hi5060 Multiple integrals

Example:

Asked: centroid of the area of the 1st quadrant area bounded by $x^2 - 8y + 4 = 0$ and $x^2 = 4y$ and $x = 0$

Solution (2D)

$x_c = \int x\,dA / \int dA$

$y_c = \int y\,dA / \int dA$

3D → A → V

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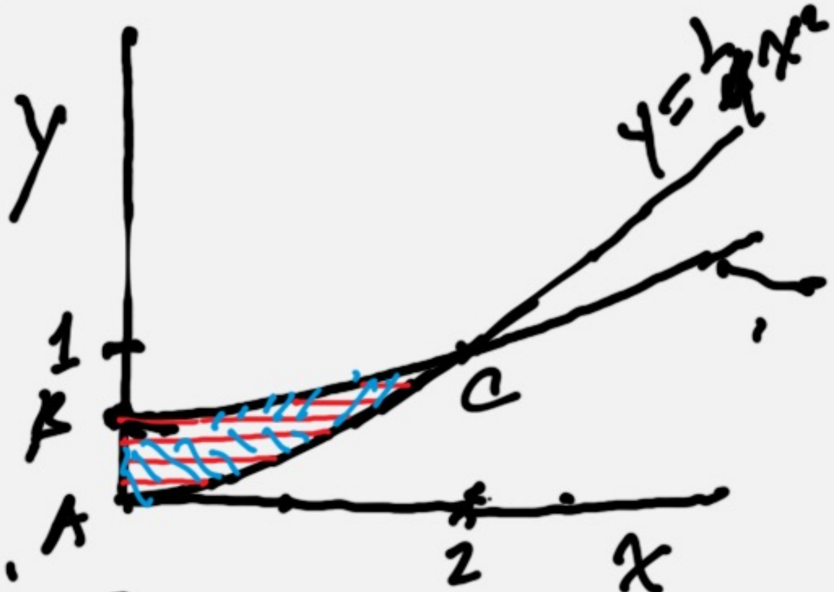

$y = \frac{1}{4}x^2 \rightarrow x = 2\sqrt{y}$

$x^2 = 4y$ $\quad y = \frac{1}{4}x^2$

$x^2 - 8y + 4 = 0$

$y = \frac{1}{2} + \frac{1}{8}x^2$

Big question: which first??

x or y??

$dA = dx\,dy$

Try x: $A = \int_0^{\frac{1}{2}}\left[\int_0^{2\sqrt{y}} 1\,dx\right]dy + \int_{\frac{1}{2}}^{1}\left[\int_{\sqrt{8y-4}}^{2\sqrt{y}} 1\,dx\right]dy$

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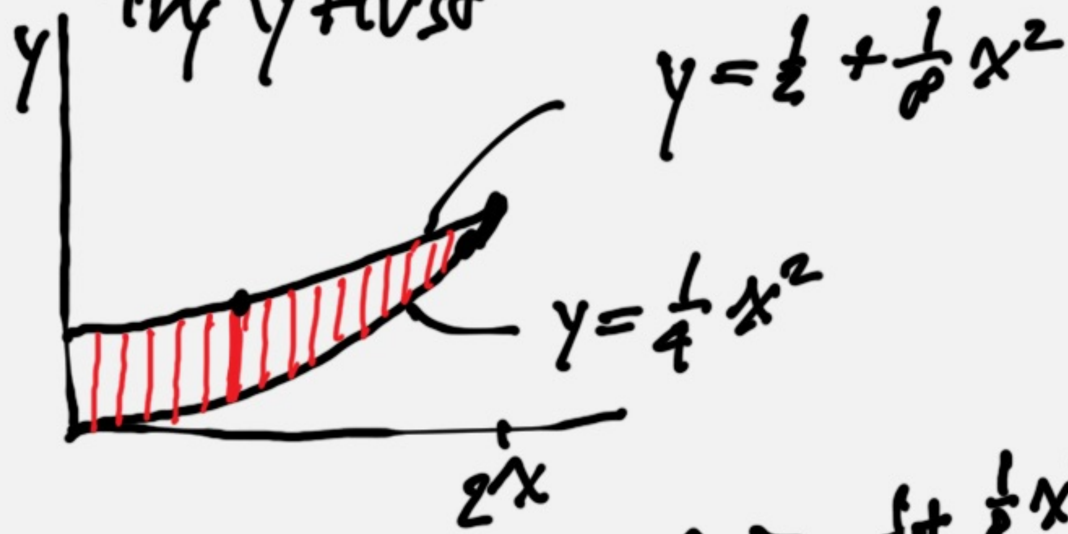
Try y first



$$A = \int_{reg} dA = \int_{reg} dx\,dy = \int_0^2 \left[\int_{\frac{1}{4}x^2}^{\frac{1}{2}+\frac{1}{8}x^2} 1\,dy \right] dx$$

$$= \int_0^2 \left[y \right]_{\frac{1}{4}x^2}^{\frac{1}{2}+\frac{1}{8}x^2} dx = \int_0^2 \left(\frac{1}{2} + \frac{1}{8}x^2 - \frac{1}{4}x^2\right) dx$$

$$= \frac{1}{2}x - \frac{1}{24}x^3 \Big|_0^2 = 1 - \frac{1}{3} = \frac{2}{3}$$

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$\int y\,dA$:

$$\int_{x=0}^{2}\left[\int_{y=\frac{1}{4}x^2}^{y=\frac{1}{8}x^2+\frac{1}{2}} y\,dy\right]dx$$

$$\int_0^2\left[\frac{1}{2}y^2\Big|_{\frac{1}{4}x^2}^{\frac{1}{8}x^2+\frac{1}{2}}\right]dx$$

$$\int_0^2 \frac{1}{2}\left(\frac{1}{8}x^2+\frac{1}{2}\right)^2-\frac{1}{2}\left(\frac{1}{4}x^2\right)^2 dx=\frac{4}{15}$$

$$y_c=\bar{y}=\int y\,dA\Big/\int dA=\frac{4}{15}\Big/\frac{2}{3}$$

$$=\frac{2}{5}$$

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In general

area: $dA = dx\,dy$ polar $dA = \rho\, d\rho\, d\theta$

Volumes: $dV = dx\,dy\,dz$ Cartesian

$dV = \rho\, d\rho\, d\theta\, dz$ Cylindrical

$dV = r^2 \sin\theta\, dr\, d\theta\, d\varphi$ spherical.

~~Mo~~ Moment of inertia

2D $I_x = \int y^2 dA$ $I_0 = \int x^2 + y^2\, dA$

3D $I_x = \int (y^2 + z^2) dV$ $I_{xy} = -\int xy\, dV$

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To do an integral $\iiint f(a,b,c)\,da\,db\,dc$ over a given area:

1) Draw the region to integrate over

2) Decide which one of a, b, c to integrate first: Usually do the one with the easiest limits first. E.g. if b has the easiest limits, do b first

$$\Rightarrow \int_{b_1(a,c)}^{b_2(a,c)} f(a,b,c)\,db$$

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After doing the integral, collapse the θ coordinate away (to say $\theta = 0$) → Redraw the region of integration without θ

Repeat

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