hi 5060 $\vec{x}' = A\vec{x}$ A constant

A not defective:

$$\vec{x} = C_1 \vec{e}_1 e^{\lambda_1 t} + C_2 \vec{e}_2 e^{\lambda_2 t} + \cdots$$

If λ complex conjugate pair

$\lambda_1 = \lambda_R + i\mu$ $\vec{e}_1 = \vec{u} + i\vec{v}$

λ_2 $\vec{e}_2$

Must clean up complex roots

$$\vec{x} = e^{\lambda_R t}\left[c_1(\vec{u}+i\vec{v})(\cos\mu t + i\sin\mu t)\right.$$

$$\left. + c_2(\vec{u}-i\vec{v})(\cos\mu t - i\sin\mu t)\right]$$

$$= e^{\lambda_R t}\Big[\Big\{\underbrace{(c_1+c_2)}_{D_1}\vec{u} + \underbrace{i(c_1-c_2)}_{D_2}\vec{v}\Big\}\cos\mu t$$

$$\Big\{\underbrace{i(c_1-c_2)}_{D_2}\vec{u} - \underbrace{(c_1+c_2)}_{D_1}\vec{v}\Big\}\sin\mu t\Big]$$

$$= \boxed{D_1 e^{\lambda_R t}(\vec{u}\cos\mu t - \vec{v}\sin\mu t) + D_2 e^{\lambda_R t}(\vec{u}\sin\mu t + \vec{v}\cos\mu t)}$$

Example: $\vec{x}\,' = \begin{pmatrix} 2 & -4 \\ 1 & 2 \end{pmatrix} \vec{x}$

$$\begin{vmatrix} 2-\lambda & -4 \\ 1 & 2-\lambda \end{vmatrix} = (2-\lambda)^2 + 4 = 0$$

$$\lambda - 2 = \pm\sqrt{-4} = \pm 2i \qquad \lambda = 2 \pm 2i$$

$$\lambda_1 = 2 + 2i \qquad A - \lambda_1 I = \begin{pmatrix} -2i & -4 \\ \cancel{1}\ 0 & \cancel{-2i}\ 0 \end{pmatrix} \begin{matrix} 1 \\ 2i \end{matrix}$$

$\lambda_r \quad \mu$

$$\begin{pmatrix} e_1 \\ e_2 \end{pmatrix} = \begin{pmatrix} -2/i \\ 1 \end{pmatrix} e_2 = \begin{pmatrix} 2i \\ 1 \end{pmatrix} e_2 \qquad \text{take } e_2 = 1$$

$$\vec{e}_1 = \begin{pmatrix} 2i \\ 1 \end{pmatrix} = \underbrace{\begin{pmatrix} 0 \\ 1 \end{pmatrix}}_{\vec{u}} + i \underbrace{\begin{pmatrix} 2 \\ 0 \end{pmatrix}}_{\vec{v}}$$

$$\vec{x} = D_1 e^{2t}\left[\begin{pmatrix}0\\1\end{pmatrix}\cos 2t - \begin{pmatrix}2\\0\end{pmatrix}\sin 2t\right] + D_2 e^{2t}\left[\begin{pmatrix}0\\1\end{pmatrix}\sin 2t + \begin{pmatrix}2\\0\end{pmatrix}\cos 2t\right]$$

$$\Omega = e^{2t}\begin{pmatrix} -2\sin 2t & 2\cos 2t \\ \cos 2t & \sin 2t \end{pmatrix}$$

Case of defective A

As a further solution, use

$$(t\vec{e}+\vec{f})e^{\lambda t} \qquad \text{P.I.I}$$

$$\vec{x}' = A\vec{x}$$

$$\vec{e}\,\cancel{e^{\lambda t}} + (t\vec{e}+\vec{f})\lambda \cancel{e^{\lambda t}} = tA\vec{e}\,\cancel{e^{\lambda t}} + A\vec{f}\cancel{e^{\lambda t}}$$

$$\boxed{(A-\lambda I)\vec{f} = \vec{e}} \longrightarrow$$

If missing two eigenvectors for λ, also use

$$\boxed{\begin{aligned}&\left(\tfrac{1}{2}t^2\vec{e} + t\vec{f} + \vec{g}\right)e^{\lambda t}\\ &(A-\lambda I)\vec{g} = \vec{f}\end{aligned}}$$

where

Example $\vec{x}\,' = A\vec{x}$

$$A = \begin{pmatrix} 2 & 5 & 6 \\ 0 & 8 & 9 \\ 0 & -1 & 2 \end{pmatrix} \quad |A - \lambda I| = \begin{vmatrix} 2-\lambda & 5 & 6 \\ 0 & 8-\lambda & 9 \\ 0 & -1 & 2-\lambda \end{vmatrix}$$

$$(2-\lambda)\left[(8-\lambda)(2-\lambda)+9\right]$$

$$\lambda^2 - 10\lambda + 25 \longrightarrow \lambda_2 = \lambda_3 = 5$$

$$(2-\lambda)(\lambda-5)^2 = 0$$

$$\lambda_1 = 2 \quad \lambda_2 = \lambda_3 = 5$$

λ_1:

$$\begin{pmatrix} 0 & 5 & 6 \\ 0 & 6 & 9 \\ 0 & -1 & 0 \end{pmatrix} \to \begin{pmatrix} 0 & -1 & 0 \\ 0 & 6 & 9 \\ 0 & 5 & 6 \end{pmatrix} \xrightarrow{6,\ -5} \begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & 9 \\ 0 & 0 & 6 \end{pmatrix}$$

$$\begin{pmatrix} 0 & -1 & 9 \\ 0 & 0 & 9 \\ 0 & 0 & 0 \end{pmatrix} \begin{matrix} \longrightarrow e_2 = 0 \\ \rightarrow e_3 = 0 \end{matrix} \qquad \vec{e}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$$c_1 \vec{e}_1 e^{2t}$$

$$\lambda_2 = \lambda_3 = 5$$

$$\begin{pmatrix} -3 & 5 & 6 \\ 0 & 3 & 9 \\ 0 & -1 & -3 \end{pmatrix} \tfrac{1}{3} \qquad \begin{pmatrix} -3 & 5 & 6 \\ 0 & 3 & 9 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\cancel{e} = -3e_1 - 15e_3 + 6e_3 = 0$$

$$e_2 = -3e_3$$

$$e_1 = -3e_3$$

$$\vec{e} = \begin{pmatrix} -3 \\ -3 \\ 1 \end{pmatrix} e_3 \Longrightarrow \vec{e}_2 = \begin{pmatrix} -3 \\ -3 \\ 1 \end{pmatrix}$$

$$c_2 \vec{e}_2 e^{5t}$$

$$\underline{(A-5I)\vec{l}=\vec{e}}$$

$$\left(\begin{array}{ccc|c} -3 & 5 & 6 & -3 \\ 0 & 3 & 9 & -3 \\ 0 & \cancel{-1} & \cancel{-3} & \cancel{1} \end{array}\right) \quad \frac{1}{3}$$

$$0 \quad 0 \quad 0$$

$$l_2 = -1 - 3l_3$$

$$-3l_1 - 5 - 15l_3 + 6l_3 = -3$$

$$l_1 = -\frac{5}{3} - 3l_3$$

Solution space

$$\begin{pmatrix} l_1 \\ l_2 \\ l_3 \end{pmatrix} = \begin{pmatrix} -3 \\ -3 \\ 1 \end{pmatrix} l_3 + \begin{pmatrix} -\frac{5}{3} \\ -1 \\ 0 \end{pmatrix}$$

take $l_3 = 0$ $\quad \vec{l} = \begin{pmatrix} -\frac{2}{3} \\ -1 \\ 0 \end{pmatrix}$

$$(A-\lambda_2 I)\vec{l} = \begin{pmatrix} 2-5 & \\ & \end{pmatrix}\begin{pmatrix} -3 \\ \\ \end{pmatrix}$$

$$\vec{x} = C_1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} e^{2t} + C_2 \begin{pmatrix} -3 \\ -3 \\ 1 \end{pmatrix} e^{5t}$$

$$+ C_3 \underbrace{\begin{pmatrix} -3t - \frac{2}{3} \\ -3t - 1 \\ t \end{pmatrix}}_{} e^{5t}$$

redefine $C_3 = 3C_3$