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$$\boxed{\begin{array}{l} A\vec{e} = \lambda\vec{e} \quad \text{with } \vec{e} \neq 0, \\ \text{then } \vec{e} \text{ is an eigenvector of } A \\ \text{and } \lambda \text{ is the corresponding} \\ \text{eigenvalue} \end{array}}$$

$$(A - \lambda I)\vec{e} = 0 \Rightarrow |A - \lambda I| = 0$$

Example: $A = \begin{pmatrix} 1 & 3 \\ 2 & 1 \end{pmatrix}$ $|A - \lambda I| = \begin{vmatrix} 1-\lambda & 3 \\ 2 & 1-\lambda \end{vmatrix}$

$(1-\lambda)^2 = 6$ $\lambda_1 = 1 + \sqrt{6}$ $\lambda_2 = 1 - \sqrt{6}$

$(A - \lambda_1 I)\vec{e} = 0$ $\begin{pmatrix} -\sqrt{6} & 3 \\ 2 & -\sqrt{6} \end{pmatrix}$ 2 $\sqrt{6}$

$\begin{pmatrix} -\sqrt{6} & 3 \\ 0 & 0 \end{pmatrix}$ the last row must always be zero, or you have the eigenvalue wrong or did G.E. wrong

Find $\vec{e} = \begin{pmatrix} e_1 \\ e_2 \end{pmatrix}$ $-\sqrt{6}e_1 + 3e_2 = 0$

$e_1 = \frac{3}{\sqrt{6}} e_2$ take e_2 to be a suitable nonzero value giving a nice vector:

there: take $e_2 = \sqrt{6}$ $\vec{e} = \begin{pmatrix} 3 \\ \sqrt{6} \end{pmatrix}$

must take the undetermined coefficient U.C. in eigenvectors are illegal.

$$\cancel{\vec{e} = \begin{pmatrix} c3 \\ c\sqrt{6} \end{pmatrix}}$$

In general, write the null space of $(A-\lambda_i I)$ and then ~~take~~ create a basis of that nullspace.

here: $N.S = \begin{pmatrix} e_1 \\ e_2 \end{pmatrix} = \begin{pmatrix} \frac{3}{\sqrt{6}} \\ 1 \end{pmatrix} e_2$

$\rightarrow$ take $e_2 = \sqrt{6}$ $\longrightarrow$ $\vec{e}_{final} = \begin{pmatrix} 3 \\ \sqrt{6} \end{pmatrix} = \vec{e}_1$

For $\lambda_2 = 1-\sqrt{6}$ $\quad A-\lambda_2 I = \begin{pmatrix} 1-1+\sqrt{6} & 3 \\ 2 & 1-1+\sqrt{6} \end{pmatrix}$

$= \begin{pmatrix} \sqrt{6} & 3 \\ 2 & \sqrt{6} \end{pmatrix} \xrightarrow{-2/\sqrt{6}} \Rightarrow \begin{pmatrix} \sqrt{6} & 3 \\ 0 & 0 \end{pmatrix} \to e_1 = -\frac{3}{\sqrt{6}} e_2$

N.S. $\begin{pmatrix} e_1 \\ e_2 \end{pmatrix} = \begin{pmatrix} -3/\sqrt{6} \\ 1 \end{pmatrix} e_2$ take $e_2 = -\sqrt{6}$

then final eigenvector $\vec{e_2} = \begin{pmatrix} 3 \\ -\sqrt{6} \end{pmatrix}$

$\lambda_1 = 1+\sqrt{6}$ $\quad \vec{e_1} = \begin{pmatrix} 3 \\ \sqrt{6} \end{pmatrix}$ $\quad \lambda_2 = 1-\sqrt{6}$ $\quad \vec{e_2} = \begin{pmatrix} 3 \\ -\sqrt{6} \end{pmatrix}$

$$A = \begin{bmatrix} 5 & 1 & 0 & 9 \\ 0 & 1 & 0 & 9 \\ 0 & 0 & 0 & 9 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$|A - \lambda I| = \begin{vmatrix} 5-\lambda & 1 & 0 & 9 \\ 0 & 1-\lambda & 0 & 9 \\ 0 & 0 & -\lambda & 9 \\ 0 & 0 & 0 & -\lambda \end{vmatrix}$$

$$= (5-\lambda)(1-\lambda)(-\lambda)(-\lambda)$$

The eigenvalues of a triangular matrix are on the main diagonal

$\lambda_1 = 5 \quad A - \lambda_1 I =$

$$\begin{pmatrix} 0 & 1 & 0 & 9 \\ 0 & -4 & 0 & 9 \\ 0 & 0 & -5 & 9 \\ 0 & 0 & 0 & -5 \end{pmatrix} \; 4 \qquad \begin{pmatrix} 0 & 1 & 0 & 9 \\ 0 & 0 & 0 & 45 \\ 0 & 0 & -5 & 9 \\ 0 & 0 & 0 & -5 \end{pmatrix}$$

$$\begin{matrix} 5 & 1 & 0 & 9 \\ 0 & 1 & 0 & 9 \\ 0 & 0 & 0 & 9 \\ 0 & 0 & 0 & 0 \end{matrix}$$

$$\begin{pmatrix} 0 & 1 & 0 & 9 \\ 0 & 0 & -5 & 9 \\ 0 & 0 & 0 & 45 \\ 0 & 0 & 0 & -5 \end{pmatrix} \; \frac{1}{9} \qquad \begin{array}{cccc} l_1 & l_2 & l_3 & l_4 \end{array}$$

$$\begin{pmatrix} 0 & 1 & 0 & 9 \\ 0 & 0 & -5 & 9 \\ 0 & 0 & 0 & 45 \\ 0 & 0 & 0 & 0 \end{pmatrix} \rightarrow 45\, l_4 = 0$$

$\rightarrow l_4 = 0 \longrightarrow l_3 = 0 \Longrightarrow l_2 = 0$

$$\begin{pmatrix} l_1 \\ l_2 \\ l_3 \\ l_4 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} l_1$$

$\vec{e}_1 = \begin{pmatrix}1\\0\\0\\0\end{pmatrix}$ $\quad \lambda_2 = 1 \quad$ $(A - I) = \begin{array}{c} \begin{pmatrix}4&1&0&9\\0&0&0&9\\0&0&-1&9\\0&0&0&-1\end{pmatrix} \\ \begin{array}{cccc} e_1 & e_2 & e_3 & e_4 \end{array} \end{array}$

$$\Rightarrow \begin{pmatrix}4&1&0&9\\0&0&-1&9\\0&0&0&9\\0&0&0&-1\end{pmatrix} \quad 9 \quad \Rightarrow \begin{pmatrix}4&1&0&9\\0&0&-1&9\\0&0&0&9\\0&0&0&0\end{pmatrix}$$

$(3)^* \rightarrow e_4 = 0 \qquad (2^*) \; -1 e_3 + 9 e_4 = 0 \rightarrow e_3 = 0$

$(1) \; 4 e_1 + e_2 + 0 * 0 + 9 * 0 = 0 \qquad e_1 = -\frac{1}{4} e_2$

N.S $\begin{pmatrix} \ell_1 \\ \ell_2 \\ \ell_3 \\ \ell_4 \end{pmatrix} = \begin{pmatrix} -\frac{1}{4} \\ 1 \\ 0 \\ 0 \end{pmatrix} \ell_2$ take $\ell_2 = 4$

$\rightarrow \vec{\ell_2} = \begin{pmatrix} -1 \\ 4 \\ 0 \\ 0 \end{pmatrix}$

last eigenvalue $\lambda_3 = \lambda_4 = 0$

$\Rightarrow$ multiple (double) eigenvalue

always at least one independent eigenvector, but for a double eigenvalue you may have up to two independent eigenvectors, (and for triple, up to 3)