EML 4930/5930 **Spring 2006**

Analysis in Mechanical Engineering II

Van Dommelen

1 Catalog Description

None.

2 Credit Hours

3

3 Prerequisites

EML 5060.

4 Textbooks

Advanced Engineering Mathematics (Hardcover) by Peter V. O'Neil. Hardcover: 1194 pages Publisher: Thomson-Engineering (Thomson Learning? Brooke/Cole?); 5th edition (July 9, 2002). ISBN: 0534400779

5 Instructor

Dr. Leon Van Dommelen:

Office hours M 10:15-11:15, R 10:30-11:30 in A242 CEB

Phone (850) 410-6324. I tend to forget to check my voice mail.

E-mail dommelen@eng.fsu.edu

Web page http://www.eng.fsu.edu/∼dommelen/index.html

Contact information See web page

6 Teaching Assistant

None

7 Schedule

Class times: MWF 9:15-10:05 am in A226 CEB (A building).

Tentative outline; first time this class is taught. Relevant sections of the book are listed below, but some material will only be in the notes (which will be scanned and put on the web.)

- 01/09/06 M 11.1 Example vector functions of one variable in science and engineering. (Position, electric field.) 11.2 Derivatives and their meaning. (Velocity, acceleration.)
- 01/11/06 W Curve length. Normal vector and curvature. Tangential and normal components.

- 01/13/06 F 11.3 Example scalar and vector fields in science and engineering. (Density, potential energy, velocity field, electrical field, gravity vector, current density.) Streamlines and lines of force. 11.4 Example gradients of a scalar in science and engineering. (Pressure, electrostatic potential, velocity potential.)

- 01/16/06 M Martin Luther King day.

- 01/18/06 W Example directional derivatives in science and engineering. (Heat flux through surfaces, shear stress on surfaces, magnetic flux.) Example level surfaces in science and engineering. (Isobars, surfaces of conductors) The normal vectors to them. Tangent planes.

- 01/20/06 F 11.5 Examples of inflow and outflow in science and engineering. (Fluid mechanics, heat transfer, electrical current.) Relation to the divergence for infinitesimal regions. Relation to the divergence for finite regions (mention.) Solid body motion. Relationship to the curl. Due:

  1. Consider a Miata cornering at maximum speed through a parabolic curve given by $y = x^2/(2R)$ where R is a constant equal to the radius of the radius of curvature at the apex $x = 0$. Write an expression for the arclength of the curve, from the apex, as a function of x. Also give the tangential unit vector to the curve as a function of x. Also find the redius of curvature as a function of x and show that the curvature is greatest at $x = 0$. (Do not try to find these quantities as a function of the arclength s; that would be prohibitively messy.)

     Now assume that the magnitude of the car acceleration that the Miata tires can support $|\vec{a}|$ is equal to one g. (In other words, the radius of the friction circle is one g.) The fastest time will be obtained by decelerating and accelerating the Miata so that the tires are always at that limit $|\vec{a}| = g$. Show that this is achieved if I drive the car so that $x = Vt$ where V is a constant. Also find V in terms of g and R. Find the tangential component of acceleration as a function of x, and from this discuss that when I have reached the apex, I should stomp on the gas, or start increasing the gas linearly with time, or quadratically with time.

  2. (11.3.17) Find and sketch the streamlines of the velocity field

     $$\vec{v} = e^z\hat{\imath} - x^2\hat{k}$$

     Next find the particular streamline that passes through the point (4,2,0).

  3. (11.3.21) Construct an electrostatic field whose field lines are straight lines. Where would you physically find such a field? Also give a few examples of common velocity fields that have straight streamlines.

- 01/23/06 M Relationship of divergence and curl to the derivative tensor of a vector field. The skew-symmetric part and solid body rotation.

- 01/25/06 W The symmetric part and rate of deformation.

- 01/27/06 F Divergence and Stokes theorems. Parametric curves and surfaces and integration.

  Due:

  1. (11.4.[1],19,29) Given the following two scalar fields and a point P:

     $$\phi = x^2 + y^2 - z^2 \quad \psi = x^2 + y^2 \qquad P = (2, 2, 2\sqrt{2})$$

     describe the shape of the surface $\phi = 0$ and of the surface $\psi = 8$. Show that point P is on both surfaces, in other words, it is one of the points where the two surfaces cut through each other. Find the gradient of ϕ at point P and also the maximum increase of ϕ with distance from point P at point P, as well as the maximum decrease.

2. Continuing the previous question, if a particle at point P moves with a velocity $\vec{v} = \hat{\imath} + \hat{\jmath}$, what is the derivative of ϕ with distance traveled for this particle at point P? At what angle do the surfaces $\phi = 0$ and $\psi = 8$ cut through each other at point P? Does this angle sound right geometrically?

3. The fundamental theorem of vector calculus, also known as Helmholtz's theorem, states that any vector field meeting certain conditions (of decaying towards infinity) can be resolved into irrotational (curl-free) and solenoidal (divergence-free) component vector fields.

   This implies that any vector field $\vec{v}$ meeting certain decay criteria can be considered to be generated by a scalar potential ϕ and a vector streamfunction ψ.

$$\vec{v} = \nabla\varphi + \nabla \times \vec{\psi}$$

   Show that the divergence of $\nabla\varphi$ must produce the divergence of $\vec{v}$ (the rate of expansion) and the curl of $\nabla \times \vec{\psi}$ produces the curl of $\vec{v}$ (the vorticity.) Hint: use that the divergence of any curl and the curl of any gradient are always zero.

4. (11.5.13) The net outward mass flow generated per unit volume in fluid mechanics is $\nabla \cdot (\rho\vec{v})$, where ρ is the density and $\vec{v}$ the velocity. Rewrite this in terms of vector derivatives of ρ and $\vec{v}$ themselves. When eliminating the pressure from the momentum equations, we end up with $\nabla \times (\rho\vec{v})$. Rewrite this too in terms of vector derivatives of ρ and $\vec{v}$ themselves.

5. (11.5.18) In 2D incompressible flow we can define a scalar streamfunction ψ so that $\vec{v} = \nabla \times (\psi\hat{k})$. In that case the Laplacian of ψ equals minus the vorticity. In 3D incompressible flow we can similarly define a vector streamfunction $\vec{\psi}$ so that $\vec{v} = \nabla \times \vec{\psi}$. This satisfies the incompressibility condition $\nabla \cdot \vec{v} = 0$ automatically since the divergence of any curl is zero. Show that

$$\nabla \times (\nabla \times \vec{\psi}) = \text{grad div } \vec{\psi} - \nabla^2\vec{\psi}$$

   (Showing this for one component is enough, since there is no prefered direction in the problem.) From this result, argue that if we still want the vorticity $\nabla \times \vec{v}$ to be minus the Laplacian of $\vec{\psi}$, we will have to choose the divergence of $\vec{\psi}$ equal to a constant. Show that if $\nabla \cdot \vec{\psi} = C$, then $\vec{\psi} = \frac{1}{3}C\vec{r} + \vec{\psi}_0$ where $\nabla \cdot \vec{\psi}_0 = 0$ and the $\frac{1}{3}C\vec{r}$ does not produce any velocity, so we may as well leave it out. So, the vector streamfunction is normally taken to be solenoidal (i.e. with zero divergence.)

- 01/30/06 M Parametric curves and surfaces and integration continued.
- 02/01/06 W Parametric curves and surfaces and integration continued. 12.1 Examples of line integrals in science and engineering: work, circulation, Ampere's law and Maxwell's.
- 02/03/06 F Continued: Faraday's law and Maxwell's. 12.4 Examples of surface integrals in science and engineering: conservation laws, Gauss' laws and Maxwell's, land surface of the earth. Archimedes. Due:

  1. (c.f. 12.8.9-16, section 12.8.1) Going back to last week's fundamental theorem of vector calculus,

$$\vec{v} = \nabla\varphi + \nabla \times \vec{\psi}$$

     show that if $\nabla \times \vec{\psi}$ produces the curl of $\vec{v}$, then the remainder *must* be the gradient of a scalar.

  2. (c.f. section 12.8.1) In fluid flow about solid, stationary bodies, the velocity must be zero on the solid surfaces. The vorticity, $\nabla \times \vec{v}$ is normally not zero on the surfaces. Explain how that is possible if $\vec{v} = 0$. Next, use Stokes' theorem to argue that the component of vorticity normal to the surface *is* zero. Vorticity lines at solid stationary surfaces follow the surface (if the vorticity is nonzero).

  3. (c.f. 12.8.9-16, section 12.8.1) We have seen in class that if a flow field is in a state of solid body rotation, the vorticity vector $\vec{\gamma}$ is constant over space and equal to twice the angular velocity vector $\vec{\omega}$. Show that the converse is *not* true. If the vorticity $\vec{\gamma}$ is constant, the velocity is not necessarily that of a solid body rotation. Hint: examine the properties of $\Delta\vec{v} = \vec{v} - \frac{1}{2}\vec{\gamma} \times \vec{r}$ and establish that Δv can be any arbitrary potential flow velocity field.

4. Consider the velocity field of Couette flow:

$$\vec{v} = \hat{\imath} C y$$

where C is a constant. Find the velocity derivative tensor, the rate of expansion, the vorticity $\vec{\gamma}$ and the strain rate tensor of this flow.

5. In the Couette flow of the previous question, since the angular velocity $\omega = 2\vec{\gamma}$ is constant, the fluid particles should continue to rotate around in time and for large times make many revolutions around their centers. Now study an arbitrary straight line of fluid particles and see how many revolutions it actually makes. In particular, show that no line of fluid particles will ever complete even half a revolution. Take a circle of fluid particles (as many people do to make the "vorticity = angular velocity" argument) and sketch how it really evolves for large times. Does it rotate many revolutions? Comment on the wisdom of using concepts from solid body mechanics for describing the motion of distorting fluids.

6. For the Couette flow of the previous questions, rotate the coordinate system 45 degrees counter-clockwise around the z-axis to new x' and y' axes and new velocity components u' and v', and show that

$$u' = \tfrac{1}{2} C (x' + y') \qquad v' = -\tfrac{1}{2} C (x' + y')$$

Show that this velocity field has a strain rate tensor that is diagonal.

7. Substract the solid body rotation from the u', v' velocity field of the previous question and show that the remaining velocity field is

$$u'_r = \tfrac{1}{2} C x' \qquad v'_r = -\tfrac{1}{2} C y'$$

Describe the distortion of an initially square fluid element in a time interval dt due to this remaining velocity field. Also describe the distortion of a little circle $x'^2 + y'^2 = \varepsilon^2$ in this velocity field.

- 02/06/06 M Derivation of the heat and continuity equations. Uniqueness of the solution of the heat equation using the energy method.

- 02/08/06 W Improperly posedness of the backward heat equation. The fundamental solution to the Laplacian in 3D. The Poisson equation.

- 02/10/06 F Solution to the Poisson equation in inifinite 3D space. Solution to the Poisson and Laplace equation in finite spaces: panel and boundary element methods.

1. (12.1.16) Given the force $\vec{F} = x\hat{\imath} + y\hat{\jmath} - xyz\hat{k}$, find the work along the path $y = x$, $z = -3x^2$ for $-1 \le x \le 3$.

2. (12.1.17) Find the work along the path $\vec{r} = (t, -2t, 5t)$ for $1 \le t \le 4$ if the component of the force tangential to the path equals $3y^3$.

3. (12.5.3) Find the center of mass of the conical shell $z = \sqrt{x^2 + y^2}$ for $x^2 + y^2 \le 9$. The mass per unit area of the shell is constant. Do not use cylindrical coordinates. For a *bonus* 10 points, give me the z-value for the problem as stated in the book.

4. (12.5.8) Find the flux of $\vec{F} = xz\hat{\imath} - y\hat{k}$ across the part of the spherical surface $x^2 + y^2 + z^2 = 4$ that is above $z = 1$. Do not use cylindrical coordinates nor polar coordinates.

5. As we have seen, the fact that a velocity field has a constant vorticity, e.g. $\nabla \times \vec{v} = \hat{k}$, does not mean that it is in a state of solid body rotation. (See the Couette flow example.) However, we can say something about the average tangential velocity around any arbitrary circle in the flow field. What? Such constant vorticity flows really exist. Assuming that z is upwards, what can you say about the speed-up of bathtub vortices in such flows? Are you sure that is right?

6. In standard fluid mechanics, the linear momentum equation for an arbitrary fixed volume V is:

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_V \vec{v}\rho\,\mathrm{d}V + \int_S \vec{v}(\rho\vec{v}\cdot\vec{n})\,\mathrm{d}S = \int_S \vec{T}\,\mathrm{d}S$$

   where S is the entire outside surface of the volume V and $\vec{T}$ is the complete stress (viscous plus pressure). In index notation:

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_V v_i\rho\,\mathrm{d}V + \int_S v_i\rho v_j n_j\,\mathrm{d}S = \int_S \sigma_{ji} n_j\,\mathrm{d}S$$

   where σ_{ji} is the complete stress tensor. Use the scalar form of the divergence theorem (see notes on Archimedes) to derive the differential linear momentum equation of fluid mechanics, which is valid pointwise.

7. In standard fluid mechanics, the angular momentum equation for an arbitrary fixed volume V is:

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_V \vec{r}\times\vec{v}\rho\,\mathrm{d}V + \int_S \vec{r}\times\vec{v}(\rho\vec{v}\cdot\vec{n})\,\mathrm{d}S = \int_S \vec{r}\times\vec{T}\,\mathrm{d}S$$

   It is very useful for sprinklers, turbines, and other fluid systems involving angular velocity. If we write the first component out in index notation:

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_V \epsilon_{1jk} r_j v_k \rho\,\mathrm{d}V + \int_S \epsilon_{1jk} r_j v_k \rho v_l n_l\,\mathrm{d}S = \int_S \epsilon_{1jk} r_j \sigma_{lk} n_l\,\mathrm{d}S$$

   where ϵ_{1jk} is a 2×2 constant matrix whose form is not needed to solve this question. Use the scalar form of the divergence theorem to derive the differential angular momentum equation of fluid mechanics, which is valid pointwise. (Note that $\partial r_j/\partial r_l$ is the Kronëcker δ_{jl}, which is zero when $l \neq j$, in other words, which forces l to equal j to get something nonzero, and then $\delta_{jj} = 1$. Also, because of the properties of matrix ϵ_{1jk}, $\epsilon_{1jk}a_{jk}$ is zero for any "symmetrix" matrix a_{jk} for which the order of the indices j and k does not make a difference.) Solve the mystery why you do not hear that much about the angular momentum equation in standard graduate fluid mechanics classes.

- 02/13/06 M Poisson integral solution for the Dirichlet problem on a sphere.
- 02/15/06 W The mean value theorem for the Laplace equation. Solution smoothness in the interior. The integral constraint for the Neumann problem. Examples of first order partial differential equations in science and engineering (population age evolution, one-dimensional inviscid flow, electrical transmission lines, vehicular traffic.)
- 02/17/06 F Solution of scalar first order PDEs in two dimensions.

1. (12.7.20) Show that for a given volume V with boundary S, the partial differential equation

$$\frac{\partial u}{\partial t} = k\nabla^2 u = \phi(x,y,z,t) \text{ in } V$$

   with boundary condition

$$\frac{\partial u}{\partial n} + hu = f \text{ on } S$$

   with $\partial u/\partial n$ the derivative of u in the direction normal to the boundary, and initial condition

$$u(x,y,z,0) = g(x,y,z)$$

   (for given ϕ, $h > 0$, f, and g) can have at most one solution.

2. Derive the fundamental solution G_0 of the Laplacian in 2D, satisfying

$$\nabla^2 G_0 = \delta(x,y)$$

3. Derive the solution of the Poisson equation in infinite two-dimensional space. Comment on the behavior of the solution at large distances.
4. Show that the solution of the Laplace equation in an arbitrary finite region in space can be written in terms of a distribution of sources and dipoles on the boundary of the region.
5. Explain why it is enough to use either sources or dipoles; we do not need both.

- 02/20/06 M Review.
- 02/22/06 W Mid Term Exam
- 02/24/06 F Last day to drop. Quasi-linear equations. Conservation laws. Shock conservation laws. Entropy condition.

1. Show that if $u(r,\theta)$ is a solution to the Laplace equation inside a unit circle, then $v(\rho,\theta) = u(r,\theta)$, with $r = 1/\rho$, is a solution of the Laplace equation outside the unit circle.
2. Derive the Poisson integral solution for the Dirichlet problem for the Laplace equation in a unit circle;
$$u(x,y) = \frac{1-r^2}{2\pi} \oint_{r=1} \frac{f}{(x-\bar{x})^2 + (y-\bar{y})^2} \, \mathrm{d}\bar{s}$$
Note that for any solution u_o to the Laplace equation outside the circle, and (x,y) a point inside the circle,
$$0 = \oint_{r=1} \left(G \frac{\partial u_0}{\partial \bar{r}} - u_o \left[\frac{\partial G}{\partial \bar{r}} - \frac{1}{2\pi} \right] \right) \mathrm{d}\bar{s}$$
(The additional $1/2\pi$ comes from converting the integral at $\bar{r} = \infty$.) Clean up the expression to polar coordinates using
$$x = r\cos(\theta) \quad y = r\sin(\theta) \quad \bar{x} = \cos(\alpha) \quad \bar{y} = \sin(\alpha)$$
and compare with literature.
3. Derive the Poisson integral solution for the Neumann problem for the Laplace equation in a circle,
$$u(r,\theta) = -\frac{1}{2\pi} \oint_{r=1} \ln\left(r^2 - 2r\cos(\theta-\alpha) + 1\right) \, \mathrm{d}\alpha + C$$
with C an undetermined constant.
4. Verify the mean value property for the Laplace equation in two dimensions.
5. (5.25) Solve
$$x u_x + y u_y = 0 \qquad u(x,1) = f(x).$$
Suppose f does not have a derivative at a point $x = \bar{x}$, then where does $u(x,y)$ not have a derivative?
6. (5.27a) Solve the McKendric-von Foerster problem
$$u_t + u_a = -\frac{cu}{L-a} \quad 0 \le a \le L \qquad u(0,t) = b(t)$$
where c and L are positive constants
7. (5.27b) Solve
$$x u_x + y u_y = 1 \qquad u = x^2 + y \text{ on } x + y = 1$$

- 02/27/06 M Propagation of small disturbances. Expansion fans. Systems of first order equations: classification, simple examples, solution of the wave equation.
- 03/01/06 W Wave equation continued. Cauchy initial value problem. D'Alembert solution.

- 03/03/06 F Diagonalizing hyperbolic systems in general. Equations of steady inviscid flows.

1. (5.29) The Cauchy problem is solving a first order equation using a given “initial condition” on some line, as we did in last week’s homeworks. However, if the initial condition to a first order PDE is given on a characteristic line, the problem is normally not solvable. Show that the following Cauchy problem with an initial condition on the characteristic line $y = x$:

$$u_x + u_y = 1 \qquad u(x,x) = x^2$$

does not have a solution.

2. (5.34) Consider the following problem for the Burgers’ equation:

$$u_t + uu_x = 0 \qquad u(x,0) = \begin{cases} 1 & 0 < x < 1 \\ 0 & x < 0 \text{ or } x > 1 \end{cases}$$

It has two possible solutions; a double shock one:

$$u_1(x,t) = \begin{cases} 0 & x < \frac{1}{2}t \\ 1 & \frac{1}{2}t < x < 1 + \frac{1}{2}t \\ 0 & 1 + \frac{1}{2}t < x \end{cases}$$

and an expansion fan/shock one:

$$u_2(x,t) = \begin{cases} 0 & x < 0 \\ x/t & 0 < x < t \\ 1 & t < x < 1 + \frac{1}{2}t \\ 0 & 1 + \frac{1}{2}t < x \end{cases}$$

(assuming that $t < 2$.)

Draw the characteristics and the shocks of each solution in an x, t-diagram.

3. Continuing the previous problem, derive the general form of the solution of the Burger’s equation in smooth regions using the method of characteristics, and show that six of seven partial solutions above are of that form, but x/t is a problem. Check directly that x/t does indeed satisfy $u_t + uu_x = 0$, so the problem must be elsewhere. Check that the problem is that all chraracteristic lines in the expansion fan have, in class notation, $C_1 = 0$, so C_1 is not a good variable to distinguish between characteristic lines. However, C_2 is different between different characteristic lines, so it is possible to write C_1 as some function of C_2 instead of vice versa. Show that in those terms, the general solution is $x = ut + f(u)$, and identify what function f is for the expansion fan.

4. Continuing the previous problem, check that both u_1 and u_2 are weak solutions; in other words, that for both all shocks continue to satisfy the conservation law requirement

$$v_s = \frac{F_2 - F_2}{u_2 - u_1}$$

assuming that the conserved quantity is $\int u \, \mathrm{d}x$, in order to determine what F is.

5. Continuing the previous problem, show that one shock in u_1 does not satisfy the entropy condition

$$F_1' > v_s > F_2'$$

while the single shock in u_2 does. For that reason, the physically correct solution is u_2.

6. Continuing the previous problem, show that for times $t > 2$, when the expansion fan has hit the shock, the shock at $x = 1 + \frac{1}{2}t$ no longer satisfies the conservation law. When the shock hits the fan, the true shock velocity will start to decrease from $\frac{1}{2}$.

- 03/06/06 M Spring Break

- 03/08/06 W Spring Break
- 03/10/06 F Spring Break

- 03/13/06 M Streamline compatibility equation for inviscid flow.
- 03/15/06 W Mach line compatibility equations. Method of characteristics. Riemannian invariants.
- 03/17/06 F Second order constant coeficient PDEs. Classification and reduction to canonical form of hyperbolic and parabolic equations.

  1. The equations of one-dimensional inviscid nonconducting flow are:

  $$\rho_t + (\rho u)_x = 0 \quad u_t + u u_x + \frac{1}{\rho} p_x = 0 \quad p_t + \gamma p u_x + u p_x = 0$$

  being, respectively, the continuity equation, Newton's second law, and a convoluted form of the energy equation. The density is ρ, p is the pressure, u the flow velocity, and γ is a constant equal to the ratio of the specific heats. Write this system in matrix form and classify it. Simplify the expressions for the eigenvalues by defining the "speed of sound" a to be

  $$a = \sqrt{\gamma \frac{p}{\rho}}$$

  2. Solve the following initial value problem:

  $$u_{tt} = u_{xx} \quad u(x,0) = 0 \quad u_t(x,0) = \begin{cases} 0 & x \le -1 \\ 1 & -1 \le x \le 1 \\ 0 & 1 \le x \end{cases}$$

  Show the solution u graphically at times $t = 0.25$, 0.5, 1, and 2. At which time does u_t at $x = 0$ switch from being 1 to being zero?

- 03/20/06 M Reduction of elliptical equations to canonical form. Cauchy-Kovalevski theorem. Well-posedness of the wave equation. Derivation of the wave equation for a string. Introduction to the method of separation of variables.
- 03/22/06 W Separation of variables continued. Sturm Liouville problems and their solution.
- 03/24/06 F Separation of variables concluded. Orthogonality.

  1. Continuing last week's problem, find the compatibility equation of one-dimensional inviscid nonconducting flow along the fluid particle paths. Interpret it physically, noting that the entropy s for an ideal gas satisfies:

  $$\mathrm{d}s = c_v \left(\frac{\mathrm{d}p}{p} - \gamma \frac{\mathrm{d}\rho}{\rho} \right)$$

  2. Find the compatibility equations of one-dimensional inviscid nonconducting flow along the acoustic wave fronts.
  3. Assuming that the entropy is not just constant along the particle paths, but the same everywhere, (which requires the absence of strong shocks,) all thermodynamic variables, including the speed of sound, are unique functions of the pressure, and can be integrated with respect to the pressure. Show that in that case, there are two more Riemannian invariants besides the entropy, and that they are equal to

  $$u \pm \frac{2}{\gamma - 1} a$$

Hint: express $\mathrm{d}p$ in terms of $\mathrm{d}a$ using

$$\mathrm{d}a^2 = \mathrm{d}\left(\gamma\frac{p}{\rho}\right) \qquad \frac{\mathrm{d}\rho}{\rho} = \frac{\mathrm{d}p}{\gamma p}$$

(the latter from $\mathrm{d}s = 0$ as in question 1.)

4. Classify the wave equation

$$u_{tt} = c^2 u_{xx}$$

and write it in characteristic coordinates. Solve the equation and convert the result back to the physical coordinates x and t. Show that the same form for the general solution is found as we got using a first order system.

5. (19.1.9) Classify and derive the general solution to

$$2u_{xx} - 10u_{xy} + 8u_{yy} + u_x - u_y = 0$$

6. (19.1.4) Classify and reduce to canonical form

$$2u_{xx} - 4u_{xy} + 2u_{yy} - y^2u_x + u_y - xu = 0$$

- 03/27/06 M Solution using D'Alembert. Mirror method. Comparison of solutions. simple derivation of the heat equation.
- 03/29/06 W Radiation boundary conditions. Separation of variables for a mixed boundary condition.
- 03/31/06 F Sturm-Liouville theory. Separation of variables with convection terms.

1. Classify

$$u_{xx} + 8u_{xy} + 25u_{yy} = 0$$

Show that when this equation is solved inside any finite domain, like a rectangle, say, the maximum value of u will occur on the boundary of that rectangle.

2. (19.3.3) While the boundary value for the Laplace equation is properly posed, the initial value problem is *not*, according to Hadamard. Consider the generic initial value problem for the Laplace equation:

$$u_{xx} + u_{yy} = 0 \quad \text{for } -\infty < x < \infty, y \geq 0$$

$$u(x,0) = f(x) \qquad u_y(x,0) = g(x)$$

Now add a small perturbation of the form $\sin(nx)/n$ (where we will let $n \to \infty$, so that it becomes zero) to g to get a perturbed solution u' satisfying

$$u'_{xx} + u'_{yy} = 0 \quad \text{for } -\infty < x < \infty, y \geq 0$$

$$u'(x,0) = f(x) \qquad u'_y(x,0) = g(x) + \frac{1}{n}\sin(nx)$$

Show that the difference in solution, $v = u' - u$ is *not* small when $n \to \infty$, although the difference in initial condition is. (The first becomes infinite, the latter zero.)

Hint: verify by direct substitution that the solution for v is

$$v = \frac{1}{n^2}\sin(nx)\sinh(ny)$$

3. (16.1.1) Show that

$$u(x,t) = \sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{n\pi ct}{L}\right)$$

satusfies the one-dimensional wave equation for any value of n.

4. Show that the solution of the previous question is of the form

$$u(x,t) = f_1(x-ct) + f_2(x+ct)$$

and identify functions $f_1(x)$ and $f_2(x)$.

5. (16.2.15) (30 pts) Solve the problem of longitudinal vibrations in a bar of length L that is strained to a strain of A and then released:

$$u_{tt} = c^2 u_{xx} \quad \text{for } 0 \le x \le L, t \ge 0$$

$$u_x(0,t) = 0 \qquad u_x(L,t) = 0$$

$$u(x,0) = Ax \qquad u_t(x,0) = 0$$

Here x is the rest position of locations on the bar; u is the displacement from the rest position; c^2 is the ratio of modulus of elasticity over density. The boundary conditions express that the stress, hence strain is zero at the ends of the bar. Solve this problem using separation of variables. Make sure you account for the fact that the boundary conditions are different from the example worked in class.

- 04/03/06 M Separation of variables with convection continued.
- 04/05/06 W Separation of variables for inhomogeneous PDE.
- 04/07/06 F Separation of variables for inhomogeneous PDE concluded.

1. Solve the last homework problem of last week using D'Alembert, after symmetrically (i.e. with no sign change) mirroring the initial condition around both ends of the bar. Sketch the solution for a time slightly greater than zero.

2. Consider the PDE:

$$u_t = k u_{xx} + U u_x + Au$$

where U is a constant convection velocity and A a constant. Define a new unknown v so that $u(x,t) = e^{cx+dt} v(x,t)$ with c and d constants to be determined. Plug this into the PDE, showing all details, and then find the values of the constants c and d for which the v_x and v terms drop out, to leave a standard heat equation for v:

$$v_t = k v_{xx}$$

3. (30 pts) Use the trick from the previous question to solve

$$u_t = u_{xx} + 6u_x \qquad \text{for } 0 \le x \le 4, t \ge 0$$

$$u(0,t) = u(4,t) = 0 \qquad u(x,0) = 1$$

Make sure to convert the initial and boundary conditions for u to ones for v. Convert v back into u and then show that your solution is the same as the one derived in class without using the trick. Make sure to show the full derivations in detail.

4. Plot the solution u against x for times $t = 0$, 0.25, 0.5, 0.75, and 1, summing at least 100,000 terms of the sum, or until the error in u is less than 0.001. Plot 200 points along each curve.

- 04/10/06 M Numerical example of an inhomogeneous PDE. Dealing with inhomogeneous boundary conditions.
- 04/12/06 W Dealing with inhomogeneous boundary conditions continued. Fourier series and complex Fourier series.
- 04/14/06 F

1. (17.2.30) (30 pts) Adapt the method followed in class (not the one in the book) to solve the inhomogeneous heat equation with Neumann boundary conditions. Go over all the steps, but be sure to emphasize which parts change due to the different boundary conditions:

$$u_t = \kappa u_{xx} + F(x,t) \qquad 0 \le x \le L, t \ge 0$$
$$u(x,0) = f(x) \qquad u_x(0,t) = 0 \quad u_x(L,t) = 0$$

In case you did not do the Neumann problem of the homework two weeks ago correct; the correct eigenfunctions were:

$$X_0(x) = 1 \qquad X_n(x) = \cos\left(\frac{n\pi x}{L}\right) \text{ for } n = 1, 2, 3, \ldots$$

and X_0 has to be done separately in orthogonality integrals and ODE solutions.

2. (17.2.31) Use the solution of the previous question to solve

$$u_t = 16u_{xx} + xt \qquad 0 \le x \le 5, t \ge 0$$
$$u(x,0) = 1 \qquad u_x(0,t) = 0 \quad u_x(5,t) = 0$$

Again, do *not* solve it in the book way. Show the derivations of the integrals.

3. Plot the solution to the previous question graphically at times 0, 0.1, 0.2, 0.3, 0.4, and 0.5.

- 04/17/06 M Review.
- 04/19/06 W Review.
- 04/21/06 F Review.

1. Consider the problem of heat conduction in a bar,

$$u_t = \kappa u_{xx} \text{ for } 0 \le x \le L, t \ge 0 \qquad u(x,0) = f(x)$$

with the temperature given at one end and the heat flux at the other end:

$$u(0,t) = T_1(t) \qquad u_x(L,t) = q_2(t)$$

Convert this problem into one with homogeneous boundary conditions. Identify the new initial condition and new PDE to solve.

2. For the previous problem, if the boundary conditions T_1 and q_1 are independent of time, compare the steady solution for u_0 with the solution you get for u_0 using a linear expression in x. In either case, write the new problem to be solved.

3. Consider the following problem of longitudinal vibrations in a bar of length L that experiences forces on the ends:

$$u_{tt} = c^2 u_{xx} \quad \text{for } 0 \le x \le L, t \ge 0$$
$$u_x(0,t) = 1 \qquad u_x(L,t) = 2$$
$$u(x,0) = \frac{x^2}{2L} \qquad u_t(x,0) = 0$$

Show that

$$u_0(x,t) = A(t) + B(t)x$$

does *not* work to get homogeneous boundary conditions, and that there is no steady long-time solution either. Show that

$$u_0(x,t) = A(t)x + B(t)x^2$$

does work. Show that if in addition you substract

$$\frac{c^2t^2}{2L}$$

from the solution, you get a problem that you already solved a few weeks ago. Write the solution for u.

4. Consider the following heat conduction problem with a homogeneous PDE and inhomogeneous BC:

$$u_t = 16u_{xx} \quad \text{for } 0 \le x \le 5, t \ge 0$$

$$u_x(0,t) = -\tfrac{1}{2}t^2 \qquad u_x(5,t) = -\tfrac{1}{2}t^2$$

$$u(x,0) = 1$$

   Reduce this problem to one you solved last week and write the solution for u.

5. For the previous problem, what would the solution have been if the boundary conditions would have been simply t^2 instead of $-\frac{1}{2}t^2$ and the initial condition would have been homogeneous? (Note that the problem is linear: you can multiply solutions by constants.)

6. (16.3.6, 30 points) Solve the wave equation

$$u_{tt} = 4u_{xx}$$

   in the infinite domain $-\infty < x < \infty$, if the initial conditions are:

$$u(x,0) = 0 \qquad u_t(x,0) = 0 \text{ for } |x| > 2$$

$$u_t(x,0) = -1 \text{ for } -2 < x < 0 \quad u_t(x,0) = 1 \text{ for } 0 < x < 2$$

   using the Fourier transform only, not D'Alembert. Explicitly evaluate the Fourier transform of the initial condition and explicitly evaluate the Fourier transform of u. Write a Fourier integral for u.

- 04/25/06: Final Tuesday 3-5 pm (ignore FSU schedule).
- 05/03/06: Grades available online

8 Goals

This course will familiarize students with applications of vector calculus and partial differential equations in mechanical engineering.

9 Course Outline

See the tentative schedule in section 7 above.

10 Methods of Instruction

Lectures, problem solving sessions, examinations.

11 Student Evaluation

The course grade will be computed as:

- Homework: 20%
- Midterm: 40%
- Final: 40%

Grading is at the discretion of the instructor.

12 Important Regulations

1. Immediately check the dates listed above for any conflicts.
2. Homework must be handed in at the *start* of the lecture at which it is due. It may *not* be handed in at the departmental office or at the end of class. Homework that is not received at the start of class on the due date cannot be made up unless permission to hand in late has been given *before* the homework is due, or it was not humanly possible to ask for such permission before the class. If there is a chance you may be late in class, hand the homework in to the instructor the day before it is due. (Shove it under his door if necessary.) This also applies to Web students: they must E-mail the homework before the time that the class starts.
3. Homework should be neat.
4. Students are bound by the rules and regulations in their University bulletin, as well as by those specified in this syllabus, and by the usual standards applied by the College of Engineering. Read your academic bulletin. Violations of the rules and regulations in your bulletin may result in reduced grades and/or other actions.
5. Students are bound by the honor code of their university. It requires you to uphold academic integrity and combat academic dishonesty. Please see your student handbook. Violations of your honor code may result in reduced grades and/or other actions.
6. *Copying* of homework, assignments, or tests is never allowed and will result in a failing or zero grade for the copied work. It will also result in a failing or zero grade of the person whose work is being copied if that person could reasonably have prevented the copying. However, *working together* is typically allowed and encouraged for most homeworks, (and sometimes for other take-home assignments,) as long as you present the final results in your own words and using your own line of reasoning. Since close similarities between solutions will reduce credit, it is better not to formally put down anything until you have figured out the problem, and then let each person write their own solution. If it is unclear whether working together is allowed on any assignment, check with the instructor beforehand.
7. Attendance is required. Exams missed, even when rescheduled from the original date and surprise tests, or homework not handed in on time due to unexcused absence or lateness will result in a zero grade for that exam and/or homework. Failure to properly complete homework, tests, assignments, etcetera due to changes in date, assignment, etcetera, that you did not know about due to unexcused absence, lateness, or inattentiveness will not be excused and cannot be made up.
8. For excused absences where the student has given advanced notice of the absence at the earliest opportunity, the instructor will work with the student to arrange for make-up work and tests.
9. The College of Engineering has a restrictive interpretation of what is considered a valid excuse for an absence. If an absence is to be excused, make sure you at least get official confirmation by phone that it will be granted beforehand.
10. The instructor will make sure that make-up tests are no simpler than the original, but he will try to make them similarly difficult. However, he cannot make allowances for increased difficulty due to the small sample size.
11. The College of Engineering has a more restrictive drop-add period than you might think based on your bulletin. Check both your bulletin and the Dean's office to determine whether drop-add will be allowed.
12. Some of these rules may not apply if you fall under the Americans with Disabilities Act. FAMU students with disabilities needing academic accommodations should contact Student Health Services for confirmation of permanent physical disability, FSU students should register with and provide documentation to the Student Disability Resource Center. Next bring a letter to the instructor from the Services or Center indicating you need academic accommodations. This should be done during the first week of classes.

13. The instructor might wave some regulation on a case-by-case basis depending on his subjective determination of fairness and appropriateness. This will occur only under exceptional circumstances and should not be assumed. Especially, never assume that a seemingly minor regulation will be waved because the instructor has waved it in the past. A second appeal to wave a minor regulation will probably indicate to the instructor that the regulation is not being taken seriously and most likely refused. Any appeal to the instructor will further be refused apriori unless it is done at the earliest possible moment by phone and/or by E-mail. Do not wait until you are back in town, say.

13 Computer Requirements

Students must have an E-mail address and daily check their E-mail. Students must be able to use a Web browser such as Netscape. The class web page can be accessed at

http://www.eng.fsu.edu/ dommelen/courses/aim2/