hi 5060) Systems of ODE

$$\begin{aligned} y_1'' &= -8y_1 + 2y_2 + l_1 \\ y_2'' &= 2y_1 - 5y_2 \end{aligned} \qquad \begin{cases} x_1 = y_1 \\ x_2 = y_1' \\ x_3 = y_2 \\ x_4 = y_2' \end{cases}$$

$$\frac{dx_1}{dt} = x_2, \quad \frac{dx_2}{dt}, \quad \frac{dx_3}{dt} = x_4, \quad \frac{dx_4}{dt}$$

$$y'' + by' + cy = f \qquad \begin{cases} y_1 = y \\ y_2 = y' \end{cases}$$

$$\begin{cases} \dfrac{dy_1}{dx} = y_2 \\ \dfrac{dy_2}{dx} + by_2 + cy_1 = f. \end{cases}$$

~~Linear first order systems~~

General form of a 1st order system

$$y_1' = f_1(y_1, y_2, y_3 \ldots, y_n, t)$$

$$y_2' = f_2(y_1, y_2, y_3 \ldots, y_n, t)$$

$$\vdots$$

$$y_n' = f_n(y_1, y_2, y_3 \ldots, y_n, t)$$

Linear 1st order system

$$y_1' = a_{11} y_1 + a_{12} y_2 + \ldots + a_{1n} y_n + g_1$$

$$y_2' = a_{21} y_1 + a_{22} y_2 + \ldots + a_{2n} y_n + g_2$$

$$a_{11}, a_{12} \ldots g_1, g_2 \ldots = a_{11}, a_{22} \ldots g_1, g_2 (t)$$

vector form

$\cancel{y' = 3y}$

$\cancel{y'' + y = 1}$

$$\underbrace{\frac{d\vec{y}}{dt} = A(t)\vec{y}}_{\text{homo}} + \underbrace{\vec{g}(t)}_{\text{inhomo term}}$$

$$\vec{y} = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix} \quad A(t) = \begin{pmatrix} a_{11}(t) & \cdots & \\ \vdots & \ddots & a_{nn}(t) \end{pmatrix} \quad \vec{g} = \begin{pmatrix} g_1(t) \\ g_2(t) \\ \vdots \\ g_n(t) \end{pmatrix}$$

Linear: if $\vec{y}_1$ is a solution

and $\vec{y}_2$ is a solution,

then so is $C_1\vec{y}_1 + C_2\vec{y}_2$

General solution of the homogeneous equation

$\vec{y}_h = C_1 \vec{y}_1 + C_2 \vec{y}_2 + \dots + C_n \vec{y}_n$

where $\vec{y}_1, \vec{y}_2, \dots \vec{y}_n$ are linearly independent. (Need to check only one time.)

Example

$$x_1' = 5x_1 + 3x_2 \qquad x_1(t)$$

$$x_2' = x_1 + 3x_2 \qquad x_2(t)$$

Solution $\vec{x} = c_1\begin{pmatrix} -e^{2t} \\ e^{2t} \end{pmatrix} + c_2\begin{pmatrix} 3e^{6t} \\ e^{6t} \end{pmatrix}$

$$\vec{x} = \underbrace{\begin{pmatrix} -e^{2t} & 3e^{6t} \\ e^{2t} & e^{6t} \end{pmatrix}}_{\Omega(t)} \underbrace{\begin{pmatrix} c_1 \\ c_2 \end{pmatrix}}_{\text{constant vector}}$$

$\Omega(t)$ a fundamental matrix

I.C. $\vec{x}(0) = \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} -1 & 3 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$

given

Solution when A is constant

Solution when A is nondefective

$$\frac{d\vec{y}}{dt} = A\vec{y} \; : \text{guess } \vec{C}e^{\lambda t} = \vec{y}$$

$$\text{P.I.T} \quad \lambda\vec{C}\cancel{e^{\lambda t}} = A\vec{C}\cancel{e^{\lambda t}}$$

$$A\vec{C} = \lambda\vec{C}$$

$\vec{C}$ must be an eigenvector of A and λ the corresponding eigenvalue

General solution of the homo equation

$$\vec{y}_h = C_1 \vec{e}_1 e^{\lambda_1 t} + C_2 \vec{e}_2 e^{\lambda_2 t} + \dots + C_n \vec{e}_n e^{\lambda_n t}$$

$\vec{e}_1, \vec{e}_2, \dots, \vec{e}_n$ independent eigenvectors

Example

$$\begin{pmatrix} y_1' \\ y_2' \end{pmatrix} = \underbrace{\begin{pmatrix} 3 & 0 \\ 5 & -4 \end{pmatrix}}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$$

$\lambda_1 = 3 \qquad \lambda_2 = -4$

$$A - \lambda_1 I = \begin{pmatrix} 0 & 0 \\ 5 & -7 \end{pmatrix} \qquad \vec{e}_1 = \begin{pmatrix} 7 \\ 5 \end{pmatrix}$$

$e_1 = \frac{7}{5} e_2$

$$A - \lambda_2 I = \begin{pmatrix} 7 & 0 \\ 5 & 0 \end{pmatrix}$$

$e_x = \frac{7}{5} e_y$

$$\vec{e}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\vec{y}_h = c_1 \begin{pmatrix} 7 \\ 5 \end{pmatrix} e^{3t} + c_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{-4t}$$

$$= \underbrace{\begin{pmatrix} 7e^{3t} & 0 \\ 5e^{3t} & e^{-4t} \end{pmatrix}}_{\Omega(t)} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$$

Clean up complex eigenvalues

Assume, say that $\lambda_1 = \lambda_r + i\mu$ $\quad i = \sqrt{-1}$

$\lambda_2 = \lambda_r - i\mu$

smart: take μ to be >0

(by taking what is λ_1)

So take $\lambda_1 = \lambda_r + i\mu$ with $\mu > 0$

$\rightarrow \vec{e}_1 = \vec{u} + i\vec{v}$ $\quad \vec{u}, \vec{v}$ real

Do not ~~compute~~ write down $\vec{e}_2, \lambda_2$

$$y_h = c_1(\vec{u} + i\vec{v})\, e^{\lambda_r} e^{i\mu t} + c_2(\vec{u} - i\vec{v})\, e^{\lambda_r} e^{-i\mu t}$$

$$= e^{\lambda_r t}\Big[c_1(\vec{u} + i\vec{v})(\cos(\mu t) + i\sin(\mu t)) + c_2(\vec{u} - i\vec{v})(\cos\mu t - i\sin\mu t)\Big]$$

$$= e^{\lambda_r t}\Big[\; c_1$$