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Linear n-th order equation

$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}} + \dots + a_1(x)\frac{dy}{dx} + a_0(x)y = g(x)$$

homogeneous part

inhomogeneous terms

$\rightarrow$ n integration constants

General solution

$$y = \underbrace{C_1 y_1(x) + C_2 y_2(x) + \dots + C_n y_n(x)}_{\text{solution of homogeneous equation}} + y_p(x)$$

particular solution

Constant coefficient equations for homogeneous eq

$a_n, a_{n-1}, \ldots, a_1, a_0$ constants

Try $y = e^{\lambda x}$ P.I.I. $y' = \lambda e^{\lambda x}$

$$\Rightarrow a_n \lambda^n \cancel{e^{\lambda x}} + a_{n-1} \lambda^{n-1} \cancel{e^{\lambda x}} + \cdots + a_1 \lambda \cancel{e^{\lambda x}} + a_0 \cancel{e^{\lambda x}} = 0$$

"characteristic equation"

$\rightarrow$ n roots

If all roots are different then

$$y = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x} \ldots + c_n e^{\lambda_n x}$$

If a double root λ, add a ~~function~~ function $x e^{\lambda x}$

If a triple root λ, also add a function $x^2 e^{\lambda x}$

Complex ~~eigen~~ roots must be cleared up.

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Example $y'' + y' - 6y = 0$ ODE

$y(0) = 3 \quad y'(0) = 1$ I.C.

solution $e^{\lambda x}$: $\lambda^2 + \lambda - 6 = 0$

$$\lambda_{1,2} = \frac{-1 \pm \sqrt{1+24}}{2} = \frac{-1 \pm 5}{2} = \begin{cases} -3 & \lambda_2 \\ 2 & \lambda_1 \end{cases}$$

$\Rightarrow$ general solution:

$$y = C_1 e^{2x} + C_2 e^{-3x} \qquad y' = 2C_1 e^{2x} - 3C_2 e^{-3x}$$

Plug in initial conditions

$$y(0) = C_1 1 + C_2 1 = 3$$

$$y'(0) = 2C_1 - 3C_2 = 1$$

$$\left(\begin{array}{cc|c} 1 & 1 & 3 \\ 2 & -3 & 1 \end{array}\right) \xrightarrow{-2} \left(\begin{array}{cc|c} 1 & 1 & 3 \\ 0 & -5 & -5 \end{array}\right)$$

$$\rightarrow C_2 = 1 \rightarrow C_1 = 2$$

$$y = 2e^{2x} + e^{-3x}$$

Example

$$y''' - 2y'' + y' = 0$$

$$\lambda^3 - 2\lambda^2 + \lambda = 0$$

$$\lambda(\lambda^2 - 2\lambda + 1) = 0$$

$$\lambda(\lambda - 1)^2 = 0 \qquad \lambda_1 = 0 \qquad \underline{\lambda_2 = \lambda_3 = 1}$$

$$y = C_1 e^{0x} + C_2 e^{1x} + C_3 x e^{1x} = C_1 + C_2 e^x + C_3 x e^x$$

$$y = C_1 + C_2 e^x + C_3 x e^x$$

$$y(1) = y'(1) = 0 \qquad y''(1) = e$$

$y(1)$: $\quad C_1 + C_2 e + C_3 e = 0$

$$y' = C_2 e^x + C_3 (x e^x + e^x)$$

$$y'' = C_2 e^x + C_3 (x e^x + 2e^x)$$

$y(1)$: $\quad C_1 + C_2 e + C_3 e = 0$

$y'(1)$: $\quad C_2 e + C_3 2e = 0$

$y''(1)$: $\quad C_2 e + C_3 3e = e$

$$\left(\begin{array}{ccc|c} 1 & e & e & 0 \\ 0 & e & 2e & 0 \\ 0 & e & 3e & e \end{array}\right) \rightarrow \left(\begin{array}{ccc|c} 1 & e & e & 0 \\ 0 & e & 2e & 0 \\ 0 & 0 & e & e \end{array}\right)$$

$C_3 = 1$

$C_2 = -2$

$C_1 = e$

$$y = e - 2e^x + x e^x$$

Example: $y'' + 4y' + 9y = 0$ $\quad i = \sqrt{-1}$

Solve: $\lambda^2 + 4\lambda + 9 = 0$

$$\lambda_{1,2} = \frac{-4 \pm \sqrt{16-36}}{2} = -2 \pm \sqrt{-5}$$

$$= -2 \pm i\sqrt{5} \qquad i = \sqrt{-1}$$

$$y = C_1 e^{(-2+i\sqrt{5})x} + C_2 e^{(-2-i\sqrt{5})x}$$

must be cleaned up

$$\boxed{\text{Euler}: \; e^{i\alpha} = \cos(\alpha) + i\sin(\alpha)}$$

$$e^{-i\alpha} = \cos(\alpha) - i\sin(\alpha)$$

$$e^{a+b} = e^a e^b \qquad a = -2x \qquad b = \pm i\sqrt{5}x$$

$$y = e^{-2x}\left[C_1 e^{i\sqrt{5}x} + C_2 e^{-i\sqrt{5}x}\right]$$

Euler

$$y = e^{-2x}\left[C_1\left\{\cos(\sqrt{5}x) + i\sin(\sqrt{5}x)\right\} + C_2\left\{\cos(\sqrt{5}x) - i\sin(\sqrt{5}x)\right\}\right]$$

$$= e^{-2x}\left[\underbrace{(C_1 + C_2)}_{\text{call it } D_1}\cos(\sqrt{5}x) + \underbrace{i(C_1 - C_2)}_{\text{call it } D_2}\sin(\sqrt{5}x)\right]$$

$$y = e^{-2x}\left[D_1\cos(\sqrt{5}x) + D_2\sin(\sqrt{5}x)\right]$$