1}{2}F(x-at)$, $\frac{1}{2}F(x+at)$, and $\int_{x-at}^{x+at} G(\xi)\,\mathrm{d}\xi$. Use graph or raster paper or a plotting package. *Use the D'Alembert solution only to plot, do not use a separation of variables solution in your software package.* Comment on the boundary conditions. At which times are they satisfied? At which times are they not meaningful? Consider all times $0 \le t < \infty$ and do not approximate.

   Make sure to include your source code if any.

7. Using the D'Alembert solution of the previous problems, find $u(0.1, 3)$. Be sure to show the value of each term in the expression.

12 04/10 F

1. Write the *complete* (Sturm-Liouville) eigenvalue problem for the eigenfunctions of 7.27. Find the eigenfunctions of that problem. Make very sure you do not miss one. Write a symbolic expression for the eigenfunctions in terms of an index, and identify all the values that that index takes.

2. Continuing the previous homework, write $f = x$ and $g = 1$ in terms of the eigenfunctions you found for the case $\ell = 1$. Be very careful with one particular eigenfunction. Note that sometimes you need to write a term in a sum or sequence out separately from the others.

3. Continuing the previous homework, substitute $u(x,t) = \sum_n u_n(t) X_n(x)$ into the PDE to convert it into an ordinary differential for each separate coefficient $u_n(t)$. Solve the ODE. Be very careful with one particular case. By writing the initial conditions in terms of the eigenfunctions, identify the integration constants. Write out a complete summary of the solution. Make sure to identify the values of your numbering index in each expression.

4. (6 pts) Reconsider the separation of variables solution you derived. Using some programming language, evaluate the found solution at 101 equally spaced x-values from 0 to ℓ at times 0, 0.25, 0.5, and 1.25. Take $\ell = 1$ and $a = 1$. Include at least 50 nonzero terms in the summations. Plot the solution at these four times. Compare with the D'Alembert solution of the previous homework, which must be the same. (Check your D'Alembert solution first against the posted solution). Show also what happens if you only include 10 terms in the summations.

   To help you get started, a Matlab program that plots the solution to problem 7.28 is provided as an example. You need both p7_28.m[1] and p7_28u.m[2]. This program is valid for the PDE and BC solved in class, with the additional data

   $$a = \tfrac{1}{2}, \quad \ell = \tfrac{1}{2}\pi, \qquad f(x) = \tfrac{1}{2}\pi - x \Rightarrow f_n = \frac{1}{(2n-1)^2}, \qquad g(x) = 0 \Rightarrow g_n = 0.$$

   These may of course not apply for your problem.

   To run the program, enter matlab and type in `p7_28`. If you do not have matlab, a free replacement is octave. Or you can use some other programming and plotting facilities.

   Include your code.

13 04/17 F

1. Refer to problem 7.19. Find a function $u_0(x,t)$ that satisfies the inhomogeneous boundary conditions. Define $v = u - u_0$. Find the PDE, BC and IC satisfied by v.

2. Find suitable eigenfunctions in terms of which v may be written, and that satisfy the homogeneous boundary conditions. Write the relevant known functions in terms of these eigenfunctions and give the expressions for their Fourier coefficients.

3. Continuing the previous homework, solve for v using separation of variables in terms of integrals of the known functions $f(x)$, $g_0(t)$, and $g_1(t)$. Write the solution for u completely.

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[1] `../p7_28.m`
[2] `../p7_28u.m`

4. Assume that $f = 0$, $k = \ell = 1$, and that $u_x = t$ at both $x = 0$ and $x = \ell$. Work out the solution completely.

5. Plot the solution numerically at some relevant times. I suspect that for large times the solution is approximately

$$u = (x - \tfrac{1}{2})t + \tfrac{1}{6}(x - \tfrac{1}{2})^3 - \tfrac{1}{8}(x - \tfrac{1}{2})$$

   Do your results agree?

14 04/24 F

1. Solve 7.26, by Laplace transforming the problem as given in time. This is a good way to practice back transform methods. Note that one factor in $\widehat{u}$ is a simpler function at a shifted value of coordinate s.

2. Solve 7.35 by Laplace transform in time. Clean up completely; only the given function may be in your answer, no Heaviside functions or other weird stuff. There is a minor error in the book's answer.

3. Consider a simple problem of unsteady, axisymmetric, heat conduction in a ring (or unsteady unidirectional flow between concentric pipes) of radii a and b:

$$u_t = \kappa \left(u_{rr} + \frac{1}{r} u_r \right) \qquad u(r, 0) = f(r) \quad u(a, t) = 0 \quad u(b, t) = 0$$

   Find the eigenvalue problem for the eigenfunctions $R(r)$. Do not try to solve it (or look under Bessell functions in our math handbook). Given an arbitrary function $g(r)$, figure out how to obtain the coefficients g_n in

$$g(r) = \sum_{\text{all } n} g_n R_n(r)$$