Calculus

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Dedication

To my much suffering students.

Contents

List of Figures

List of Tables

Preface

To the Student

This is a collection of notes on calculus.

Sections marked with "[NC]" in their title are not part of the material covered in "Analysis in Mechanical Engineering." Read them if you are interested.

The usual "Why this book?" blah-blah will eventually be found in a note at the back of this book, {N.1} A version history is in note {N.2}.

This is a living document. At this stage it is a rough initial concept. That is reflected in the version number. Maybe at some later time it becomes something useful for a general audience to read.

Acknowledgments

Many examples originate from the book by Ayres & Mendelson, [1], that I use as basic text book in my class.

This book owes its formatting to my book "Quantum Mechanics for Engineers."

Thank you all.

Comments and Feedback

If you find an error, please let me know. There seems to be an unending supply of them. As one author described it brilliantly, "the hand is still writing though the brain has long since disengaged."

Also let me know if you find points that are unclear to the intended readership, mechanical engineering graduate students with a typical exposure to mathematics, or equivalent.

The same for sections that cannot be understood without delving back into earlier material. All within reason of course. If you pick a random starting word somewhere in the book and start reading from there, you most likely will be

completely lost. But sections are intended to be fairly self-contained, and you should be able read one without backing up through all of the text.

General editorial comments are also welcome.

Chapter 1

Graphs

1.1   Introduction

Graphs are important for engineers for a number of reasons:

- Understanding relationships between variables.

- Summarizing data.

- Representing data (like in a Moody diagram).

- Interpolating data.

- Understanding the overal nature of data. See warming.jpg[1] for an example that you simply could not appreciate by looking at a list of numbers.

- ...

Look for:
- Intercepts. Intercepts with the x-axis satisfy $y = 0$. Intercepts with the y-axis satisfy $x = 0$.
- A symmetry line exists if the curve is the same at both sides of the line. More precisely, a symmetry line acts as a mirror that mirrors the curve into itself. The y-axis is a symmetry line if the sign of x does not make a difference. The x-axis is one if the sign of y does not make a difference. The $45°$ line $y = x$ is one if swapping x and y does not make a difference.
- Symmetry points. Every point on the curve must have match at the exact opposite side of a symmetry point. Mathematically, if $\vec{r}_1$ is on the curve, then so must be $\vec{r}_S - (\vec{r}_1 - \vec{r}_S)$. The origin is a symmetry point if $y(-x) = -y(x)$, i.e. if function $y(x)$ is antisymmetric.

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[1]`http://www.eng.fsu.edu/~dommelen/calculus/warming.jpg`

- Singular points:
    - corners where the direction of the curve changes by an angle less than $180°$,
    - cusps where it changes $180°$,
    - crossings where the curve crosses itself,
    - positions of infinite curvature,
    - ...

  If y or any of its derivatives is infinite or not uniquely defined, the curve has a singularity at that point.
- A vertical asymptote $x_{\mathrm{va}} = A$ exists if $y \to \pm\infty$ for $x \to A$.
- A horizontal asymptote $y = A$ exists if $A = \lim_{x\to\pm\infty} y$ exists.
- Behavior for $x \to \pm\infty$ (e.g. $y \sim |x|^p$ for some p).
- An oblique asymptote $y_{\mathrm{oa}} = Ax + B$ exists if $A = \lim_{x\to\pm\infty} y'$ and $b = \lim_{x\to\pm\infty} y - Ax$ exist. (Or more simply if $\lim_{x\to\pm\infty} y - Ax - B = 0$.)
- Extent in x (the range of x-values of the curve) and extent in y (the range of y-values of the curve). If y is a given function of x, then the x-extent is the x-values for which y can be computed, but the y extent may not be so simple.
- Minima and maxima. A global maximum/minimum is the highest/lowest value of y that can be found anywhere. You should find both the value of the maximum/minimum and its location(s). A local maximum/minimum is the highest/lowest value that can be found in a small vicinity around the localtion of the local maximum/minimum. Normally, you first find the local maxima/minima, and then, based on consideration of the *entire* graph, decide whether they are also global ones. The derivative changes sign at a maximum/minimum *if* defined at both sides of the maximum/minimum. So look for both zero derrivatives and singular points.
- Concavity is upward if $y'' > 0$, downward if $y'' < 0$.
- Inflection points are points where the concavity changes sign.

See [1, Chapters 13-15]

1.2   Example

From [1, p. 128, 13a]

Asked: Draw the graph of

$$xy = \left(x^2 - 9\right)^2$$

1.2.1   Using reasoning

$$xy = \left(x^2 - 9\right)^2$$

Instead of starting to crunch numbers, look at the pieces first:
Factor $x^2 - 9 = (x - 3)(x + 3)$ is a parabola with zeros at $x = \pm 3$:
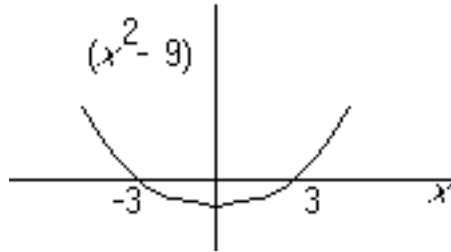
Squaring gives a quartic with double zeros at $x = \pm 3$:
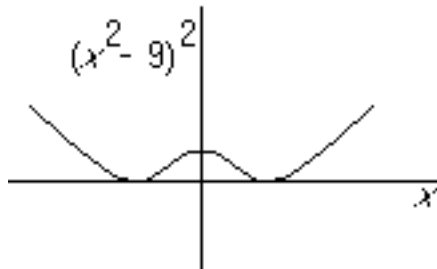
Dividing by x will produce a simple pole at $x = 0$ and also a sign change at negative x:
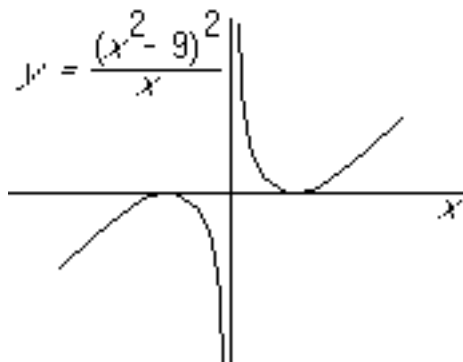
Function $y(x)$:
- has an x-extent $x \neq 0$ and a y-extend $-\infty < y < \infty$;
- is odd (symmetric with respect to the origin);
- has a relative maximum at -3 of finite curvature: $y \propto (x + 3)^2$;
- has a relative minimum at 3 of finite curvature: $y \propto (x - 3)^2$;

- has a vertical asymptote at $x = 0$ with asymptotic behavior: $y \sim 81/x$ for $|x| \to 0$;
- behaves asymptotically as $y \sim x^3$ for $x \to \pm\infty$;
- is concave up for $x > 0$, down for $x < 0$. (Should really prove this, I guess.)

1.2.2   Using brute force

$$ y = \frac{(x^2 - 9)^2}{x} $$

Hence
- intercepts with x-axis are at $x = \pm 3$;
- no intercepts with the y axis;
- y is an odd function of x (symmetric about the origin);
- for $x \downarrow 0$, $y \to \infty$ (vertical asymptote);
- for $x \uparrow 0$, $y \to -\infty$ (singularity is an odd, simple pole);
- for $x \to \pm\infty$, $y \sim x^3 \to \pm\infty$.

$$ y' \equiv \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{(x^2 - 9)\,(3x^2 + 9)}{x^2} $$

Hence,
- $y' > 0$ for $-\infty < x < -3$ (y increases from $-\infty$);
- $y' = 0$ for $x = -3$ (local maximum, $y = 0$);
- $y' < 0$ for $-3 < x < 0$ (y decreases towards $-\infty$);
- $y' = -\infty$ for $x = 0$ (singular point, vertical asymptote);
- $y' < 0$ for $0 < x < -3$ (decreases from ∞);
- $y' = 0$ for $x = 3$ (local minimum, $y = 0$);
- $y' > 0$ for $3 < x < \infty$ (increases to ∞).
Also,
- $y' \to \infty$ when $x \to \pm\infty$ (no horizontal or oblique asymptotes);
- all derivatives exist, except at $x = 0$, which has no point on the curve (no corners, cusps, infinite curvature, or other singular points);
- probably no inflection points.

$$ y'' = \frac{6x^4 + 162}{x^3} $$

Hence
- really no inflection points (since there is no point at $x = 0$);
- cocave downward for $x < 0$, upward for $x > 0$.
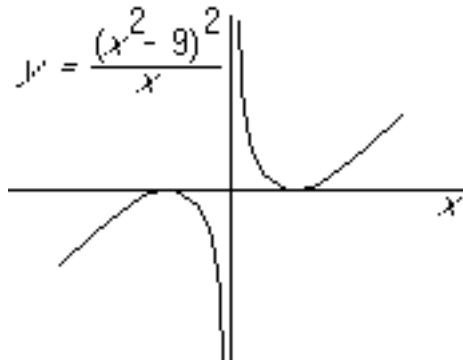
Hence the x- and y-extends are as before.

1.3 Example

From [1, p. 128, 13g]
  Asked: Graph

$$y = x\sqrt{x-1}$$

1.3.1 Solution

$$y = x\sqrt{x-1}$$

Factor $\sqrt{x-1}$ is $\sqrt{x}$ shifted one unit towards the right.
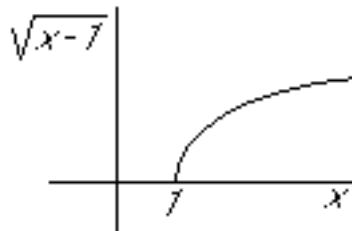
Multiplying by x magnifies it by a factor ranging from 1 to ∞:
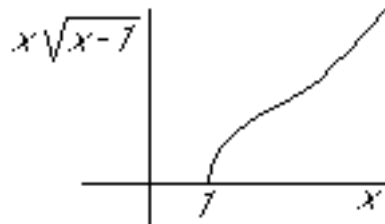
Function $y(x)$:

- has an x-extent $x \geq 1$ and a y-extent $y \geq 0$;
- behaves asymptotically as $y \sim x^{3/2}$ for $x \to \infty$;
- is monotonous:

$$y' = \frac{\mathrm{d}y}{\mathrm{d}x} = \sqrt{x-1} + \frac{x}{2\sqrt{x-1}} = \frac{2x-2+x}{2\sqrt{x-1}} = \frac{3x-2}{2\sqrt{x-1}} > 0;$$

- has vertical slope at $x = 1$;
- is concave down for smaller x, concave up for larger x;
- the inflection point is at

$$y'' = \frac{3x-4}{4(x-1)^{3/2}} = 0$$

giving $x = 4/3$.

Chapter 2

Optimization

2.1 Introduction

Optimization is important for engineers for a number of reasons:

- A best design finds the maximum of some benefit function.

- Drag reduction minimizes drag.

- Potential energy minimization finds the stationary state of a system. That is the basis for true finite element methods.

- Much of economics is based on finding the extrema of cost or benefi functions.

- ...

Key ideas:
- zero partial derivatives at an interior extremum
- Lagrangian multipliers can account for constraints

2.2 Example

From [1, p. 116, 30]
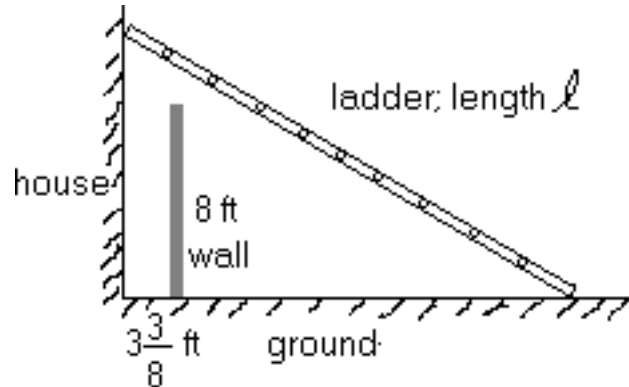
Given: A free standing wall, located $3\frac{3}{8}$ ft from the side of a house.

Asked: What is the length ℓ of the shortest ladder that can reach the house (over the free standing wall).

2.2.1   Definition
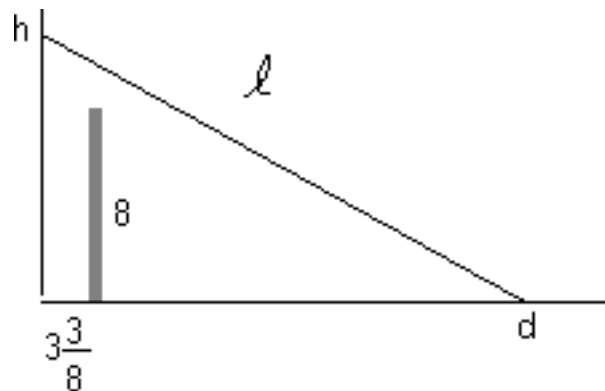
Two degrees of freedom: say h and d

One *inequality* constraint: the ladder must be above the free standing wall.

2.2.2   Reduction

The shortest ladder hits the free standing wall:
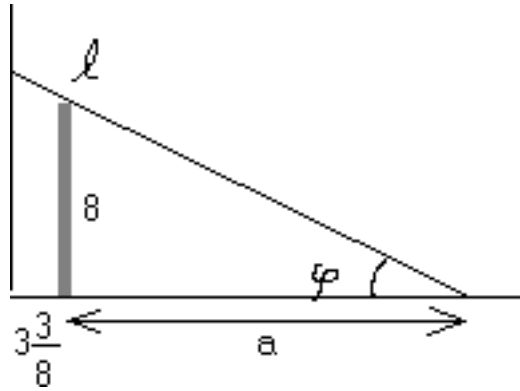
One degree of freedom left: φ.

2.2.3   Further reduction
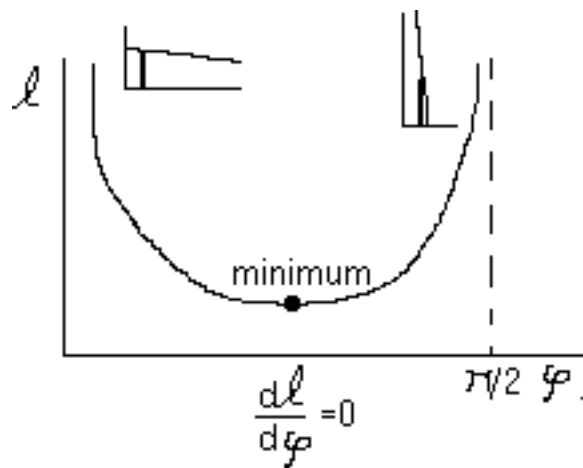
At the minimum:

$$\frac{\mathrm{d}\ell}{\mathrm{d}\varphi} = 0$$

2.2.4   Finding the length
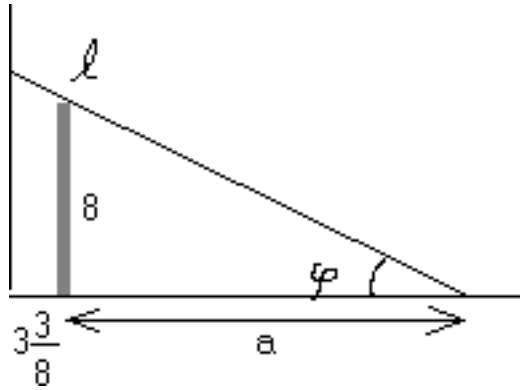
First find a:

$$a = \frac{8}{\tan \varphi}.$$

Then:

$$\ell = \frac{3\frac{3}{8} + a}{\cos \varphi} = \frac{3\frac{3}{8}}{\cos \varphi} + \frac{8}{\sin \varphi} \tag{2.1}$$

2.2.5   Finding the optimum angle

$$\frac{\mathrm{d}\ell}{\mathrm{d}\varphi} = \frac{3\frac{3}{8}}{\cos^2 \varphi} \sin \varphi - \frac{8}{\sin^2 \varphi} \cos \varphi = 0.$$

$$\frac{27}{8 \cos^2 \varphi} \sin \varphi = \frac{8}{\sin^2 \varphi} \cos \varphi$$

$$\tan^3 \varphi = \frac{64}{27} \qquad \Rightarrow \qquad \varphi_{\min} = 0.9273 \text{ radians}$$

2.2.6   Finding the optimum length

From (2.1)

$$\ell_{\min} = 15.625 \text{ ft}$$

2.3 General Approach
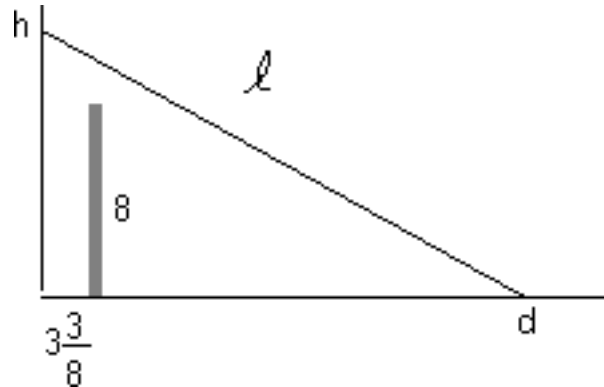
If you do not know a priori that the ladder hits the wall, you can follow the general approach.

There are now two degrees of freedom. They are conveniently taken to be h and d.

Then the length of the ladder is, (from Pythagoras),

$$\ell(h, d) = \sqrt{h^2 + d^2}$$

We now need to figure out what values of h and d produce the shortest ladder.

2.3.1 Formulation
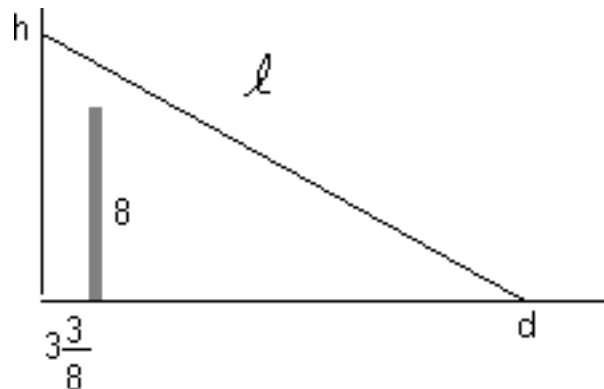
Note that by the definition of the problem, $h > 8$ and $d > 3\frac{3}{8}$. But these constraints are not precise. For example, $h = 8.001$ and $d = 3\frac{3}{8} + 0.001$ would obviously have the ladder go through the wall.

There is a precise constraint, that the ladder cannot pass through the wall. If b is the height of the point on the ladder straight above the wall, then similar triangles give:

$$\frac{b}{d - 3\frac{3}{8}} = \frac{h}{d} \quad (= \tan \phi)$$

The constraint is that $b \geq 8$, so:

$$h\frac{d - 3\frac{3}{8}}{d} \geq 8 \qquad \Rightarrow \qquad h[d - 3\frac{3}{8}] - 8d \geq 0$$

Note that this is in general an *inequality* constraint. Equality occurs when the ladder hits the wall.

So the problem is to minimize

$$\ell(h, d) = \sqrt{h^2 + d^2}$$

(from Pythagoras), subject to the inequality constraint

$$h[d - 3\frac{3}{8}] - 8d \geq 0$$

To solve this, first plot the possible h and d values:
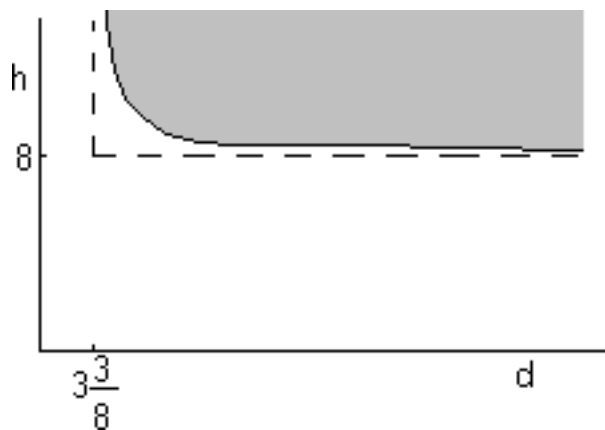
Plotting the curve where equality occurs in the constraint gives the bottom of the grey region above. On that curve, the ladder hits the wall. If you go above the curve, into the grey region, h becomes bigger and the ladder then moves above the wall.

Now you must figure out whether the shortest ladder occurs in the strict *interior* of the grey region or on its boundary. Try the interior first.

2.3.2 Interior minima
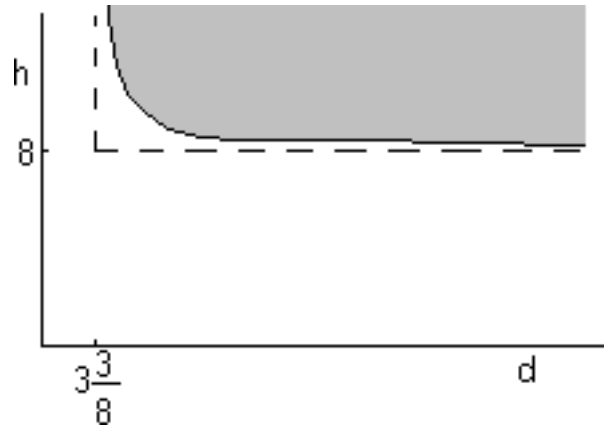
For a minimum, at least locally, in the strict *interior* of the grey region, the partial derivatives must be zero.

$$\frac{\partial \ell}{\partial d} = (h^2 + d^2)^{-1/2}d = 0 \qquad \frac{\partial \ell}{\partial h} = (h^2 + d^2)^{-1/2}h = 0$$

But these two requirements can only be true if $d = h = 0$, and that point is not in the grey region. So there is *no* interior minimum (or maximum, for that matter).

2.3.3 Boundary minima
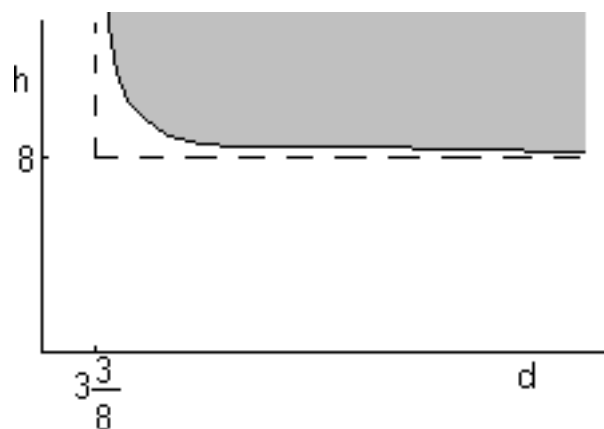
Since the minimum is not in the interior, it must be on the boundary of the grey region. Now obviously for infinite h or d or both you do not have the shortest ladder. So the minimum cannot be on the boundary at infinity. It must be on the curve where the ladder just hits the wall.

But how do we find the minimum on this line? The partial derivatives of ℓ are not zero at this point. (Just check that out in the previous subsection.)

The trick is to define an artificial third variable λ, called a "Lagrangian multiplier," corresponding to the constraint. (In the most general case, this Lagrangian multiplier has no particular physical meaning.) Then define a new function f to replace ℓ in the minimization:

$$f = \sqrt{h^2 + d^2} + \lambda(h[d - 3\tfrac{3}{8}] - 8d).$$

Note that λ multiplies whatever is zero according to the constraint.

Now it turns out that you can find the desired minimum by finding an unconstrained stationary point to this function f:

$$\partial f/\partial d = (h^2 + d^2)^{-1/2}d + \lambda(h - 8) = 0$$

$$\partial f/\partial h = (h^2 + d^2)^{-1/2}h + \lambda(d - 3\tfrac{3}{8}) = 0$$

$$\partial f/\partial \lambda = h[d - 3\tfrac{3}{8}] - 8d = 0$$

From the first two equations

$$(h^2 + d^2)^{-1/2}d = -\lambda(h - 8) \qquad (h^2 + d^2)^{-1/2}h = -\lambda(d - 3\tfrac{3}{8})$$

or taking the ratio of these two equations,

$$\frac{d}{h} = \frac{h - 8}{d - 3\tfrac{3}{8}}$$

Solving the constraint for h and putting it in the above gives after simplification:

$$(d - 3\tfrac{3}{8})^3 = 8 \times 27 \qquad \Rightarrow \qquad d = \tfrac{75}{8} \qquad \Rightarrow \qquad h = \tfrac{75}{6} \qquad \Rightarrow \qquad \ell = \tfrac{125}{8}$$

If you would have more than one constraint, there is one separate Lagrangian multiplier for each one. For example, if you take care of the boundary conditions in a finite element computation this way, you will get one for each boundary condition at each boundary point.

Chapter 3

Approximations

3.1 Introduction

Why use approximation:

- Because it is needed. In fact *everything* you do in real life is an approximation. Real life is proably determined by some sort of quantum mechanics. But we do not know what it is. And we would definitely not know how to solve it even if we knew what it was.

- To reduce effort.

- To increase accuracy. You might be able to solve approximate equations more accurately than more accurate equations, producing a better result.

- To get more insight in the problem. If you have computed say an incompressible flow of interest, the only real thing you can say about it is that indeed, it is free from singularities. You can wax about "vortices," "fingers," "intestines," etcetera, that you seem to see in the flow, but that you can also see in the clouds in the sky. If you start computing the flow for various parameters, you may start getting somewhere to an inkling of insight. However, doing that is limited by what the computer can do. And the interpretation will always have ambiguity. However, if you start looking at limiting processes of your parameters, you are suddenly getting somewhere. You can now define meaningful "boundary layers", "vortex layers", "shocks." etcetera that are not just arbitrary interpretations but have rigorous mathematical definitions.

- ...

This chapter looks at simple approximations using Taylor series (which of course always includes linearization.)

3.2   Example

From [1, p. 402, 10b]

Asked: The Maclaurin series of $\sin^2 x$.

3.2.1   Identification

General Taylor series:

$$\begin{aligned} f(x) &= f(a) + f'(a)\frac{x-a}{1!} + f''(a)\frac{(x-a)^2}{2!} + \dots \\ &= \sum_{n=0}^{\infty} f^{(n)}(a)\frac{(x-a)^n}{n!} \end{aligned}$$

This is a power series (a is a given constant.) Maclaurin series: $a = 0$.

Approach:

- note that $a = 0$;
- identify the derivatives;
- evaluate them at $a = 0$;
- put in the formula;
- identify the terms for any value of n.

3.2.2   Results

$$\begin{aligned} f(x) &= \sin^2 x & f(0) &= 0 \\ f'(x) &= 2\sin x \cos x & f'(0) &= 0 \\ f''(x) &= 2\cos^2 x - 2\sin^2 x = 2 - 4\sin^2 x & f''(0) &= 2 \\ f'''(x) &= -8\sin x \cos x = -4f'(x) & f'''(0) &= 0 \\ f''''(x) &= -4f''(x) & f''''(0) &= -8 \\ f^{(5)}(x) &= -4f'''(x) & f^{(5)}(0) &= 0 \\ f^{(6)}(x) &= -4f''''(x) = (-4)^2 f''(x) & f^{(6)}(0) &= 32 \\ \vdots & & \vdots \end{aligned}$$

$$\begin{aligned} \sin^2 x &= f(0) + f'(0)\frac{x-a}{1!} + f''(0)\frac{(x-a)^2}{2!} + \dots \\ &= 2\frac{x^2}{2!} - 8\frac{x^4}{4!} + 32\frac{x^6}{6!} + \dots \end{aligned}$$

General expression:

When $n = 2k$ with $k \geq 1$: $f^{(n)} = 2(-4)^{ek-1}$ Otherwise: $f^{(n)}e = 0$

$$\sin^2 x = \sum_{k=1}^{\infty} 2(-4)^{k-1}\frac{x^{2k}}{(2k)!}$$

3.2.3 Other way

Write $\sin^2 x = \frac{1}{2} - \frac{1}{2}\cos(2x)$ and look up the Maclaurin series for the cosine. (No fair.)

3.3 Example

From [1, p. 404, 30]

Asked: The area below $y = \sin x^2$ for $0 \le x \le 1$.

3.3.1 Identification
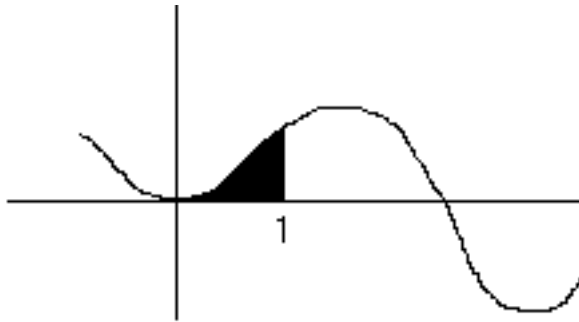
$$\int_0^1 \sin x^2 \, dx$$

Analytically? Actually, the integral is equivalent to $\int (\sin(x)/x) \, dx$, which cannot be written in terms of elementary functions.

But since the x range is not large, we will try approximating $\sin x^2$ using a Taylor series.

3.3.2 Finish

The Taylor series of $\sin x^2$ is that of $\sin x$ with x replaced by x^2. So:

$$\begin{aligned}
\int_0^1 \sin x^2 \, dx &= \int_0^1 \frac{x^2}{1!} - \frac{x^6}{3!} + \frac{x^{10}}{5!} + \dots \\
&= \frac{1}{3} - \frac{1}{3!7} + \frac{1}{5!11} \\
&= .3103 \pm 0.0008
\end{aligned}$$

The error estimate is rigorous since the series is an *alternating* one whose terms get smaller monotoneously.

Chapter 4

Limits

4.1   Introduction

Taylor series often do not work because the functions involved are not analytic at the point of interest. For example, this is common if the behavior of interest is at large time, or for large values of some other parameter, like the Reynolds number Re of a flow. (In fact, for flows in infinite domains it normally also occurs for small Reynolds numbers.

Finding nonanalytic limits is then needed. Applications are very similar to those of Taylor series:

- Because it is needed.

- To reduce effort.

- To increase accuracy.

- For making estimates of how importants something is.

- To get more insight. For example, consider the laminar flow past a flat plate if the plate is aligned with the incoming flow velocity U. For finite Reynolds numbers, there is little more you can say than that the flow velocity will be zero at the plate, and U far away from the plate. To get more insight than that, you can ask: "What is the limit of the velocity for infinite Reynolds number Re, assuming that you keep the streamwise location x fixed, as well as keep the ratio $\eta = y\sqrt{\text{Re}}/x$ fixed, where y the distance from the wall?" The answer is $Uf'(\eta)$ where f' is the Blasius function tabulated in any real book on fluid mechanics. (If you instead keep y itself fixed at a nonzero value, the limit is U, which is not very interesting.) Most of my theoretical research in fluids (as opposed to in numerical methods) really simply finds limits like this.

- ...

4.2   Example

From [1, p. 227, 10v]
   Asked:

$$\lim_{x \to -\infty} x^2 e^x$$

4.2.1   Observations

$$\lim_{x \to -\infty} x^2 e^x$$

You must first look whether the limit is trivial:

$$x^2 \to \infty \qquad e^x \to 0$$

Since the product of infinity times zero is unknown, this limit is nontrivial.

4.2.2   L'Hopital

L'Hopital can be used if you create a ratio of quantities that both become zero or both become infinite. (For example, you would not want to apply L'Hopital on $\lim_{x \to 0} 3/2$.)

$$\lim_{x \to -\infty} \frac{x^2}{e^{-x}} = \lim_{x \to -\infty} \frac{\left(x^2\right)'}{\left(e^{-x}\right)'}$$

Now both top and bottom become infinite. So L'Hopital can be applied, by differentiating top and bottom separately:

$$\lim_{x \to -\infty} \frac{\left(x^2\right)'}{\left(e^{-x}\right)'} = \lim_{x \to -\infty} \frac{2x}{-e^{-x}} =$$

Still infinity over infinity, so differentiate once more

$$\lim_{x \to -\infty} \frac{2x}{-e^{-x}} = \lim_{x \to -\infty} \frac{2}{e^{-x}} = 0$$

4.2.3   Better

Using some insight is always better than just crunching it out. First simplify things for yourself by defining $u = -x$. Then u goes to plus infinity instead of minus infinity like x. Then

$$\lim_{x \to -\infty} x^2 e^x \lim_{u \to \infty} \frac{u^2}{e^u}$$

and that is zero because e^u is much greater than any power of u for large positive u. (To see that, just look at the Taylor series:

$$e^u = 1 + \frac{u}{1!} + \frac{u^2}{2!} + \frac{u^3}{3!} + \dots$$

The u^3 term is much larger than u^2 for large u and the other terms make e^u larger still.)

So you could replace x^2 by x^{100} and the limit would still be zero.

4.3 Example

From [1, p. 228, 10z]

Asked:

$$\lim_{x \to 0} (x - \arcsin x) \csc^3 x$$

4.3.1 Grinding it out

In

$$\lim_{x \to 0} (x - \arcsin x) \csc^3 x$$

x and $\arcsin x$ become zero, but $\csc x$ becomes infinite. The total is undefined.

The simplest way to make a ratio suitable for l'Hopital is to use that $\csc x \equiv 1/\sin x$:

$$\lim_{x \to 0} (x - \arcsin x) \csc^3 x = \lim_{x \to 0} \frac{x - \arcsin x}{\sin^3 x}$$

Differentiate top and bottom

$$\lim_{x \to 0} \frac{1 - (1 - x^2)^{-1/2}}{3 \sin^2 x \cos x}$$

Still zero over zero, so differentiate again

$$\lim_{x \to 0} \frac{-x(1 - x^2)^{-3/2}}{6 \sin x \cos^2 x - 3 \sin^3 x}$$

Still zero over zero, so differentiate again

$$\lim_{x \to 0} \frac{-(1 - x^2)^{-3/2} - 3x^2(1 - x^2)^{-3/2}}{6 \cos^3 x - 21 \sin^2 x \cos x} = \tfrac{1}{6}$$

If I did not make any mistakes, I guess.

4.3.2   Using insight

Since $\sin x \approx x$ for small x, $\sin^3 x \approx x^3$. Also looking at a mathematical hand-book, $\arcsin x \approx x + \frac{1}{6}x^3 + \ldots$. So:

$$\lim_{x \to 0} \frac{x - \arcsin x}{\sin^3 x} \approx \frac{-\frac{1}{6}x^3}{x^3} = -\frac{1}{6}$$

(Note that we needed to keep the cubic term in the Taylor series for $\arcsin x$ since the term x dropped out.)

Chapter 5

Surface and Volume Integrals

5.1 Introduction

Multiple integrals are used to find various engineering quantities:

- Areas (cost, heat losses, ...):

$$\text{2D Cartesian:} \quad \mathrm{d}A = \mathrm{d}x\mathrm{d}y$$

$$\text{2D polar:} \quad \mathrm{d}A = \rho\mathrm{d}\rho\mathrm{d}\theta$$

- Volumes (weight, ...):

$$\text{3D Cartesian:} \quad \mathrm{d}V = \mathrm{d}x\mathrm{d}y\mathrm{d}z$$

$$\text{Cylindrical:} \quad \mathrm{d}V = \rho\mathrm{d}\rho\mathrm{d}\theta\mathrm{d}z$$

$$\text{Spherical:} \quad \mathrm{d}V = r^2 \sin\phi\mathrm{d}r\mathrm{d}\phi\mathrm{d}\theta$$

- Centroids (center of gravity, center of pressure, ...)

$$\bar{x} = \int x\mathrm{d}A \left/ \int \mathrm{d}A \right. \qquad \bar{x} = \int x\mathrm{d}V \left/ \int \mathrm{d}V \right.$$

- Moments of inertia (solid body dynamics, center of pressure, ...)

$$I_x = \int y^2\mathrm{d}A \qquad I_0 = \int x^2 + y^2\mathrm{d}A$$

$$I_x = \int y^2 + z^2\mathrm{d}V \qquad I_{xy} = -\int xy\mathrm{d}V$$

- ...

If the material is not homogenous, you may have to put an additional density-like factor in those integrals.

Procedure:

- Draw the region to be integrated over.
- When integrating, say $\int \int \int f(a,b,c) \; \mathrm{d}a\mathrm{d}b\mathrm{d}c$, you have to decide whether you want to do a, b, or c first. Usually, you do the co-ordinate with the easiest limits of integration first.
- If you decide to do, say, b first, (i.e. you want to integrate

$$\int_{b_1}^{b_2} f(a,b,c) \; \mathrm{d}b$$

  first), the limits of integration b_1 and b_2 must be identified from the graph at *arbitrary* a and c, and are normally functions of a and c: $b_1 = b_1(a,c)$, $b_2 = b_2(a,c)$.
- After integrating over, say, b, the remaining double integral should no longer depend on b in any way. Nor does the region of integration: redraw it without the b coordinate. In other words, project it onto the a, c-plane. Then integrate over the next easiest coordinate in the same way.
- If you change integration variables from a, b, c to p, q, r, the integral becomes $\int \int \int f(p,q,r)J \; \mathrm{d}p\mathrm{d}q\mathrm{d}r$ with the "Jacobian"

$$J = \left| \begin{vmatrix} \frac{\partial a}{\partial p} & \frac{\partial a}{\partial q} & \frac{\partial a}{\partial r} \\ \frac{\partial b}{\partial p} & \frac{\partial b}{\partial q} & \frac{\partial b}{\partial r} \\ \frac{\partial c}{\partial p} & \frac{\partial c}{\partial q} & \frac{\partial c}{\partial r} \end{vmatrix} \right|$$

  Here the inner bars indicate the determinant of the matrix of deriva-tives and the outer bars the absolute value of that. (Sometimes it is easier to take the inverse of the Jacobian of the inverse transforma-tion.)

5.2   Example

From [1, p. 487, 14e]

   Asked: Find the centroid of the first-quadrant area bounded by $x^2 - 8y + 4 = 0$ and $x^2 = 4y$ and $x = 0$.

5.2.1 Region
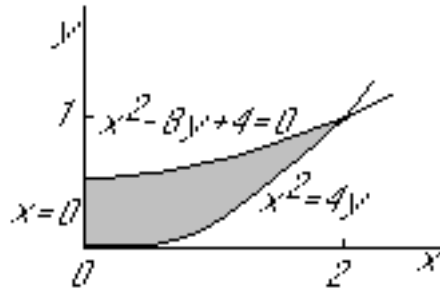
5.2.2 Approach

Integrate x first?
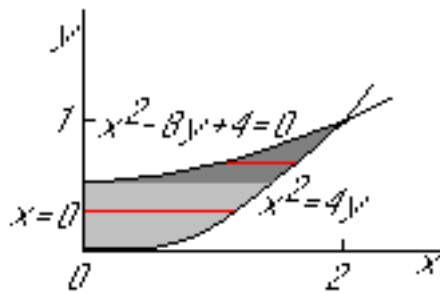
The integral would have to be split up into the light and dark areas since the lower boundary of integration is $x_1 = 0$ in the light region and $x_1 = \sqrt{8y - 4}$ in the dark region.

So integrate y first!
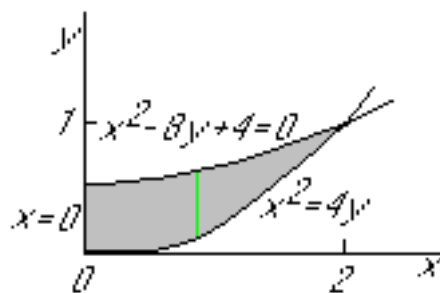
The boundaries of integration will be

$$y_1 = \tfrac{1}{4}x^2 \qquad y_2 = \tfrac{1}{8}x^2 + \tfrac{1}{2}$$

After integration over y, the remaining region of integration over x will be a line segment:

$$x_1 = 0 \qquad x_2 = 2$$

5.2.3   Results
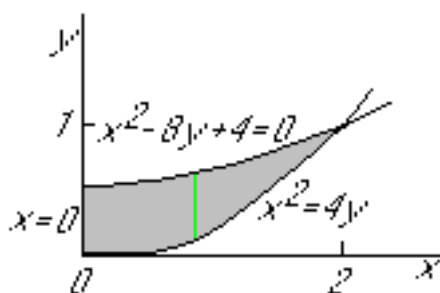
For $A = \int \mathrm{d}A = \int \int \mathrm{d}x\mathrm{d}y$:

$$A = \int_{x=0}^{x=2} \left[\int_{y=\frac{1}{4}x^2}^{y=\frac{1}{8}x^2+\frac{1}{2}} \mathrm{d}y \right] \, \mathrm{d}x$$

$$= \int_{x=0}^{2} \left[y \Big|_{y=\frac{1}{4}x^2}^{y=\frac{1}{8}x^2+\frac{1}{2}} \right] \, \mathrm{d}x$$

$$= \int_{x=0}^{2} \left[\left(\tfrac{1}{8}x^2 + \tfrac{1}{2}\right) - \left(\tfrac{1}{4}x^2\right) \right] \mathrm{d}x$$

$$= \int_{x=0}^{2} \left[\left(\tfrac{1}{2} - \tfrac{1}{8}x^2\right) \right] \mathrm{d}x = \tfrac{2}{3}$$
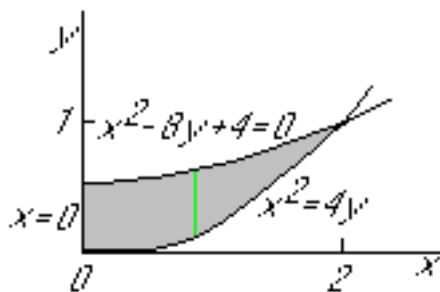
For $A\bar{x} = \int x\mathrm{d}A = \int \int x\mathrm{d}x\mathrm{d}y$:

$$A = \int_{x=0}^{x=2} \left[\int_{y=\frac{1}{4}x^2}^{y=\frac{1}{8}x^2+\frac{1}{2}} x \, \mathrm{d}y \right] \mathrm{d}x$$

where x is constant in the integration;

$$= \int_{x=0}^{2} \left[xy \Big|_{y=\frac{1}{4}x^2}^{y=\frac{1}{8}x^2+\frac{1}{2}} \right] \mathrm{d}x$$

$$= \int_{x=0}^{2} \left[(\tfrac{1}{8}x^3 + \tfrac{1}{2}x) - (\tfrac{1}{4}x^3) \right] \mathrm{d}x$$

$$= \int_{x=0}^{2} \left[(\tfrac{1}{2}x - \tfrac{1}{8}x^3 \right] \mathrm{d}x = \tfrac{1}{2}$$

Hence $\bar{x} = \frac{1}{2}/\frac{2}{3} = \frac{3}{4}$.
For $A\bar{y} = \int y\mathrm{d}A = \int \int y\mathrm{d}x\mathrm{d}y$:

$$A = \int_{x=0}^{x=2} \left[\int_{y=\frac{1}{4}x^2}^{y=\frac{1}{8}x^2+\frac{1}{2}} y \, \mathrm{d}y \right] \mathrm{d}x$$

$$= \int_{x=0}^{2} \left[\tfrac{1}{2}y^2 \Big|_{y=\frac{1}{4}x^2}^{y=\frac{1}{8}x^2+\frac{1}{2}} \right] \mathrm{d}x$$

$$= \int_{x=0}^{2} \left[\tfrac{1}{2}(\tfrac{1}{8}x^2 + \tfrac{1}{2})^2 - \tfrac{1}{2}(\tfrac{1}{4}x^2)^2 \right] \mathrm{d}x$$

$$= \int_{x=0}^{2} \left[(\tfrac{1}{8} + \tfrac{1}{16}x^2 - \tfrac{3}{128}x^2 \right] \mathrm{d}x = \tfrac{4}{15}$$

Hence $\bar{y} = \frac{4}{15}/\frac{2}{3} = \frac{2}{5}$.

5.3 Example

From [1, p. 507, 21c]

Asked: Find the centroid of the first octant region inside $x^2 + y^2 = 9$ and below $x + z = 4$.

5.3.1   Approach

The region inside $x^2 + y^2 = 9$ is the inside of a cylinder of radius 3 around the z-axis. The equation $x + z = 4$ describes a plane through the y-axis under 45 degrees with the x-axis:
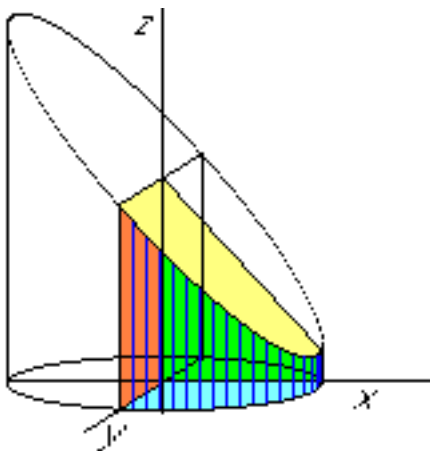
Use cylindrical coordinates r, θ, and z:

$$x = r \cos \theta \qquad y = r \sin \theta$$

Integrate z first:
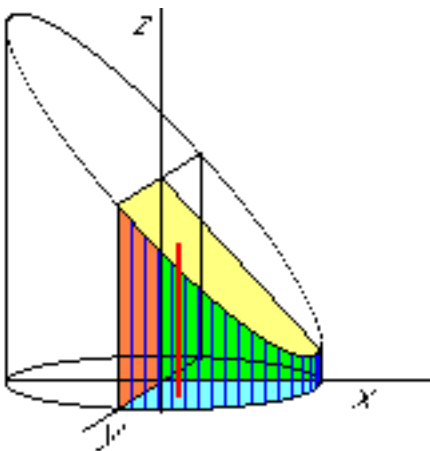
(Why not r first? Why not θ?). Boundaries are

$$z_1 = 0 \qquad z_2 = 4 - x = 4 - r \cos \theta$$

Next integrate θ and r:
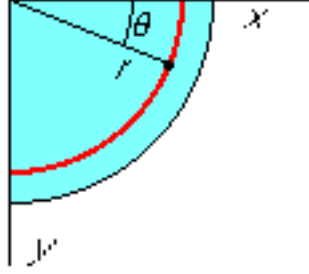
$$\theta_1 = 0 \qquad \theta_2 = \tfrac{1}{2}\pi$$

$$r_1 = 0 \qquad r_2 = 3$$

5.3.2   Results

For the volume $V = \int\int\int dV = \int\int\int r \, dzdrd\theta$:

$$V = \int_{\theta=0}^{\pi/2} \int_{r=0}^{3} \left[\int_{z=0}^{4-r\cos\theta} r \, dz \right] drd\theta$$

$$= \int_{\theta=0}^{\pi/2} \left[\int_{r=0}^{3} (4 - r\cos\theta)rdr \right] d\theta$$

$$= \int_{\theta=0}^{\pi/2} 18 - 9\cos\theta d\theta = 9(\pi - 1)$$

For $V\bar{x} = \int\int\int x \, dV = \int\int\int xrdzdrd\theta$:

$$V\bar{x} = \int_{\theta=0}^{\pi/2} \int_{r=0}^{3} \left[\int_{z=0}^{4-r\cos\theta} r^2 \cos\theta \, dz \right] drd\theta$$

$$= \int_{\theta=0}^{\pi/2} \left[\int_{r=0}^{3} 4r^2\cos\theta - r^3\cos^2\theta dr \right] d\theta$$

$$= \int_{\theta=0}^{\pi/2} 36\cos\theta - \frac{81}{4}\cos^2\theta d\theta = \frac{9}{16}(64 - 9\pi)$$

hence $\bar{x} = (64 - 9\pi)/16(\pi - 1)$

Etcetera.

Chapter 6

Combined Changes in Variables

6.1 Introduction

Combined changes in variables are common. For example:

- Error estimates.

- Changes for a moving particle in a field.

- Changes in scalar quantities depending on several variables,

- ...

The key concept is the total differential. For any function $f = f(x, y, z)$,

$$\mathrm{d}f = \frac{\partial f}{\partial x}\mathrm{d}x + \frac{\partial f}{\partial y}\mathrm{d}y + \frac{\partial f}{\partial z}\mathrm{d}z$$

Of course, f could be a vector.

6.2 Example

From [1, p. 422, 27a]

Given:

$$\omega = \sqrt[3]{\frac{g}{b}}$$

The maximum error in g is 1%, the maximum error in b is 0.5%.

Asked: The maximum percentage error in ω.

6.2.1 Identification

Given is that the maximum error in g is 1% and the maximum error in b is 0.5%. That means that the *relative* errors are:

$$\frac{\delta g}{g} \le 0.01 \qquad \frac{\delta b}{b} \le 0.005$$

where δg and δb are the *absolute* errors. Errors are always positive.

Error manipulation rules:

1. During addition and substraction of variables, add their absolute errors;
2. During multiplication or division, add their relative errors;
3. During exponentiation, multiply the relative error by the absolute power.

6.2.2 Results

Consider first the relative change in g/b due to changes $\mathrm{d}g$ in g and $\mathrm{d}b$ in b. The rule for differentiating a ratio implies:

$$\frac{\mathrm{d}(g/b)}{(g/b)} = \frac{b}{g}\left(\frac{b\mathrm{d}g - g\mathrm{d}b}{b^2}\right) = \frac{\mathrm{d}g}{g} - \frac{\mathrm{d}b}{b}$$

Note that if you do not know the sign of the errors, you can only say that the final result is no bigger than

$$\left|\frac{\mathrm{d}g}{g}\right| + \left|\frac{\mathrm{d}b}{b}\right|$$

which is simply the rule for adding relative errors if you take a ratio or product of variables.

Hence the greatest possible relative error in (g/b) is:

$$\frac{\delta(g/b)}{(g/b)} = 0.01 + 0.005 = 0.015$$

But we need the relative error in $\sqrt[3]{g/b}$ instead of in g/b. Denoting g/b by u for now, the rule for differentiating a power gives

$$\frac{\mathrm{d}u^{1/3}}{u^{1/3}} = \frac{\frac{1}{3}u^{-2/3}\mathrm{d}u}{u^{1/3}} = \frac{1}{3}\frac{\mathrm{d}u}{u}$$

That is simply the rule of multiplying the relative error by the absolute power when exponentiating.

Hence

$$\frac{\delta\omega}{\omega} = \tfrac{1}{3} \times 0.015 = 0.005 = 0.5\%$$

6.3 Example

From [1, p. 423, 29]

Given: A circular cylinder of changing radius r and height h. At a given time, $r = 6$ inch, $\dot{r} = 0.2$ in/sec, $h = 8$ in, $\dot{h} = -0.4$ in/sec.

Asked: $\dot{V}$ and $\dot{A}$ at that time.

6.3.1 Solution

$$V = \pi r^2 h \qquad A = 2\pi r h + 2\pi r^2$$

The total differential gives

$$\mathrm{d}V = \frac{\partial V}{\partial h}\mathrm{d}h + \frac{\partial V}{\partial r}\mathrm{d}r$$

where differential changes become time derivatives if you divide by $\mathrm{d}t$. So identifying the partial derivatives gives:

$$\dot{V} = \pi r^2 \dot{h} + \pi 2 r h \dot{r} = 15.08 \text{ in}^3/\text{sec}$$

Similarly:

$$\dot{A} = 2\pi r \dot{h} + (2\pi h + 4\pi r)\,\dot{r} = 10.05 \text{ in}^2/\text{sec}$$

Appendix A

Addenda

This appendix describes a number of additional topics. They did not seem important enough to warrant including them in the main text. An addition is always a distraction; at the minimum you have to worry about whether you need to worry about it.

Appendix D

Derivations

This appendix gives various derivations. Sometimes you need to see the derivation to judge whether a result is applicable in given circumstances. And some people like to see the derivation period.

Appendix N

Notes

This appendix collects various notes on the material. This sort of material is often given in footnotes at the bottom of the text. However, such a footnote is distracting. You tend to read them even if they are probably not really that important to you. Also, footnotes have to be concise, or they make a mess of the main text.

N.1   Why this book?

See the preface.

N.2   History and wish list

- Sep. 1, 2013 The first version of this manuscript was posted.

Bibliography

[1] F. Ayres and E. Mendelson. *Calculus.* Schaum's Outline Series. McGraw-Hill, 5th edition, 2009.

[2] M.R. Spiegel and J. Liu. *Mathematical Handbook of Formulas and Tables.* Schaum's Outline Series. McGraw-Hill, second edition, 1999.

Web Pages

Below is a list of relevant web pages.

1. Wikipedia[1]

   A valuable source source of information on about every loose end, though somewhat uneven. Some great, some confusing, some overly technical.

---

[1] `http://wikipedia.org`

Notations

The below are the simplest possible descriptions of various symbols, just to help you keep reading if you do not remember/know what they stand for.

Watch it. There are so many ad hoc usages of symbols, some will have been overlooked here. Always use common sense first in guessing what a symbol means in a given context.

· A dot might indicate

- A dot product between vectors, if in between them.
- A time derivative of a quantity, if on top of it.

And also many more prosaic things (punctuation signs, decimal points, ...).

× Multiplication symbol. May indicate:

- An emphatic multiplication.
- Multiplication continued on the next line / from the previous line.
- A vectorial product between vectors. In index notation, the i-th component of $\vec{v} \times \vec{w}$ equals

$$(\vec{v} \times \vec{w})_i = v_{\overline{\imath}} w_{\overline{\overline{\imath}}} - v_{\overline{\overline{\imath}}} w_{\overline{\imath}}$$

where $\overline{\imath}$ is the index following i in the sequence $123123\ldots$, and $\overline{\overline{\imath}}$ the one preceding it (or second following). Alternatively, evaluate the determinant

$$\vec{v} \times \vec{w} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ v_x & v_y & v_z \\ w_x & w_y & w_z \end{vmatrix}$$

! Might be used to indicate a factorial. Example: $5! = 1 \times 2 \times 3 \times 4 \times 5 = 120$.

The function that generalizes $n!$ to noninteger values of n is called the gamma function; $n! = \Gamma(n+1)$. The gamma function generalization is

due to, who else, Euler. (However, the fact that $n! = \Gamma(n+1)$ instead of $n! = \Gamma(n)$ is due to the idiocy of Legendre.) In Legendre-resistant notation,

$$n! = \int_0^\infty t^n e^{-t} \, \mathrm{d}t$$

Straightforward integration shows that 0! is 1 as it should, and integration by parts shows that $(n+1)! = (n+1)n!$, which ensures that the integral also produces the correct value of $n!$ for any higher integer value of n than 0. The integral, however, exists for any real value of n above -1, not just integers. The values of the integral are always positive, tending to positive infinity for both $n \downarrow -1$, (because the integral then blows up at small values of t), and for $n \uparrow \infty$, (because the integral then blows up at medium-large values of t). In particular, Stirling's formula says that for large positive n, $n!$ can be approximated as

$$n! \sim \sqrt{2\pi n}\, n^n e^{-n} \left[1 + \ldots\right]$$

where the value indicated by the dots becomes negligibly small for large n. The function $n!$ can be extended further to any complex value of n, except the negative integer values of n, where $n!$ is infinite, but is then no longer positive. Euler's integral can be done for $n = -\frac{1}{2}$ by making the change of variables $\sqrt{t} = u$, producing the integral $\int_0^\infty 2e^{-u^2} \, \mathrm{d}u$, or $\int_{-\infty}^\infty e^{-u^2} \, \mathrm{d}u$, which equals $\sqrt{\int_{-\infty}^\infty e^{-x^2} \, \mathrm{d}x \int_{-\infty}^\infty e^{-y^2} \, \mathrm{d}y}$ and the integral under the square root can be done analytically using polar coordinates. The result is that

$$-\frac{1}{2}! = \int_{-\infty}^\infty e^{-u^2} \, \mathrm{d}u = \sqrt{\pi}$$

To get $\frac{1}{2}!$, multiply by $\frac{1}{2}$, since $n! = n(n-1)!$.

| May indicate:

- The magnitude or absolute value of the number or vector, if enclosed between a pair of them.

- The determinant of a matrix, if enclosed between a pair of them.

- The norm of the function, if enclosed between two pairs of them.

$\sum$ Summation symbol. Example: if in three dimensional space a vector $\vec{f}$ has components $f_1 = 2$, $f_2 = 1$, $f_3 = 4$, then $\sum_{\text{all } i} f_i$ stands for $2 + 1 + 4 = 7$.

$\int$ Integration symbol, the continuous version of the summation symbol. For example,

$$\int_{\text{all } x} f(x) \, \mathrm{d}x$$

46

is the summation of $f(x)\,\mathrm{d}x$ over all little fragments $\mathrm{d}x$ that make up the entire x-range.

$\rightarrow$ May indicate:

- An approaching process. $\lim_{\varepsilon \to 0}$ indicates for practical purposes the value of the expression following the lim when ε is extremely small. Similarly, $\lim_{r \to \infty}$ indicates the value of the following expression when r is extremely large.

- The fact that the left side leads to, or implies, the right-hand side.

$\vec{}$ Vector symbol. An arrow above a letter indicates it is a vector. A vector is a quantity that requires more than one number to be characterized. Typical vectors in physics include position $\vec{r}$, velocity $\vec{v}$, linear momentum $\vec{p}$, acceleration $\vec{a}$, force $\vec{F}$, moment $\vec{M}$, etcetera.

$'$ May indicate:

- A derivative of a function. Examples: $1' = 0$, $x' = 1$, $\sin'(x) = \cos(x)$, $\cos'(x) = -\sin(x)$, $(e^x)' = e^x$.

- A small or modified quantity.

∇ The spatial differentiation operator nabla. In Cartesian coordinates:

$$\nabla \equiv \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right) = \hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}$$

Nabla can be applied to a scalar function f in which case it gives a vector of partial derivatives called the gradient of the function:

$$\operatorname{grad} f = \nabla f = \hat{i}\frac{\partial f}{\partial x} + \hat{j}\frac{\partial f}{\partial y} + \hat{k}\frac{\partial f}{\partial z}.$$

Nabla can be applied to a vector in a dot product multiplication, in which case it gives a scalar function called the divergence of the vector:

$$\operatorname{div} \vec{v} = \nabla \cdot \vec{v} = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}$$

or in index notation

$$\operatorname{div} \vec{v} = \nabla \cdot \vec{v} = \sum_{i=1}^{3} \frac{\partial v_i}{\partial x_i}$$

Nabla can also be applied to a vector in a vectorial product multiplication, in which case it gives a vector function called the curl or rot of the vector. In index notation, the i-th component of this vector is

$$(\text{curl}\,\vec{v})_i = (\text{rot}\,\vec{v})_i = (\nabla \times \vec{v})_i = \frac{\partial v_{\bar{\bar{\imath}}}}{\partial x_{\bar{\imath}}} - \frac{\partial v_{\bar{\imath}}}{\partial x_{\bar{\bar{\imath}}}}$$

where $\bar{\imath}$ is the index following i in the sequence $123123\ldots$, and $\bar{\bar{\imath}}$ the one preceding it (or the second following it).

The operator ∇^2 is called the Laplacian. In Cartesian coordinates:

$$\nabla^2 \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

In non Cartesian coordinates, don't guess; look these operators up in a table book.

* A superscript star normally indicates a complex conjugate. In the complex conjugate of a number, every i is changed into a $-$i.

$<$ Less than.

$>$ Greater than.

$\equiv$ Emphatic equals sign. Typically means "by definition equal" or "everywhere equal."

$\sim$ Indicates approximately equal. Normally the approximation applies when something is small or large. Read it as "is approximately equal to."

$\propto$ Proportional to. The two sides are equal except for some unknown constant factor.

Γ (Gamma) May indicate:

- The Gamma function. Look under "!" for details.

Δ (capital delta) May indicate:

- An increment in the quantity following it.
- Often used to indicate the Laplacian ∇^2.

δ (delta) May indicate:

- With two subscripts, the "Kronecker delta", which by definition is equal to one if its two subscripts are equal, and zero in all other cases.

- Without two subscripts, the "Dirac delta function", which is infinite when its argument is zero, and zero if it is not. In addition the infinity is such that the integral of the delta function over its single nonzero point is unity. The delta function is not a normal function, but a distribution.

  One often important way to create a three-dimensional delta function in spherical coordinates is to take the Laplacian of the function $-1/4\pi r$. In two dimensions, take the Laplacian of $\ln(r)/2\pi$ to get a delta function.

- Often used to indicate a small amount of the following quantity, or of a small change in the following quantity. There are nuanced differences in the usage of δ, ∂ and d that are too much to go in here.

- Often used to indicate a second small quantity in addition to ε.

∂ (partial) Indicates a vanishingly small change or interval of the following variable. For example, $\partial f/\partial x$ is the ratio of a vanishingly small change in function f divided by the vanishingly small change in variable x that causes this change in f. Such ratios define derivatives, in this case the partial derivative of f with respect to x.

ε (variant of epsilon) May indicate:

- A very small quantity.

η (eta) May be used to indicate a y-position.

Θ (capital theta) Used in this book to indicate some function of θ to be determined.

θ (theta) May indicate:

- In spherical coordinates, the angle from the chosen z axis, with apex at the origin.

- a z-position.

- A generic angle, like the one between the vectors in a cross or dot product.

ϑ (variant of theta) An alternate symbol for θ.

λ (lambda) May indicate:

- Wave length.
- An eigenvalue.
- Some multiple of something.

ξ (xi) May indicate:

- An x-position.

π (pi) May indicate:

- The area of a circle of unit radius. Value 3.141 592...
- Half the perimeter of a circle of unit radius. Value 3.141 592...
- A 180° angle expressed in radians. Note that $e^{\pm i\pi} = -1$. Value 3.141 592...

ρ (rho) May indicate:

- Scaled radial coordinate.
- Radial coordinate.

τ (tau) May indicate:

- A time or time interval.

Φ (capital phi) May indicate:

- Some function of ϕ to be determined.

ϕ (phi) May indicate:

- In spherical coordinates, the angle around the chosen z axis. Increasing ϕ by 2π encircles the z-axis exactly once.
- A phase angle.
- Something equivalent to an angle.

φ (variant of phi) May indicate:

- A change in angle ϕ.
- An alternate symbol for ϕ.

ω (omega) May indicate:

- Angular frequency.

A May indicate:

- Some generic matrix or operator.
- Some constant.
- Area.

a May indicate:

- Acceleration.
- Start point of an integration interval.
- Some coefficient.
- Some constant.

absolute May indicate:

- The absolute value of a real number a is indicated by $|a|$. It equals a is a is positive or zero and $-a$ if a is negative.
- The absolute value of a complex number a is indicated by $|a|$. It equals the length of the number plotted as a vector in the complex plane. This simplifies to above definition if a is real.

adjoint The adjoint A^H or $A^\dagger$ of a matrix is the complex-conjugate transpose of the matrix.

Alternatively, it is the matrix you get if you take it to the other side of an inner product. (While keeping the value of the inner product the same regardless of whatever two vectors or functions may be involved.)

"Hermitian" matrices are "self-adjoint;" they are equal to their adjoint. "Skew-Hermitian" matrices are the negative of their adjoint.

"Unitary" matrices are the inverse of their adjoint. Unitary matrices generalize rotations and reflections of vectors. Unitary operators preserve inner products.

Fourier transforms are unitary operators on account of the Parseval equality that says that inner products are preserved.

angle According to trigonometry, if the length of a segment of a circle is divided by its radius, it gives the total angular extent of the circle segment. More precisely, it gives the angle, in radians, between the line from the center to the start of the circle segment and the line from the center to the end of the segment. The generalization to three dimensions is called the "solid angle;" the total solid angle over which a segment of a spherical surface extends, measured from the center of the sphere, is the area of that segment divided by the square radius of the sphere.

B May indicate:

- A generic second matrix.
- Some constant.

b May indicate:

- End point of an integration interval.
- Some coefficient.
- Some constant.

basis A basis is a minimal set of vectors or functions that you can write all other vectors or functions in terms of. For example, the unit vectors $\hat{\imath}$, $\hat{\jmath}$, and $\hat{k}$ are a basis for normal three-dimensional space. Every three-dimensional vector can be written as a linear combination of the three.

C May indicate:

- A third matrix.
- A constant.

Cauchy-Schwartz inequality The Cauchy-Schwartz inequality puts a limitation on the magnitude of inner products. In particular, it says that for any vectors $\vec{v}$ and vec w

$$|\vec{v}^H \vec{w}| \leq |\vec{v}||\vec{w}|$$

For example, if $\vec{v}$ and $\vec{w}$ are real vectors, the inner product is the dot product and we have

$$\vec{v} \cdot \vec{w} = |\vec{v}||\vec{w}| \cos \theta$$

where $|\vec{v}|$ is the length of vector $\vec{v}$ and $|\vec{w}|$ the one of $\vec{w}$, and θ is the angle in between the two vectors. Since a cosine is less than one in magnitude, the Cauchy-Schwartz inequality is therefore true for vectors.

cos The cosine function, a periodic function oscillating between 1 and -1 as shown in [2, pp. 40-...].

curl The curl of a vector field $\vec{v}$ is defined as curl $\vec{v} = $ rot $\vec{v} = \nabla \times \vec{v}$.

d Indicates a vanishingly small change or interval of the following variable. For example, dx can be thought of as a small segment of the x-axis.

derivative A derivative of a function is the ratio of a vanishingly small change in a function divided by the vanishingly small change in the independent variable that causes the change in the function. The derivative of $f(x)$ with respect to x is written as df/dx, or also simply as f'. Note that the derivative of function $f(x)$ is again a function of x: a ratio f' can be found at every point x. The derivative of a function $f(x, y, z)$ with respect to x is written as $\partial f/\partial x$ to indicate that there are other variables, y and z, that do not vary.

determinant The determinant of a square matrix A is a single number indicated by $|A|$. If this number is nonzero, $A\vec{v}$ can be any vector $\vec{w}$ for the right choice of $\vec{v}$. Conversely, if the determinant is zero, $A\vec{v}$ can only produce a very limited set of vectors. But if it can produce a vector w, it can do so for multiple vectors $\vec{v}$.

There is a recursive algorithm that allows you to compute determinants from increasingly bigger matrices in terms of determinants of smaller matrices. For a 1×1 matrix consisting of a single number, the determinant is simply that number:

$$|a_{11}| = a_{11}$$

(This determinant should not be confused with the absolute value of the number, which is written the same way. Since we normally do not deal with 1×1 matrices, there is normally no confusion.) For 2×2 matrices, the determinant can be written in terms of 1×1 determinants:

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = +a_{11} \begin{vmatrix} \\ a_{22} \end{vmatrix} - a_{12} \begin{vmatrix} \\ a_{21} \end{vmatrix}$$

so the determinant is $a_{11}a_{22} - a_{12}a_{21}$ in short. For 3×3 matrices, we have

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} =$$

$$+a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

and we already know how to work out those 2×2 determinants, so we now know how to do 3×3 determinants. Written out fully:

$$a_{11}(a_{22}a_{33} - a_{23}a_{32}) - a_{12}(a_{21}a_{33} - a_{23}a_{31}) + a_{13}(a_{21}a_{32} - a_{22}a_{31})$$

For 4×4 determinants,

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{vmatrix} =$$

$$+a_{11}\begin{vmatrix} a_{22} & a_{23} & a_{24} \\ a_{32} & a_{33} & a_{34} \\ a_{42} & a_{43} & a_{44} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} & a_{24} \\ a_{31} & a_{33} & a_{34} \\ a_{41} & a_{43} & a_{44} \end{vmatrix}$$

$$+a_{13}\begin{vmatrix} a_{21} & a_{22} & a_{24} \\ a_{31} & a_{32} & a_{34} \\ a_{41} & a_{42} & a_{44} \end{vmatrix} - a_{14}\begin{vmatrix} a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \\ a_{41} & a_{42} & a_{43} \end{vmatrix}$$

Etcetera. Note the alternating sign pattern of the terms.

As you might infer from the above, computing a good size determinant takes a large amount of work. Fortunately, it is possible to simplify the matrix to put zeros in suitable locations, and that can cut down the work of finding the determinant greatly. We are allowed to use the following manipulations without seriously affecting the computed determinant:

1. We may "transpose"the matrix, i.e. change its columns into its rows.

2. We can create zeros in a row by subtracting a suitable multiple of another row.

3. We may also swap rows, as long as we remember that each time that we swap two rows, it will flip over the sign of the computed determinant.

4. We can also multiply an entire row by a constant, but that will multiply the computed determinant by the same constant.

Applying these tricks in a systematic way, called "Gaussian elimination" or "reduction to lower triangular form", we can eliminate all matrix coefficients a_{ij} for which j is greater than i, and that makes evaluating the determinant pretty much trivial.

div(ergence) The divergence of a vector field $\vec{v}$ is defined as div $\vec{v} = \nabla \cdot \vec{v}$.

e May indicate:

- The basis for the natural logarithms. Equal to $2.71\,281\,828\,459...$ This number produces the "exponential function" e^x, or $\exp(x)$, or in words "e to the power x", whose derivative with respect to x is again e^x. If a is a constant, then the derivative of e^{ax} is ae^{ax}. Also, if a is an ordinary real number, then e^{ia} is a complex number with magnitude 1.

e^{iax} Assuming that a is an ordinary real number, and x a real variable, e^{iax} is a complex function of magnitude one. The derivative of e^{iax} with respect to x is iae^{iax}

eigenvector A vector $\vec{v}$ is an eigenvector of a matrix A if $\vec{v}$ is nonzero and $A\vec{v} = \lambda\vec{v}$ for some number λ called the corresponding eigenvalue.

exponential function A function of the form $e^{\cdots}$, also written as $\exp(\ldots)$. See function and e.

F May indicate:

- The anti-derivative of some function f.
- Some function.

f May indicate:

- A generic function.
- A fraction.
- Frequency.

function A mathematical object that associates values with other values. A function $f(x)$ associates every value of x with a value f. For example, the function $f(x) = x^2$ associates $x = 0$ with $f = 0$, $x = \frac{1}{2}$ with $f = \frac{1}{4}$, $x = 1$ with $f = 1$, $x = 2$ with $f = 4$, $x = 3$ with $f = 9$, and more generally, any arbitrary value of x with the square of that value x^2. Similarly, function $f(x) = x^3$ associates any arbitrary x with its cube x^3, $f(x) = \sin(x)$ associates any arbitrary x with the sine of that value, etcetera.

One way of thinking of a function is as a procedure that allows you, whenever given a number, to compute another number.

functional A functional associates entire functions with single numbers. For example, the expectation energy is mathematically a functional: it associates any arbitrary wave function with a number: the value of the expectation energy if physics is described by that wave function.

g May indicate:

- A second generic function.

grad(ient) The gradient of a scalar f is defined as grad $f = \nabla f$.

$\Im$ The imaginary part of a complex number. If $c = c_r + \mathrm{i}c_i$ with c_r and c_i real numbers, then $\Im(c) = c_i$. Note that $c - c^* = 2\mathrm{i}\Im(c)$.

i May indicate:

- The number of a particle.
- A summation index.
- A generic index or counter.

Not to be confused with i.

i The standard square root of minus one: $\mathrm{i} = \sqrt{-1}$, $\mathrm{i}^2 = -1$, $1/\mathrm{i} = -\mathrm{i}$, $\mathrm{i}^* = -\mathrm{i}$.

index notation A more concise and powerful way of writing vector and matrix components by using a numerical index to indicate the components. For Cartesian coordinates, we might number the coordinates x as 1, y as 2, and z as 3. In that case, a sum like $v_x + v_y + v_z$ can be more concisely written as $\sum_i v_i$. And a statement like $v_x \neq 0, v_y \neq 0, v_z \neq 0$ can be more compactly written as $v_i \neq 0$. To really see how it simplifies the notations, have a look at the matrix entry. (And that one shows only 2 by 2 matrices. Just imagine 100 by 100 matrices.)

iff Emphatic "if." Should be read as "if and only if."

integer Integer numbers are the whole numbers: $\ldots, -2, -1, 0, 1, 2, 3, 4, \ldots$.

inverse (Of matrices.) If a matrix A converts a vector $\vec{v}$ into a vector $\vec{w}$, then the inverse of the matrix, A^{-1}, converts $\vec{w}$ back into $\vec{v}$.

in other words, $A^{-1}A = AA^{-1} = I$ with I the unit, or identity, matrix.

The inverse of a matrix only exists if the matrix is square and has nonzero determinant.

j May indicate:

- A summation index.
- A generic index or counter.

k May indicate:

- A generic summation index.

l May indicate:

- The azimuthal quantum number.
- A generic summation index.

ℓ May indicate:

- A length.

lim Indicates the final result of an approaching process. $\lim_{\varepsilon \to 0}$ indicates for practical purposes the value of the following expression when ε is extremely small.

linear combination A very generic concept indicating sums of objects times coefficients. For example, a position vector $\vec{r}$ is the linear combination $x\hat{\imath} + y\hat{\jmath} + z\hat{k}$ with the objects the unit vectors $\hat{\imath}$, $\hat{\jmath}$, and $\hat{k}$ and the coefficients the position coordinates x, y, and z.

matrix A table of numbers.

As a simple example, a two-dimensional matrix A is a table of four numbers called a_{11}, a_{12}, a_{21}, and a_{22}:

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

unlike a two-dimensional (ket) vector $\vec{v}$, which would consist of only two numbers v_1 and v_2 arranged in a column:

$$\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$$

(Such a vector can be seen as a "rectangular matrix" of size 2×1, but let's not get into that.)

In index notation, a matrix A is a set of numbers $\{a_{ij}\}$ indexed by two indices. The first index i is the row number, the second index j is the

57

column number. A matrix turns a vector $\vec{v}$ into another vector $\vec{w}$ according to the recipe

$$w_i = \sum_{\text{all } j} a_{ij} v_j \quad \text{for all } i$$

where v_j stands for "the j-th component of vector $\vec{v}$," and w_i for "the i-th component of vector $\vec{w}$."

As an example, the product of A and $\vec{v}$ above is by definition

$$\left(\begin{array}{cc} a_{11} & a_{12} \\ a_{21} & a_{22} \end{array} \right) \left(\begin{array}{c} v_1 \\ v_2 \end{array} \right) = \left(\begin{array}{c} a_{11}v_1 + a_{12}v_2 \\ a_{21}v_1 + a_{22}v_2 \end{array} \right)$$

which is another two-dimensional ket vector.

Note that in matrix multiplications like the example above, in geometric terms we take dot products between the rows of the first factor and the column of the second factor.

To multiply two matrices together, just think of the columns of the second matrix as separate vectors. For example:

$$\left(\begin{array}{cc} a_{11} & a_{12} \\ a_{21} & a_{22} \end{array} \right) \left(\begin{array}{cc} b_{11} & b_{12} \\ b_{21} & b_{22} \end{array} \right) = \left(\begin{array}{cc} a_{11}b_{11} + a_{12}b_{21} & a_{11}b_{12} + a_{12}b_{22} \\ a_{21}b_{11} + a_{22}b_{21} & a_{21}b_{12} + a_{22}b_{22} \end{array} \right)$$

which is another two-dimensional matrix. In index notation, the ij component of the product matrix has value $\sum_k a_{ik} b_{kj}$.

The zero matrix is like the number zero; it does not change a matrix it is added to and turns whatever it is multiplied with into zero. A zero matrix is zero everywhere. In two dimensions:

$$\left(\begin{array}{cc} 0 & 0 \\ 0 & 0 \end{array} \right)$$

A unit matrix is the equivalent of the number one for matrices; it does not change the quantity it is multiplied with. A unit matrix is one on its "main diagonal" and zero elsewhere. The 2 by 2 unit matrix is:

$$\left(\begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array} \right)$$

More generally the coefficients, $\{\delta_{ij}\}$, of a unit matrix are one if $i = j$ and zero otherwise.

The transpose of a matrix A, A^T, is what you get if you switch the two indices. Graphically, it turns its rows into its columns and vice versa. The Hermitian "adjoint" A^H is what you get if you switch the two indices and

then take the complex conjugate of every element. If you want to take a matrix to the other side of an inner product, you will need to change it to its Hermitian adjoint. "Hermitian matrices"are equal to their Hermitian adjoint, so this does nothing for them.

See also "determinant" and "eigenvector."

M May indicate:

- Molecular mass. See separate entry.
- Mirror operator.
- Figure of merit.

m May indicate:

- Number of rows in a matrix.
- A generic summation index or generic integer.

n May indicate:

- Number of columns in a matrix.
- A generic summation index or generic integer.
- A natural number.

and maybe some other stuff.

natural Natural numbers are the numbers: $1, 2, 3, 4, \ldots$.

normal A normal operator or matrix is one that has orthonormal eigenfunctions or eigenvectors. Since eigenvectors are not orthonormal in general, a normal operator or matrix is abnormal! Normal matrices are matrices that commute with their adjoint.

opposite The opposite of a number a is $-a$. In other words, it is the additive inverse.

perpendicular bisector For two given points P and Q, the perpendicular bisector consists of all points R that are equally far from P as they are from Q. In two dimensions, the perpendicular bisector is the line that passes through the point exactly half way in between P and Q, and that is orthogonal to the line connecting P and Q. In three dimensions, the perpendicular bisector is the plane that passes through the point exactly half way in between P and Q, and that is orthogonal to the line connecting

P and Q. In vector notation, the perpendicular bisector of points P and Q is all points R whose radius vector $\vec{r}$ satisfies the equation:

$$(\vec{r} - \vec{r}_P) \cdot (\vec{r}_Q - \vec{r}_P) = \tfrac{1}{2}(\vec{r}_Q - \vec{r}_P) \cdot (\vec{r}_Q - \vec{r}_P)$$

(Note that the halfway point $\vec{r} - \vec{r}_P = \tfrac{1}{2}(\vec{r}_Q - \vec{r}_P)$ is included in this formula, as is the half way point plus any vector that is normal to $(\vec{r}_Q - \vec{r}_P)$.)

phase angle Any complex number can be written in "polar form" as $c = |c|e^{i\alpha}$ where both the magnitude $|c|$ and the phase angle α are real numbers. Note that when the phase angle varies from zero to 2π, the complex number c varies from positive real to positive imaginary to negative real to negative imaginary and back to positive real. When the complex number is plotted in the complex plane, the phase angle is the direction of the number relative to the origin. The phase angle α is often called the argument, but so is about everything else in mathematics, so that is not very helpful.

In complex time-dependent waves of the form $e^{i(\omega t - \phi)}$, and its real equivalent $\cos(\omega t - \phi)$, the phase angle ϕ gives the angular argument of the wave at time zero.

q May indicate:

- Charge.
- Heat flux density.

R May indicate:

- Some function of r to be determined.
- Some function of (x, y, z) to be determined.
- Some radius.

$\Re$ The real part of a complex number. If $c = c_r + ic_i$ with c_r and c_i real numbers, then $\Re(c) = c_r$. Note that $c + c^* = 2\Re(c)$.

r May indicate:

- The radial distance from the chosen origin of the coordinate system.
- r_i often indicates the i-th Cartesian component of the radius vector $\vec{r}$.
- Some ratio.

60

$\vec{r}$ The position vector. In Cartesian coordinates (x, y, z) or $x\hat{\imath} + y\hat{\jmath} + z\hat{k}$. In spherical coordinates $r\hat{\imath}_r$. Its three Cartesian components may be indicated by r_1, r_2, r_3 or by x, y, z or by x_1, x_2, x_3.

reciprocal The reciprocal of a number a is $1/a$. In other words, it is the multiplicative inverse.

rot The rot of a vector $\vec{v}$ is defined as curl $\vec{v} = $ rot $\vec{v} = \nabla \times \vec{v}$.

scalar A quantity characterized by a single number.

sin The sine function, a periodic function oscillating between 1 and -1 as shown in [2, pp. 40-]. Good to remember: $\cos^2 \alpha + \sin^2 \alpha = 1$.

Stokes' Theorem This theorem, first derived by Kelvin and first published by someone else I cannot recall, says that for any reasonably smoothly varying vector $\vec{v}$,

$$\int_A (\nabla \times \vec{v}) \, \mathrm{d}A = \oint \vec{v} \cdot \mathrm{d}\vec{r}$$

where the first integral is over any smooth surface area A and the second integral is over the edge of that surface. How did Stokes get his name on it? He tortured his students with it, that's how!

symmetry Symmetries are operations under which an object does not change. For example, a human face is almost, but not completely, mirror symmetric: it looks almost the same in a mirror as when seen directly. The electrical field of a single point charge is spherically symmetric; it looks the same from whatever angle you look at it, just like a sphere does. A simple smooth glass (like a glass of water) is cylindrically symmetric; it looks the same whatever way you rotate it around its vertical axis.

t May indicate:

- Time.

triple product A product of three vectors. There are two different versions:

- The scalar triple product $\vec{a} \cdot (\vec{b} \times \vec{c})$. In index notation,

$$\vec{a} \cdot (\vec{b} \times \vec{c}) = \sum_i a_i (b_{\overline{\imath}} c_{\overline{\overline{\imath}}} - b_{\overline{\overline{\imath}}} c_{\overline{\imath}})$$

where $\overline{\imath}$ is the index following i in the sequence 123123..., and $\overline{\overline{\imath}}$ the one preceding it. This triple product equals the determinant $|\vec{a}\vec{b}\vec{c}|$ formed with the three vectors. Geometrically, it is plus or minus the volume of the parallelepiped that has vectors $\vec{a}$, $\vec{b}$, and $\vec{c}$ as edges.

Either way, as long as the vectors are normal vectors and not operators,

$$\vec{a} \cdot (\vec{b} \times \vec{c}) = \vec{b} \cdot (\vec{c} \times \vec{a}) = \vec{c} \cdot (\vec{a} \times \vec{b})$$

and you can change the two sides of the dot product without changing the triple product, and/or you can change the sides in the vectorial product with a change of sign.

- The vectorial triple product $\vec{a} \times (\vec{b} \times \vec{c})$. In index notation, component number i of this triple product is

$$a_{\bar{i}}(b_i c_{\bar{i}} - b_{\bar{i}} c_i) - a_{\bar{\bar{i}}}(b_{\bar{\bar{i}}} c_i - b_i c_{\bar{\bar{i}}})$$

which may be rewritten as

$$a_i b_i c_i + a_{\bar{i}} b_i c_{\bar{i}} + a_{\bar{\bar{i}}} b_i c_{\bar{\bar{i}}} - a_i b_i c_i - a_{\bar{i}} b_{\bar{i}} c_i - a_{\bar{\bar{i}}} b_{\bar{\bar{i}}} c_i$$

In particular, as long as the vectors are normal ones,

$$\vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a} \cdot \vec{c})\vec{b} - (\vec{a} \cdot \vec{b})\vec{c}$$

u May indicate:

- The first velocity component in a Cartesian coordinate system.
- An integration variable.

V May indicate:

- Volume.

v May indicate:

- The second velocity component in a Cartesian coordinate system.
- Magnitude of a velocity (speed).

$\vec{v}$ May indicate:

- Velocity vector.
- Generic vector.

vector A quantity characterized by a list of numbers. A vector $\vec{v}$ in index notation is a set of numbers $\{v_i\}$ indexed by an index i. In normal three-dimensional Cartesian space, i takes the values 1, 2, and 3, making the vector a list of three numbers, v_1, v_2, and v_3. These numbers are called the three components of $\vec{v}$.

vectorial product An vectorial product, or cross product is a product of vectors that produces another vector. If

$$\vec{c} = \vec{a} \times \vec{b},$$

it means in index notation that the i-th component of vector $\vec{c}$ is

$$c_i = a_{\bar{\imath}} b_{\bar{\bar{\imath}}} - a_{\bar{\bar{\imath}}} b_{\bar{\imath}}$$

where $\bar{\imath}$ is the index following i in the sequence $123123\ldots$, and $\bar{\bar{\imath}}$ the one preceding it. For example, c_1 will equal $a_2 b_3 - a_3 b_2$.

w May indicate:

- The third velocity component in a Cartesian coordinate system.
- Weight factor.

$\vec{w}$ Generic vector.

X Used in this book to indicate a function of x to be determined.

x May indicate:

- First coordinate in a Cartesian coordinate system.
- A generic argument of a function.
- An unknown value.

Y Used in this book to indicate a function of y to be determined.

y May indicate:

- Second coordinate in a Cartesian coordinate system.
- A second generic argument of a function.
- A second unknown value.

Z Used in this book to indicate a function of z to be determined.

z May indicate:

- Third coordinate in a Cartesian coordinate system.
- A third generic argument of a function.
- A third unknown value.

Index

65