Ris060: Ordinary differential equations:

only a single independent variable

→ function of a single variable

involve derivatives

Order: order of the highest derivative

Order 1 $\quad x^2 y' + y^3 = x^{-1} \quad \left(y' = \frac{dy}{dx}\right)$

order 2 $\quad y'' + y = 0 \quad \left(y'' = \frac{d^2y}{dx^2}\right)$

$\boxed{\text{First order ODE}}$

$$\frac{dy}{dx} = f(x, y).$$ (when solved for dy/dx)

Separable equations

$$\frac{dy}{dx} = f(x)g(y)$$

solve:

$$\int \frac{dy}{g(y)} = \int f(x)dx$$

Required

→ integration constant

(1st order: 1, second order: 2, ...)

$y' + y = 2$

$y' = 2 - y = \frac{dy}{dx}$

$-\frac{d(y-2)}{y-2}$

$\int \frac{dy}{2-y} = \int dx \to -\ln|y-2| = x + C$

$e^{LHS} = e^{RHS}$

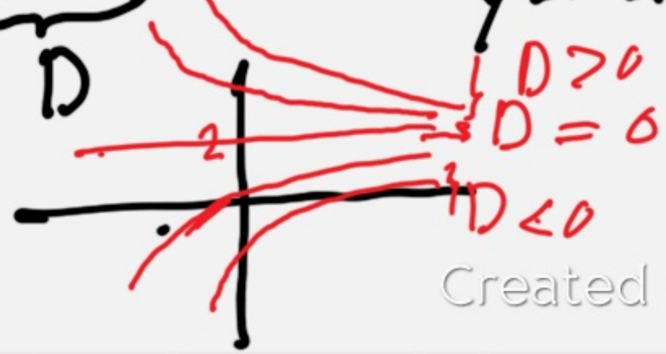
$\ln|y-2| = -x - C$

$e^{a+b} = e^a e^b$

$|y-2| = e^{-x-c} = e^{-c} e^{-x}$

$y - 2 = \underbrace{\pm e^{-c}}_{D} e^{-x}$

$y = 2 + De^{-x}$



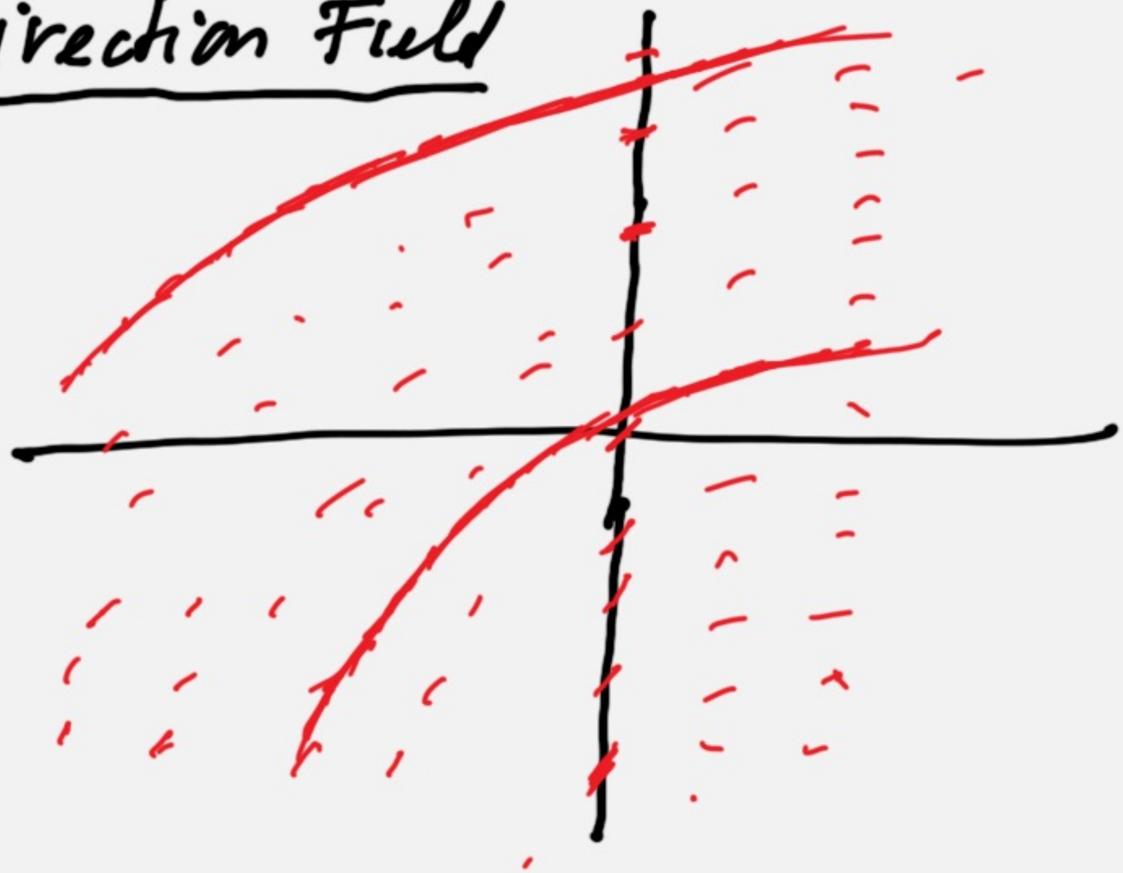
To find D, must have an "initial condition" e.g. $y=1$ at $x=0$

$y = 2 + De^{-x}$ $\quad y(0) = 1 = 2 + D\cancel{e^{-0}}$

$\Rightarrow D = -1$ $\quad y = 2 - e^{-x}$

Direction Field

$$\frac{dy}{dx} = f(x,y)$$



in y

Linear equations

$$\underbrace{\frac{dy}{dx} + p(x)\,y}_{\substack{\text{linear in } y \\ \text{homogeneous} \\ \text{part}}} = \underbrace{q(x)}_{\substack{\text{independent of } y \\ \text{inhomogeneous} \\ \text{part}}}$$

Required procedure

integration constant

1) Solve the homogeneous equation
2) Use "variation of parameter" to solve the full (inhomogeneous) equation.

Example

$$\underbrace{y' + \sec(x) y} = \cos(x)$$

$$e^a e^b = e^{a+b}$$

$$-\ln a = \ln \frac{1}{a}$$

1) homogeneous:

$$y_h' + \sec(x)\, y_h = 0$$

$$y_h' = -\sec(x)\, y_h$$

$$\frac{dy_h}{y_h} = -\sec(x)\,dx = -\frac{dx}{\cos(x)} \qquad \frac{1+\sin x}{\cos(x)}$$

$$\ln|y_h| = -\ln\left|\frac{1}{\cos(x)} + \tan(x)\right| + C$$

$$e^{LHS} = e^{RHS}$$

$$|y_h| = -e^C \left|\frac{\cos x}{1+\sin x}\right|$$

$$y_h = D\,\frac{\cos x}{1+\sin x}$$

2) $y_h = D \dfrac{\cos x}{1+\sin x}$ $\qquad y = E(x)\dfrac{\cos x}{1+\sin x}$

P.I.T Plug it in

$$y' + \frac{1}{\cos x}\, y = \cos x$$

$$y' = E' \frac{\cos x}{1+\sin x} + E \frac{-\sin x - \sin^2 x - \cos^2 x}{(1+\sin x)^2}$$

$$E' \frac{\cos x}{1+\sin x} \cancel{+ E \frac{-\sin x - \sin^2 x - \cos^2 x}{(1+\sin x)^2}}$$

$$\cancel{+ E \frac{\cos x}{1+\sin x} \frac{1}{\cos x}} = \cos x$$

$$E' \frac{\cos x}{1+\sin x} = \cos x \qquad E' = 1+\sin x$$

$$E = \int 1+\sin x \, dx =$$

$$E = x - \cos x + E_0$$

$$y = E \frac{\cos x}{1+\sin x}$$

$$= \underbrace{\frac{x\cos x - \cos^2 x}{1+\sin x}}_{\text{particular solution}} + \underbrace{E_0 \frac{\cos x}{1+\sin x}}_{\text{homogeneous solution}}$$