7.20m

1 7.20m, §1 Asked

Asked: Find the unsteady temperature distribution in the bar below for arbitrary position and time if the initial temperature distribution f at time zero, the heat flux g_0 out of the left end, the temperature g_1 of the right end, and the added heat q are given.

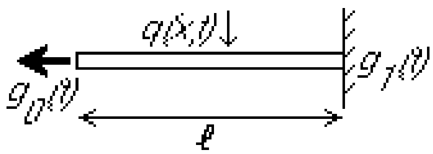

Figure 1: Heat conduction in a bar.

2 7.20m, §2 P.D.E. Model

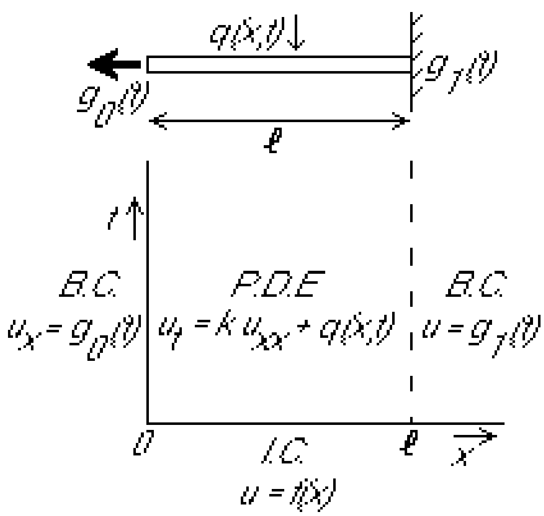

Figure 2: Heat conduction in a bar.

- Finite domain $\bar{\Omega}$: $0 \leq x \leq \ell$
- Unknown temperature $u = u(x, t)$
- Constant κ, so a linear constant coefficient PDE.

- Parabolic
- Inhomogeneous
- One initial condition
- One Neumann boundary condition
- One Dirichlet boundary condition
- All of f, g_0, g_1, and q are given functions.

We would like to use separation of variables to write the solution in a form that looks roughly like:

$$u(x,t) = \sum_n u_n(t) X_n(x)$$

Here the X_n would be the eigenfunctions. The u_n cannot be eigenfunctions since the time axis is semi-infinite. Also, Sturm-Liouville problems require boundary conditions at both ends, not initial conditions.

However, eigenfunctions must have homogeneous boundary conditions, so if u was written as a sum of eigenfunctions, it could not satisfy the given inhomogeneous boundary conditions. Fortunately, we can apply a trick to get around this problem.

3 7.20m, §3 Boundary Condition Fix

To get rid of the inhomogeneous boundary conditions at $x = 0$ and $x = \ell$, use the following trick:

Trick: Find any function u_0 that satisfies the inhomogeneous boundary conditions at $x = 0$ and $x = \ell$ and substract it from u. The remainder, call it v, will have homogeneous boundary conditions.

So, we try to find a $u_0(x,t)$ that satisfies the same boundary conditions as $u(x,t)$:

$$u_{0x}(0,t) = g_0(t) \qquad u_0(\ell,t) = g_1(t)$$

This u_0 *does not* have to satisfy the PDE nor IC, which allows us to take something simple for it.

A linear function of x works:

$$u_0(x,t) = A(t) + B(t)x$$

If we require this to satisfy the two boundary conditions for u above, we get

$$B(t) = g_0(t) \qquad A(t) + B(t)\ell = g_1(t)$$

The solution is $B(t) = g_0(t)$ and $A(t) = g_1(t) - B(t)\ell$. So our u_0 is

$$\boxed{u_0(x,t) = g_1(t) + g_0(t)(x - \ell)}$$

Please keep in mind what we know, and what we do not know. Since we (supposedly) have been given functions $g_0(t)$ and $g_1(t)$, function u_0 is from now on a *known* quantity, as above. I put a box around it so that we can later find it back.

You could use something more complicated than a linear function if you like to make things difficult for yourself. Go ahead and use $A(t)\mathrm{erf}(x) + B(t)J_0(x)$ if you really love to integrate error functions and Bessel functions. It will work. I prefer a linear function myself, though. (For some problems, you may need a quadratic instead of a linear function.)

Under certain conditions, there may be a better choice than a low order polynomial in x. If the problem has steady boundary conditions and a simple steady solution, go ahead and take u_0 to be that steady solution. It will work great. However, in this case the boundary conditions are not steady; we are assuming that g_0 and g_1 are arbitrary given functions of time.

Having found u_0, define a new unknown v as the remainder when u_0 is substracted from u:

$$v \equiv u - u_0$$

We now solve the problem by finding v. When we have found v, we simply add u_0, already known, back in to get u.

To do so, first, of course, we need the problem for v to solve. We get it from the problem for u by everywhere replacing u by $u_0 + v$. Let's take the picture of the problem for u in front of us and start converting.

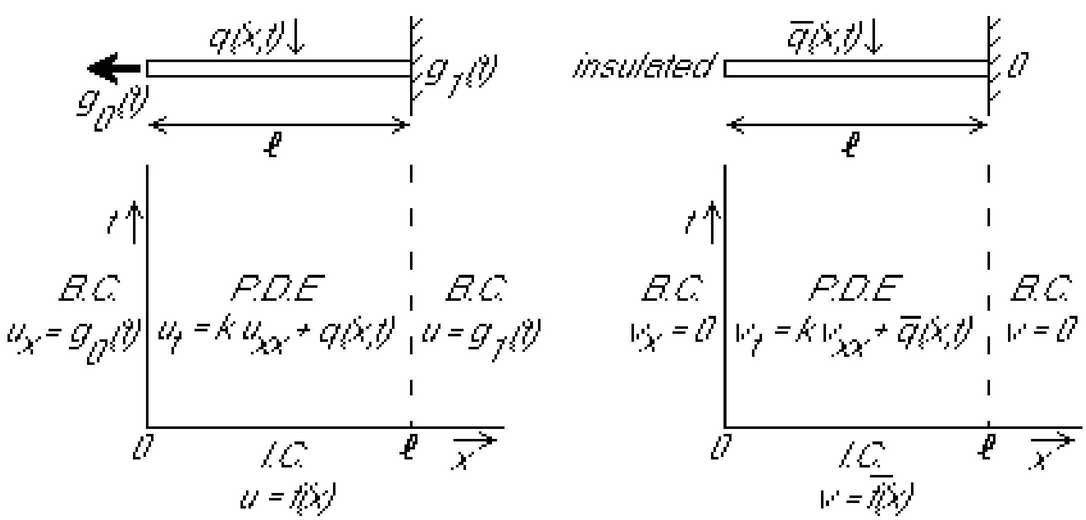

Figure 3: Heat conduction in a bar.

First take the boundary conditions at $x = 0$ and $x = \ell$:

$$u_x(0,t) = g_0(t) \qquad u(\ell,t) = g_1(t)$$

Replacing u by $u_0 + v$:

$$u_{0x}(0,t) + v_x(0,t) = g_0(t) \qquad u_0(\ell,t) + v(\ell,t) = g_1(t)$$

But since by construction $u_{0x}(0,t) = g_0$ and $u_0(\ell,t) = g_1$,

$$v_x(0,t) = 0 \qquad v(\ell,t) = 0$$

Note the big thing: while the boundary conditions for v are similar to those for u, they are *homogeneous*. We will get a Sturm-Liouville problem in the x-direction for v where we did not for u. That is what u_0 does for us.

We continue finding the rest of the problem for v. We replace u by $u_0 + v$ into the PDE $u_t = \kappa u_{xx} + q$,

$$u_{0t} + v_t = \kappa(u_{0xx} + v_{xx}) + q$$

and take all u_0 terms to the right hand side:

$$v_t = \kappa v_{xx} + \bar{q}$$

where $\bar{q} = \kappa u_{0xx} + q - u_{0t}$, or, written out

$$\boxed{\bar{q}(x,t) = q(x,t) - g_1'(t) - g_0'(t)(x - \ell)}$$

Hence $\bar{q}$ is now a *known* function, just like q.

The final part of the problem for u that we have not converted yet is the initial condition. We replace u by $u_0 + v$ in $u(x,0) = f(x)$,

$$u_0(x,0) + v(x,0) = f(x)$$

and take u_0 to the other side:

$$v(x,0) = \bar{f}(x)$$

where $\bar{f}(x)$ is $f(x) - u_0(x,0)$, or written out:

$$\boxed{\bar{f}(x) = f(x) - g_1(0) - g_0(0)(x - \ell)}$$

Again, $\bar{f}$ is now a known function.

The problem for v is now the same as the one for u, except that the boundary conditions are homogeneous and functions f and q have changed into known functions $\bar{f}$ and $\bar{q}$.

Using separation of variables, we can find the solution for v in the form:

$$v(x,t) = \sum_n v_n(t) X_n(x).$$

We already know how to do that! (Don't worry, we will go over the steps anyway.) Having found v, we will simply add u_0 to find the asked temperature u.

4 7.20m, §4 Eigenfunctions

To find the eigenfunctions X_n, substitute a trial solution $v = T(t)X(x)$ into the *homogeneous part* of the PDE, $v_t = \kappa v_{xx} + \bar{q}$. Remember: ignore the inhomogeneous part $\bar{q}$ when finding the eigenfunctions. Putting $v = T(t)X(x)$ into $v_t = \kappa v_{xx}$ produces:

$$T'X = \kappa T X''$$

Separate variables:

$$\frac{T'}{\kappa T} = \frac{X''}{X} = \text{ constant } = -\lambda$$

As always, λ cannot depend on x since the left hand side does not. Also, λ cannot depend on t since the middle does not. So λ must be a constant.

We then get the following Sturm-Liouville problem for any eigenfunctions $X(x)$:

$$-X'' = \lambda X \qquad X'(0) = 0 \qquad X(\ell) = 0$$

The last two equations are the boundary conditions on v which we made homogeneous.

This is the exact same eigenvalue problem that we had in problem 7.28b, so I can just take the solution from there. The eigenfunctions are:

$$\boxed{\lambda_n = \frac{(2n-1)^2\pi^2}{4\ell^2} \qquad X_n = \cos\left(\frac{(2n-1)\pi x}{2\ell}\right) \qquad (n = 1, 2, 3, \ldots)}$$

5 7.20m, §5 Solve the Problem

We again expand *everything* in the problem for v in a Fourier series:

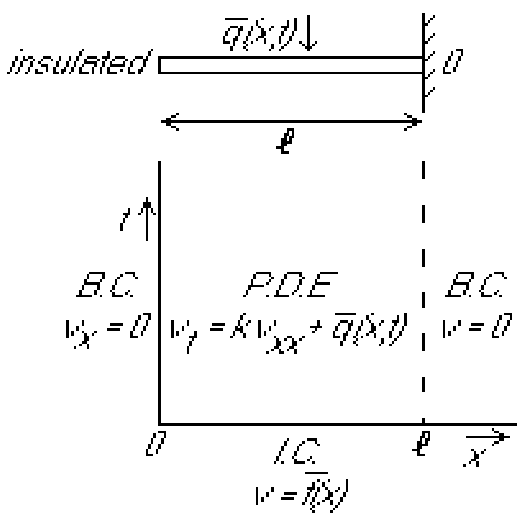

Figure 4: Heat conduction in a bar.

We write

$$v = \sum_{n=1}^{\infty} v_n(t) X_n(x) \quad \bar{f} = \sum_{n=1}^{\infty} \bar{f}_n X_n(x) \quad \bar{q} = \sum_{n=1}^{\infty} q_n(t) X_n(x)$$

Since $\bar{q}(x)$ and $\bar{f}(x)$ are known functions, we can find their Fourier coefficients from orthogonality:

$$\bar{f}_n = \frac{\int_0^\ell \bar{f}(x) X_n(x)\,\mathrm{d}x}{\int_0^\ell X_n^2(x)\,\mathrm{d}x}$$

$$\bar{q}_n(t) = \frac{\int_0^\ell \bar{q}(x,t) X_n(x)\,\mathrm{d}x}{\int_0^\ell X_n^2(x)\,\mathrm{d}x}$$

or with the eigenfunctions written out

$$\boxed{\bar{f}_n = \frac{\int_0^\ell \bar{f}(x) \cos((2n-1)\pi x/2\ell)\,\mathrm{d}x}{\int_0^\ell \cos^2((2n-1)\pi x/2\ell)\,\mathrm{d}x}}$$

$$\boxed{\bar{q}_n(t) = \frac{\int_0^\ell \bar{q}(x,t) \cos((2n-1)\pi x/2\ell)\,\mathrm{d}x}{\int_0^\ell \cos^2((2n-1)\pi x/2\ell)\,\mathrm{d}x}}$$

The integrals in the bottom equal $\frac{1}{2}\ell$.

So the Fourier coefficients $\bar{f}_n$ are now known constants, and the $\bar{q}_n(t)$ are now known functions of t. Though in actual application, numerical integration may be needed to find them. During finals, I usually make the functions f, g_0 and g_1 simple enough that you can do the integrals analytically.

Now write the PDE $v_t = \kappa v_{xx} + \bar{q}$ using the Fourier series:

$$\sum_{n=1}^{\infty} \dot{v}_n(t) X_n(x) = \kappa \sum_{n=1}^{\infty} v_n(t) X_n''(x) + \sum_{n=1}^{\infty} q_n(t) X_n(x)$$

Looking in the previous section, the Sturm-Liouville ODE was $-X'' = \lambda X$, so the PDE simplifies to:

$$\sum_{n=1}^{\infty} \dot{v}_n(t) X_n(x) = -\kappa \sum_{n=1}^{\infty} \lambda_n v_n(t) X_n(x) + \sum_{n=1}^{\infty} q_n(t) X_n(x)$$

It will always simplify or you made a mistake.

For the sums to be equal for any x, the coefficients of every individual eigenfunction must balance. So we get

$$\dot{v}_n(t) + \kappa \lambda_n v_n(t) = q_n(t)$$

We have obtained an ODE for each v_n. It is again constant coefficient, but inhomogeneous.

Solve the homogeneous equation first. The characteristic polynomial is

$$k + \kappa \lambda_n = 0$$

so the homogeneous solution is

$$v_{nh} = A_n e^{-\kappa\lambda_n t}$$

For the inhomogeneous equation, since we do not know the actual form of the functions q, undetermined constants is not a possibility. So we use variation of parameter:

$$v_n = A_n(t) e^{-\kappa\lambda_n t}$$

Plugging into the ODE produces

$$A_n' e^{-\kappa\lambda_n t} + 0 = q_n(t) \quad \Longrightarrow \quad A_n' = q_n(t) e^{\kappa\lambda_n t}$$

We integrate this equation to find A_n. I could write the solution using an indefinite integral:

$$A_n(t) = \int q_n(t) e^{\kappa\lambda_n t} \, \mathrm{d}t$$

But that has the problem that the integration constant is not explicitly shown, which makes it impossible to apply the initial condition. It is better to write the anti-derivative using an integral with limits plus an explicit integration constant as:

$$A_n(t) = \int_{\tau=0}^{t} q_n(\tau) e^{\kappa\lambda_n \tau} \, \mathrm{d}\tau + A_{n0}$$

You can check using the Leibnitz rule for differentiation of integrals (or really, just the fundamental theorem of calculus,) that the derivative is exactly what it should be. (Also, the lower limit does not really have to be zero; you could start the integration from 1, if it would be simpler. The important thing is that the upper limit is the independent variable t.)

Putting the found solution for $A_n(t)$ into

$$v_n = A_n(t) e^{-\kappa\lambda_n t}$$

we get, cleaned up:

$$v_n(t) = \int_{\tau=0}^{t} q_n(\tau) e^{-\kappa\lambda_n (t-\tau)} \, \mathrm{d}\tau + A_{n0} e^{-\kappa\lambda_n t}$$

We still need to find the integration constant A_{n0}. To do so, write the IC $v(x,0) = \bar{f}(x)$ using Fourier series:

$$\sum_{n=0}^{\infty} v_n(0) X_n(x) = \sum_{n=0}^{\infty} \bar{f}_n X_n(x)$$

This gives us initial conditions for the v_n:

$$v_n(0) = \bar{f}_n = A_{n0}$$

the latter from above, and hence

$$v_n(t) = \int_{\tau=0}^{t} q_n(\tau) e^{-\kappa\lambda_n (t-\tau)} \, \mathrm{d}\tau + \bar{f}_n e^{-\kappa\lambda_n t}$$

or writing out the eigenvalue:

$$\boxed{v_n(t) = \int_{\tau=0}^{t} q_n(\tau) e^{-\kappa(2n-1)^2\pi^2(t-\tau)/4\ell^2} \, \mathrm{d}\tau + \bar{f}_n e^{-\kappa(2n-1)^2\pi^2 t/4\ell^2}}$$

We have v_n in terms of known quantities, so we are done.

6 7.20m, §6 Total

Collecting all the boxed formulae together, the solution is found by first computing the coefficients $\bar{f}_n$ from:

$$\bar{f}_n = \frac{2}{\ell}\int_0^\ell \bar{f}(x)\cos((2n-1)\pi x/2\ell)\,\mathrm{d}x \qquad (n = 1, 2, 3, \ldots)$$

where

$$\bar{f}(x) = f(x) - g_1(0) - g_0(0)(x-\ell)$$

Also compute the functions $\bar{q}_n(t)$ from:

$$\bar{q}_n(t) = \frac{2}{\ell}\int_0^\ell \bar{q}(x,t)\cos((2n-1)\pi x/2\ell)\,\mathrm{d}x \qquad (n = 1, 2, 3, \ldots)$$

where

$$\bar{q}(x,t) = q(x,t) - g_1'(t) - g_0'(t)(x-\ell)$$

Then the temperature is:

$$\begin{aligned} u(x,t) &= g_1(t) + g_0(t)(x-\ell) \\ &+ \sum_{n=1}^{\infty}\left[\int_{\tau=0}^{t} q_n(\tau)e^{-\kappa(2n-1)^2\pi^2(t-\tau)/4\ell^2}\,\mathrm{d}\tau + \bar{f}_n e^{-\kappa(2n-1)^2\pi^2 t/4\ell^2}\right]\cos((2n-1)\pi x/2\ell) \end{aligned}$$

7 7.20m, §7 More Fun

We can, if we want, write the solution for v in other ways that may be more efficient numerically. The solution was, rewritten more concisely in terms of the eigenvalues and eigenfunctions:

$$v(x,t) = \sum_n \left[\int_{\tau=0}^{t} \bar{q}_n(\tau)e^{-\kappa\lambda_n(t-\tau)}\,\mathrm{d}\tau + \bar{f}_n e^{-\kappa\lambda_n t}\right] X_n(x).$$

The first part is due to the inhomogeneous term $\bar{q}$ in the PDE, the second due to the initial condition $v(x,0) = \bar{f}(x)$

Look at the second term first, let's call it v_f,

$$v_f \equiv \sum_n \bar{f}_n e^{-\kappa\lambda_n t} X_n(x).$$

We can substitute in the orthogonality relationship for $\bar{f}(x)$:

$$v_f = \sum_n \frac{\int_0^\ell \bar{f}(\xi)X_n(\xi)\,\mathrm{d}\xi}{\int_0^\ell X_n^2(\zeta)\,\mathrm{d}\zeta}\, e^{-\kappa\lambda_n t} X_n(x)$$

and change the order of the terms to get:

$$v_f = \int_0^\ell \left[\sum_n \frac{X_n(\xi)X_n(x)}{\int_0^\ell X_n^2(\zeta)\,\mathrm{d}\zeta}\, e^{-\kappa\lambda_n t} \right] \bar{f}(\xi) d\xi$$

We define a shorthand symbol for the term within the square brackets:

$$G(x,t,\xi) \equiv \sum_n \frac{X_n(\xi)X_n(x)}{\int_0^\ell X_n^2(\zeta)\,\mathrm{d}\zeta}\, e^{-\kappa\lambda_n t}$$

Since this does not depend on what function $\bar{f}(x)$ is, we can evaluate G once and for all. For any $\bar{f}(x)$, the corresponding temperature is then simply found by integration:

$$v_f(x,t) = \int_0^\ell G(x,t,\xi)\bar{f}(\xi) d\xi$$

Function $G(x,t,\xi)$ by itself is the temperature $v(x,t)$ if $\bar{f}$ is a single spike of heat initially located at $x = \xi$. Mathematically, G is the solution for v if $\bar{f}(x)$ is the "delta function" $\delta(x-\xi)$.

Now look at the first term in v, due to $\bar{q}$, let's call it v_q:

$$v_q \equiv \sum_n \int_{\tau=0}^t \bar{q}_n(\tau) e^{-\kappa\lambda_n(t-\tau)}\,\mathrm{d}\tau X_n(x)$$

We plug in the orthogonality expression for $\bar{q}_n(\tau)$:

$$v_q = \sum_{n=0}^\infty \int_{\tau=0}^t \frac{\int_0^\ell \bar{q}(\xi,\tau)X_n(\xi)\,\mathrm{d}\xi}{\int_0^\ell X_n^2(\zeta)\,\mathrm{d}\zeta}\, e^{-\kappa\lambda_n(t-\tau)}\,\mathrm{d}\tau X_n(x)$$

and rewrite

$$v_q = \int_{\tau=0}^t \int_0^\ell \left[\sum_n \frac{X_n(\xi)X_n(x)}{\int_0^\ell X_n^2(\zeta)\,\mathrm{d}\zeta}\, e^{-\kappa\lambda_n(t-\tau)} \right] \bar{q}(\xi,\tau)\,\mathrm{d}\xi\,\mathrm{d}\tau$$

We see that

$$v_q(x,t) = \int_{\tau=0}^t \int_0^\ell G(x,t-\tau,\xi)\bar{q}(\xi,\tau)\,\mathrm{d}\xi\,\mathrm{d}\tau$$

where the function G is exactly the same as it was before. However, $G(x,t-\tau,\xi)$ describes the temperature due to a spike of heat added to the bar at a time τ and position ξ; it is called the Green's function.

The fact that solving the initial value problem ($\bar{f}$), also solves the inhomogeneous PDE problem ($\bar{q}$) is known as the Duhamel principle. The idea behind this principle is that fuction $\bar{q}(x,t)$ can be "sliced up" as a cake. The contribution of each slice $\tau \le t \le \tau + \mathrm{d}\tau$ of the cake to the solution v can be found as an initial value problem with $\bar{q}(x,\tau)\,\mathrm{d}\tau$ as the initial condition at time τ.