7.28

1 7.28, §1 Asked

Asked: Find the unsteady pressure field $u(x,t)$ in a pipe with one end closed and the other open to the athmospere.

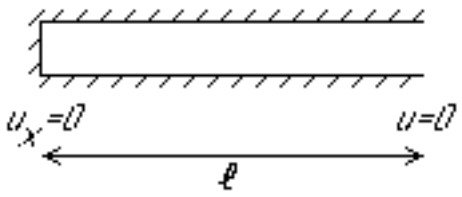

Figure 1: Acoustics in a pipe.

2 7.28, §2 P.D.E. Model

Always draw the depend-variables picture first.

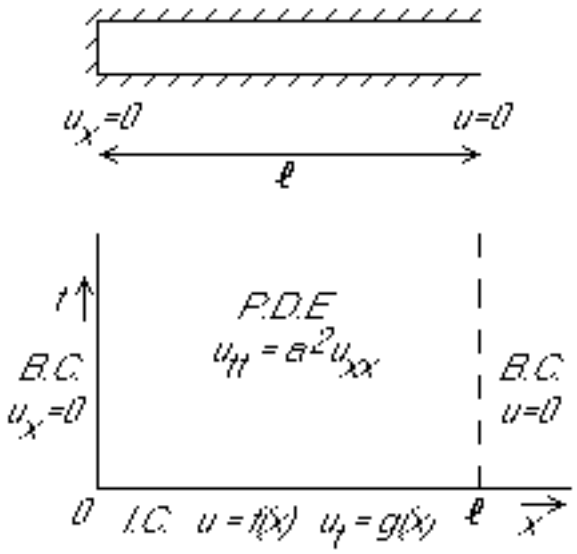

Figure 2: Dependent variables.

Observations:

- Unknown pressure $u = u(x, t)$.
- Constant speed of sound a.
- Finite domain $\bar{\Omega}$: $0 \leq x \leq \ell$.
- Hyperbolic PDE (waves, singularities, evolution).
- Two initial conditions.
- One homogeneous Neumann boundary condition at $x = 0$ and one homogeneous Dirichlet condition at $x = \ell$.

We will try to find a solution of this problem in the form

$$u = \sum_n u_n(t) X_n(x)$$

i.e. a sum, in which each term is a product of a function of x only times a function of t only.

3 7.28, §3 Eigenfunctions

To find the solution in the form,

$$u(x, t) = \sum_n u_n(t) X_n(x)$$

we try substituting an individual term of this general form into the PDE. In particular, we substitute a trial solution $u = T(t)X(x)$ into the homogeneous P.D.E. $u_{tt} = a^2 u_{xx}$. This gives:

$$T''(t)X(x) = a^2 T(t)X''(x)$$

Now we can take the terms depending on t only to one side of the equation, and the ones depending on x only to the other side:

$$\frac{T''(t)}{a^2 T(t)} = \frac{X''(x)}{X(x)}$$

This trick is why this solution procedure is called the "method of separation of variables."

While the right hand side, $X''(x)/X(x)$, does not depend on t, you would think that it would depend on the position x; both X and X'' change when x changes. But actually, X''/X does *not* change with x; after all, if we change x, it does nothing to t, so the left hand side does not change. And since the right hand side is the same, it too does not change. So the right hand side does not depend on either x or t; it must be a constant. By convention, we call the constant $-\lambda$:

$$\frac{T''}{a^2 T} = \frac{X''}{X} = \text{ constant } = -\lambda$$

If we also require X to satisfy the same homogeneous boundary conditions as u, i.e., that at $x = 0$, its x-derivative is zero, and that at $x = \ell$, X itself is zero, we get the following problem for X:

$$X'' + \lambda X = 0 \qquad X'(0) = 0 \qquad X(\ell) = 0$$

This is an ordinary differential equation boundary value problem.

Note that this problem is completely homogeneous: $X(x) = 0$ satisfies both the PDE *and* the boundary conditions. This is similar to the eigenvalue problem for vectors $A\vec{v} = \lambda\vec{v}$, which is certainly always true when $\vec{v} = 0$. But for the eigenvalue problem, we are interested in *nonzero* vectors $\vec{v}$ so that $A\vec{v} = \lambda v$, which only occurs for special values $\lambda_1, \lambda_2, \ldots$ of λ.

Similarly, we are interested only in *nonzero solutions X(x)* of the above ODE and boundary conditions. Eigenvalue problems for functions such as the one above are called "Sturm-Liouville problems." The biggest differences from matrix eigenvalue problems are:

- There are infinitely many eigenvalues $\lambda_1, \lambda_2, \ldots$ and corresponding eigenfunctions $X_1(x)$, $X_2(x)$, ... rather than just n eigenvalues and eigenvectors.
- We cannot write a determinant to find the eigenvalues. Instead we must solve the problem using our methods for solving ODE.

Fortunately, the above ODE is simple: it is a constant coefficient one, so we write its characteristic polynomial:

$$k^2 + \lambda = 0 \quad \Longrightarrow \quad k = \pm\sqrt{-\lambda} = \pm \mathrm{i}\sqrt{\lambda}$$

We must now find *all* possible eigenvalues λ and corresponding eigenfunctions that satisfy the required boundary conditions. We must look at all possibilities, one at a time.

Case $\lambda < 0$:

Since $k = \pm\sqrt{-\lambda}$

$$X = Ae^{\sqrt{-\lambda}x} + Be^{-\sqrt{-\lambda}x}$$

We try to satisfy the boundary conditions:

$$X'(0) = 0 = A\sqrt{-\lambda} - B\sqrt{-\lambda} \quad \Longrightarrow \quad B = A$$

$$X(\ell) = 0 = A\left(e^{\sqrt{-\lambda}\ell} + e^{-\sqrt{-\lambda}\ell}\right) \quad \Longrightarrow \quad A = 0$$

So $A = B = 0$; there are *no* nontrivial solutions for $\lambda < 0$.

Case $\lambda = 0$:

Since $k_1 = k_2 = 0$ we have a multiple root of the characteristic equation, and the solution is

$$X = Ae^{0x} + Bxe^{0x} = A + Bx$$

We try to satisfy the boundary conditions again:

$$X'(0) = 0 = B \qquad X(\ell) = 0 = A$$

So $A = B = 0$; there are again no nontrivial solutions.

Case $\lambda > 0$:

Since $k = \pm\sqrt{-\lambda} = \pm \mathrm{i}\sqrt{\lambda}$, the solution of the ODE is after cleanup:

$$X = A \sin\left(\sqrt{\lambda}x\right) + B \cos\left(\sqrt{\lambda}x\right)$$

We try to satisfy the first boundary condition:

$$X'(0) = 0 = A\sqrt{\lambda}$$

Since we are looking at the case $\lambda > 0$, this can only be true if $A = 0$. So, we need

$$X = B \cos\left(\sqrt{\lambda}x\right)$$

We now try to also satisfy the second boundary condition:

$$X(\ell) = 0 = B \cos\left(\sqrt{\lambda}\ell\right) = 0$$

For a nonzero solution, B may not be zero, so the cosine must be zero. For positive argument, a cosine is zero at $\frac{1}{2}\pi, \frac{3}{2}\pi, \ldots$, so that our eigenvalues are

$$\sqrt{\lambda_1} = \frac{\pi}{2\ell}, \sqrt{\lambda_2} = \frac{3\pi}{2\ell}, \sqrt{\lambda_3} = \frac{5\pi}{2\ell}, \ldots$$

The same as for eigenvectors, for our eigenfunctions we *must choose* the one undetermined parameter B. Choosing each $B = 1$, we get the eigenfunctions:

$$X_1 = \cos\left(\frac{\pi x}{2\ell}\right), X_2 = \cos\left(\frac{3\pi x}{2\ell}\right), X_3 = \cos\left(\frac{5\pi x}{2\ell}\right), \ldots$$

Total:

The only eigenvalues for this problem are the positive ones above, with the corresponding eigenfunctions. If we want to evaluate them on a computer, we need a general formula for them. You can check that it is:

$$\lambda_n = \frac{(2n-1)^2\pi^2}{4\ell^2} \qquad X_n = \cos\left(\frac{(2n-1)\pi x}{2\ell}\right) \qquad (n = 1, 2, 3, \ldots)$$

Just try a few values for n. We have finished finding the eigenfunctions.

4 7.28, §4 Solve the Other ODE?

If you look back to the beginning of the previous section, you may wonder about the function $T(t)$. It satisfied

$$\frac{T''}{a^2T} = -\lambda$$

Now that we have found the values for λ from the X-problem, we could solve this ODE too, and find functions $T_1(t), T_2(t), \ldots$. Many people do exactly that. However, if you want to follow the crowd, please keep in mind the following:

1. The values of λ can only be found from the Sturm-Liouville problem for X. The problem for T is *not* a Sturm-Liouville problem and *can never* produce the correct values for λ.
2. The functions $T(t)$ do *not* satisfy the same initial conditions at time $t = 0$ as u does.
3. Finding T is useless if the PDE is inhomogeneous; it simply does not work. (Unless you add still more artificial tricks to the mix, as the book does.)

We will just ignore the entire T. Instead in the next section we will systematically solve the problem for u without tricks using our found eigenfunctions. What we do there will always work. If you want to try to take a shortcut for an homogeneous PDE, well, the responsibility and risk are yours alone. Someday I will stop seeing students getting themselves in major trouble this way at the final, but it may not be this year.

5 7.28, §5 Solve the PDE

So what is the procedure for solving the original problem for the pressure u having found the eigenfunctions X_n? It is to write everything in terms of eigenfunctions. And if I say everything, I mean *everything.*

We first write our solution $u(x,t)$ in terms of the eigenfunctions:

$$u(x,t) = \sum_{n=1}^{\infty} u_n(t) X_n(x)$$

The coefficients $u_n(t)$ are called the "Fourier coefficients" of u. The sum is called the "Fourier series" for u.

We know our eigenfunctions $X_n(x)$, but not yet our Fourier coefficients $u_n(t)$. In fact, the $u_n(t)$ are what is still missing; if we know the $u_n(t)$, we can find the pressure u we want by doing the sum above. On a computer probably, if we want to get high accuracy. Or just the first few terms by hand, if we accept some numerical error.

Second, we also write our PDE, $u_{tt} = a^2 u_{xx}$, in terms of the eigenfunctions:

$$\sum_{n=1}^{\infty} \ddot{u}_n(t) X_n(x) = a^2 \sum_{n=1}^{\infty} u_n(t) X_n''(x)$$

This PDE will *always* simplify; that is how the method of separation of variables works. Look up the Sturm-Liouville problem for X_n in the previous section; it was $X_n''(x) = -\lambda_n X_n(x)$. So we can get rid of the x-derivatives in the PDE:

$$\sum_{n=1}^{\infty} \ddot{u}_n(t) X_n(x) = a^2 \sum_{n=1}^{\infty} \left(-\lambda_n u_n(t)\right) X_n(x)$$

Now if two functions are equal, all their Fourier coefficients must be equal, so we have, for any value of n,

$$\ddot{u}_n(t) = -a^2 \lambda_n u_n(t) \qquad (n = 1, 2, 3, \ldots)$$

That no longer contains x at all: *the PDE has become a set of ODE in t only.* And we (hopefully) know how to solve those! Getting rid of x is really what the method of separation variables does for us.

We can solve the ODE above easily. It is a constant coefficient one, with a characteristic equation $k^2 = -a^2 \lambda_n$, hence $k = \pm ia\sqrt{\lambda}_n$, giving

$$u_n(t) = C_{1n} e^{ia\sqrt{\lambda}_n t} + C_{2n} e^{-ia\sqrt{\lambda}_n t}$$

or after cleaning up,

$$u_n(t) = D_{1n} \cos\left(a\sqrt{\lambda}_n t\right) + D_{2n} \sin\left(a\sqrt{\lambda}_n t\right)$$

So, we have already found our pressure a bit more precisely:

$$u(x,t) = \sum_{n=1}^{\infty} \left[D_{1n} \cos\left(a\sqrt{\lambda}_n t\right) + D_{2n} \sin\left(a\sqrt{\lambda}_n t\right)\right] X_n(x)$$

but we still need to figure out what the integration constants D_{1n} and D_{2n} are.

6 7.28, §6 Satisfy the IC

In the previous section, we found the general solution of the PDE with the given boundary conditions to be

$$u(x,t) = \sum_{n=1}^{\infty} \left[D_{1n} \cos\left(a\sqrt{\lambda}_n t\right) + D_{2n} \sin\left(a\sqrt{\lambda}_n t\right)\right] X_n(x)$$

Now we want to find the coefficients D_{1n} and D_{2n} for all values of n from the initial conditions. That will fully determine the solution.

To do so, we must first write our initial condition $u(x,0) = f(x)$ and $u_t(x,0) = g(x)$ in terms of the eigenfunctions. Writing the Fourier series for the two functions as

$$f(x) = \sum_{n=1}^{\infty} f_n X_n(x) \qquad g(x) = \sum_{n=1}^{\infty} g_n X_n(x).$$

and using the Fourier series for u above, the two initial conditions become

$$\sum_{n=1}^{\infty} D_{1n} X_n(x) = \sum_{n=1}^{\infty} f_n X_n(x)$$

$$\sum_{n=1}^{\infty} a\sqrt{\lambda}_n D_{2n} X_n(x) = \sum_{n=1}^{\infty} g_n X_n(x).$$

The Fourier coefficients must again be equal, so we conclude that the coefficients we are looking for are

$$D_{1n} = f_n \qquad D_{2n} = \frac{g_n}{a\sqrt{\lambda}_n}$$

The Fourier series for u becomes now

$$u(x,t) = \sum_{n=1}^{\infty} \left[f_n \cos\left(a\sqrt{\lambda}_n t\right) + \frac{g_n}{a\sqrt{\lambda}_n} \sin\left(a\sqrt{\lambda}_n t\right) \right] X_n(x)$$

where

$$\lambda_n = \frac{(2n-1)^2\pi^2}{4\ell^2} \qquad X_n = \cos\left(\frac{(2n-1)\pi x}{2\ell}\right)$$

So, if we can find the Fourier coefficients f_n and g_n of functions $f(x)$ and $g(x)$, we are done.

Now $f(x)$ and $g(x)$ are, supposedly, given functions, but how do we find their Fourier coefficients? The answer is the following important formula:

$$\boxed{f_n = \frac{\int_0^l f(x) X_n(x) \mathrm{d}x}{\int_0^l X_n(x)^2 \mathrm{d}x}}$$

which is called the "orthogonality relation". Even if $f(x) = 1$, say, we still need to do those integrals. The same for g of course:

$$g_n = \frac{\int_0^l g(x) X_n(x) \mathrm{d}x}{\int_0^l X_n(x)^2 \mathrm{d}x}$$

We are done! Or at least, we have done as much as we can do until someone tells us the actual functions $f(x)$ and $g(x)$. If they do, we just do the integrals above to find all the f_n and g_n, (maybe analytically or on a computer), and then we can sum the expression for $u(x,t)$ for any x and t that strikes our fancy.

Note that we did not have to do anything with the boundary conditions $u_x(0,t) = 0$ and $u(\ell,t) = 0$; since every eigenfunction X_n satisfies them, the expression for u above automatically also satisfies these homogeneous boundary conditions.

7 7.28, §7 Comparison

Separation of variables solution found as:

$$u = \sum_{n=1}^{\infty} \left[f_n \cos\left(\frac{(2n-1)\pi a t}{2\ell}\right) + \frac{2\ell g_n}{(2n-1)\pi a} \sin\left(\frac{(2n-1)\pi a t}{2\ell}\right) \right] \cos\left(\frac{(2n-1)\pi x}{2\ell}\right)$$

- Shows the natural frequencies (tones) to be $\pi a/2\ell$, $3\pi a/2\ell$,...
- Shows the energy in each harmonic.
- Not restricted to the 1D wave equation.

D'Alembert:

$$u(x,t) = \frac{\bar{f}(x-at) + \bar{f}(x+at)}{2} + \frac{1}{2a}\int_{x-at}^{x+at} \bar{g}(\xi)\,\mathrm{d}\xi$$

- I can evaluate the pressure at any point without doing big sums.
- Shows how wave fronts propagate.
- Shows regions of influence and dependence.

8 7.28, §8 Comparison Of SOV and Eigenvalue Problems

The similarity between eigenvalue problems and Sturm-Liouville is not a coincidence. This section explores it a bit deeper:

8.1 Functions as vectors

First I want to convince you that there is no big difference between vectors and functions. A vector $\vec{f}$ is usually shown in the form of an arrow:

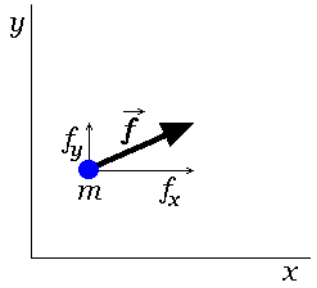

Figure 3: The classical picture of a vector.

However, the same vector may instead be represented as a spike diagram, by plotting the value of the components versus the component index:

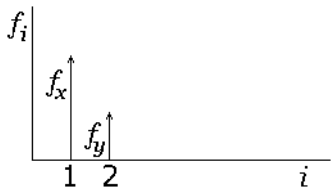

Figure 4: Spike diagram of a vector.

In the same way as in two dimensions, a vector in three dimensions, or, for that matter, in thirty dimensions, can be represented by a spike diagram:

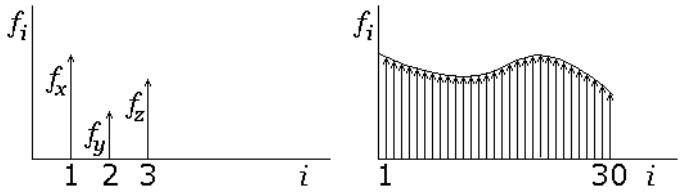

Figure 5: More dimensions.

For a large number of dimensions, and in particular in the limit of infinitely many dimensions, the large values of i can be rescaled into a continuous coordinate, call it x. For example, x might be defined as i divided by the number of dimensions. In any case, the spike diagram becomes a function $f(x)$:

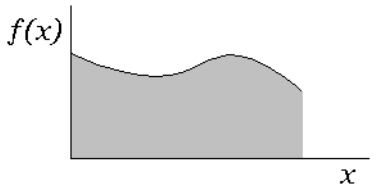

Figure 6: Infinite dimensions.

The spikes are usually not shown:

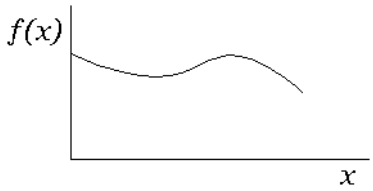

Figure 7: The classical picture of a function.

In this way, a function is just a vector in infinitely many dimensions.

8.2 The dot product

The dot product of vectors is an important tool. It makes it possible to find the length of a vector, by multiplying the vector by itself and taking the square root. It is also used to check if two vectors are orthogonal: if their dot product is zero, they are. In this subsection, the dot product is defined for functions.

The usual dot product of two vectors $\vec{f}$ and $\vec{g}$ can be found by multiplying components with the same index i together and summing that:

$$\vec{f} \cdot \vec{g} \equiv f_1 g_1 + f_2 g_2 + f_3 g_3$$

Figure 8 shows multiplied components using equal colors.

Note the use of numeric subscripts, f_1, f_2, and f_3 rather than f_x, f_y, and f_z; it means the same thing. Numeric subscripts allow the three term sum above to be written more compactly as:

$$\vec{f} \cdot \vec{g} \equiv \sum_{\text{all } i} f_i g_i$$

The Σ is called the "summation symbol."

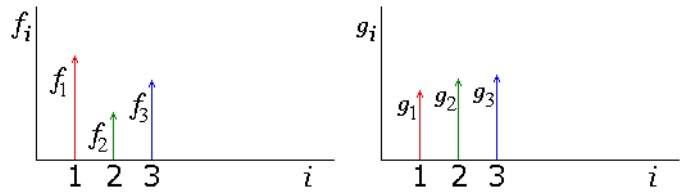

Figure 8: Forming the dot product of two vectors.

The dot (or "inner") product of functions is defined in exactly the same way as for vectors, by multiplying values at the same x position together and summing. But since there are infinitely many x-values, the sum becomes an integral:

$$f \cdot g = \int_{\text{all } x} f(x)g(x)\,\mathrm{d}x \tag{1}$$

as illustrated in figure 9.

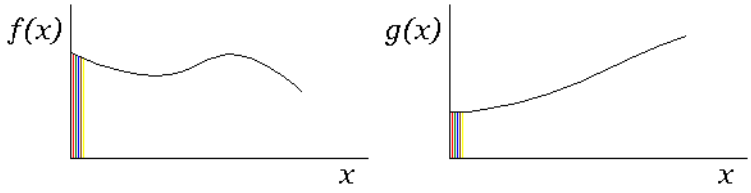

Figure 9: Forming the inner product of two functions.

8.3 Application to the present problem

Now let us have a look at the separation of variables procedure for $u_{tt} = a^2 u_{xx}$. First of all, let us write this problem like $u_{tt} = Au$ where A is the operator

$$A = a^2 \frac{\partial^2}{\partial x^2}$$

I want to convince you that the operator A is really like a matrix. Indeed, a matrix A can transform any arbitrary vector $\vec{f}$ into a different vector $A\vec{f}$:

$$\vec{f} \quad \xrightarrow{\text{matrix } A} \quad \vec{g} = A\vec{f}$$

Similarly, the operator A above transforms a function into another function, a^2 times its second order derivative:

$$f(x) \quad \xrightarrow{\text{operator } A} \quad g(x) = Af(x) = a^2 f''(x)$$

Now I want to convince you that the operator A above is not just a matrix, but a symmetric matrix. If a matrix A is symmetric then for any two vectors $\vec{f}$ and $\vec{g}$,

$$\vec{f}^T(A\vec{g}) = (A\vec{f})^T\vec{g}$$

or in other words, the dot product between $\vec{f}$ and $A\vec{g}$ is the same as that between $A\vec{f}$ and $\vec{g}$. For our operator A we have:

$$f \cdot (Ag) = \int_{\text{all } x} f(x)a^2 g''(x)\,\mathrm{d}x$$

and using two integrations by parts, (and the homogeneous boundary conditions,) this can be transformed into $Af \cdot g$. So A must be symmetric.

Now it is no longer that surprising that we have only *real* eigenvalues; that is a general property of symmetric matrices.

And remember also that symmetric matrices have orthogonal eigenvectors, so the eigenfunctions X_1, X_2, X_3, ... of operator A are going to be orthogonal. And a complete set, so we can write *any* function of x in terms of these eigenfunctions, including the initial conditions f, g, and the solution u:

$$f = \sum f_n X_n \qquad g = \sum g_n X_n \qquad u = \sum u_n X_n$$

How do we get the f_n given f? Well, we usually do not normalize the eigenfunctions to "length" one, so the unit function in the direction of an X_n is:

$$e_n = \frac{1}{\sqrt{X_n \cdot X_n}} X_n$$

and the component of f in the direction of this unit function, call it c_n, is found by taking a dot product:

$$c_n = \frac{X_n \cdot f}{\sqrt{X_n \cdot X_n}}$$

The component vector of f in the direction of X_n is then c_n times the unit function:

$$\frac{X_n \cdot f}{\sqrt{X_n \cdot X_n}} e_n = \frac{X_n \cdot f}{X_n \cdot X_n} X_n$$

so the Fourier coefficient f_n is

$$f_n = \frac{X_n \cdot f}{X_n \cdot X_n}$$

That is our old "orthogonality relation."

Another way to understand the orthogonality relation is in terms of transformation matrices. Note that the f_n are just the components of function f when written in terms of the eigenfunctions. We get them by evaluating $P^{-1}f$, where the transformation matrix P consists of

the eigenfunctions as columns, and P^{-1} is found as the transpose matrix (ignoring the lack of normalization of the eigenfunctions):

$$P = (X_1, X_2, X_3, \ldots) \qquad P^{-1} = \begin{pmatrix} X_1 \\ X_2 \\ X_3 \\ \vdots \end{pmatrix}$$

It is seen that $P^{-1}f$ then works out to dot products between the X_n and f.