Analysis in Mechanical Engineering
EML 5060
Homework

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Contents

1 Calculus I

In this class,

- *Questions must be answered in order asked.*
- *Solutions must be neat.*
- *Solutions must be exact and fully simplified.*
- *You must use the given symbols.*
- *You must show all reasoning.*
- *Copying is never allowed, even when working together.*

1. Plot $y^3 = 6x^2 - x^3$ following class procedures[a b c]. [1, Maximum and Minimum Values, Curve Sketching]
2. Plot $y = x^4/(1 - x^2)$ following class procedures[a b]. [1, Maximum and Minimum Values, Curve Sketching]
3. Plot $y = \ln(e^{x+1} - 1)$ following class procedures[a b]. [1, Maximum and Minimum Values, Curve Sketching]

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[a] Include a plot

[b] Discuss intercepts, extents, symmetries, asymptotes, asymptotic behavior for $x \to \pm\infty$, local/global maxima/minima, concavity, inflection points, poles, cusps, corners, and other singularities

[c] Note that $\lim_{x\to\infty} = \lim_{u\downarrow 0}$ if $u = 1/x$. Alternatively, use the Taylor series approximation $(1+\varepsilon)^p \sim 1 + p\varepsilon$ for $\varepsilon \to 0$

2 Calculus II

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1. A company charges orders as follows
   - Orders of 50,000 items or less are charged at \$30 per thousand.
   - For orders above 50,000, the charge per thousand is reduced by $37\frac{1}{2}$¢ for each thousand above 50,000. (This reduced charge per thousand is applied to the *complete* order.)

   Plot the receipts of this company versus order size and analyze the graph for all features. What order size maximizes the receipts of the company? Give the reasons why the financial management of this company is clearly incompetent.[a]. [1, Maximum and Minimum Values]
2. Inside a conical tent of height h and radius of the base r, a "living space" is to be partioned in the shape of a circular cylinder with a flat top of radius R and height H. (a) Find the living space with the largest possible volume. (b) Find the living space with the largest curved surface.[b]. [1, Maximum and Minimum Values]
3. Find the MacLaurin series for $\sin(x^5)$. Hint: you may not want to crunch this out. Explain why not. Use a suitable trick instead. [1, Taylor and Maclaurin Series]
4. Write out the Taylor series for $\cos x$ around $\pi/3$ using the exact values of $\cos(\pi/3)$ and $\sin(\pi/3)$. Now assume that you approximate the Taylor series by its first three terms. What is the exact expression for the error in that approximation. How can you approximate this error at values of x close to $\pi/3$? Use the approximate error to find the distance r from $\pi/3$ so that the error is no more than 0.000 05 for all x for which $|x - \frac{1}{3}\pi| < r$. [1, Taylor and Maclaurin Series]
5. Find $\lim_{x\to 0^+} \frac{\ln \cot x}{e^{csc^2 x}}$. [1, L'Hôpital's Rule]
6. Find $\lim_{x\to\pi/2}(\sec^3 x - \tan^3 x)$. [1, L'Hôpital's Rule]
7. Variable z is given in terms of the measurable variables f and g as

   $$\frac{1}{z} = \frac{1}{f} + \frac{1}{g}$$

   The values of f and g and their uncertainties are:

   $$f = 4 \pm 0.01 \qquad g = 8 \pm 0.02$$

   What are the maximum relative and absolute errors in the computed z? Are you stunned by the value of the relative error in z? Explain why not. [1, Total Differential]

[a] Include a plot

[b] Include graphs of the functions being maximized and a picture of the tent and cylinder

3 Calculus III

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1. A particle moves in the first quadrant along the parabola $y^2 = 12x$. The x-component of velocity is $v_x = 15$. At the point (3,6), what are the velocity vector, including its magnitude and angle with the positive x-axis, and the acceleration vector, including its magnitude and angle with the positive x-axis?
2. Find I_x for the area between the curves

$$y = x \qquad y = 4x - x^2$$

   Exact answers only, please. Since the integrand y^2 does not depend on x, it would seem logical to integrate x first. Comment on that. [1, Centroids and Moments of Inertia]
3. Find the volume of the region bounded by

$$z = 0 \qquad x^2 + y^2 = 4x \qquad x^2 + y^2 = 4z$$

   Use cylindrical coordinates r, θ (or ϕ if you want), and z around the z-axis. What variable is obviously the one to integrate first? For the second integration, discuss each possibility and explain which is the best choice. Use pictures to make your points.[1, Triple Integrals]
4. Try to do the previous question using Cartesian coordinates x, y and z instead of cylindrical ones. Work it out at least as far as a single-variable integral, and find the relevant parts in the Math handbook to find its anti-derivative. Use pictures to make your points.

4 Linear Algebra I

In this class,

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1. Show the vectors

$$\vec F = \sqrt{2}\hat\imath + \hat\jmath - 6\hat k \qquad \vec G = 8\hat\imath + 2\hat k$$

as neatly and accurately as possible in a 3D left-handed coordinate system with the x-axis to the right and the z axis upwards in the plane of the paper. Now, using a different color, construct graphically, as accurately as possible, $2\vec F$, $3\vec G$, $\vec F+\vec G$, $\vec F-\vec G$ and also demonstrate them graphically in a neat plot, as accurately as possible. Compute these quantities also algebraically. Does that look about right in the graph? Compute $||\vec F||$ and $||\vec G||$. After evaluating the results in your calculator, do they agree with what you measure in the graph (ignoring that $\vec F$ is sticking a bit out of the paper)? Evaluate the angle between vectors $\vec F$ and $\vec G$ using a dot product. Does that angle seem about right? How would the above work out for the vectors

$$\vec v = (\sqrt{2}, 1, -6) \qquad \vec w = (8, 0, 2)$$

Double-check you are using the correct vectors while doing all the above. Make sure you have answered everything. As always, give *exact, cleaned up*, answers. As always, explain all reasoning! [2, 7.1-2].

2. A flat mirror passes through the points

$$\vec r_A = (-2, 1, 6) \quad \vec r_B = (2, 1, -7) \quad \vec r_C = (4, 2, 1)$$

Find a vector $\vec N$, with integer components, that is normal to the plane of the mirror using the appropriate vector combinations and product. Find a simple scalar equation for the plane of the mirror. Check that A, B, and C satisfy it. Hints: $\vec r_B - \vec r_A$ and $\vec r_C - \vec r_a$ are two vectors *in* the plane, and you need a vector *normal* to the plane. How would you get it? Double-check you are using the correct vectors while doing all the above. Make sure you have answered everything. [2, 7.1-2,5 ex. 9].

3. Continuing the previous question, assume that a laser beam moves in the positive $\vec e = \hat\imath$ direction before it is reflected by the mirror. Find the unit vector $\vec e\,'$ in the direction of the reflected beam. Make a picture of the reflection. Hints: Snell's law of reflection may be formulated as follows: reflection leaves the component, [2, 7.3, ex. 6-7], of $\vec e$ in the direction parallel to the mirror the same. However, it inverts the component normal to the mirror. So to get $\vec e\,'$, first consider the expression for the component of $\vec e$ in the direction normal to the mirror. You

know that if $\vec{n}$ is a unit vector normal to the mirror, then this component is $\vec{e}\cdot\vec{n}$, [2, 7.3, ex. 6]. The *vector* component is the scalar one multiplied by the unit vector $\vec{n}$, [2, 7.3, ex. 7]. To invert this vector component subtract it *twice* from $\vec{e}$. Once to zero the component and a second time to add the negative component. Therefore you see that:

$$\vec{e}\,' = \vec{e} - 2(\vec{e}\cdot\vec{n})\vec{n}$$

Do not compute $\vec{n}$ explicitly, as this would bring in a nasty square root. Just substitute $\vec{n} = \vec{N}/|\vec{N}|$ in the above expression.) You must use the procedures that require the *least* amount of algebra to answer all questions. Double-check you are using the correct vectors while doing all the above. Make sure you have answered everything.

4. Given

$$\vec{r}_A = (-2,1,6) \quad \vec{r}_B = (2,1,-7) \quad \vec{r}_C = (4,2,1) \quad \vec{r}_D = (2,2,2)$$

Using appropriate vector products, find the area of the triangle with sides AB and AC, of the parallelogram with sides AB and AC, and the volume of the parallelepiped with sides AB, AC, and AD. Also find the angle between AD and the parallelogram. You must use the vector procedures that require the *least* amount of algebra to answer all questions. (To find the angle between a line and a plane, first find the angle between the line and a line normal to the plane, in the range from 0 to $\pi/2$ and then take the complement of that.) Double-check you are using the correct vectors while doing all the above. Make sure you have answered everything. [2, 7.4].

5.
   1. Find the vector expression for the line through the point $(2,-3,5)$ that is normal to the plane $7x - 4y + 2z - 8 = 0$. Reduce to two scalar equations for the position coordinates of this line.
   2. Find the plane through the point (1,2,3) that is parallel to the vectors $2\hat{\imath} + \hat{\jmath} - \hat{k}$ and $3\hat{\imath} + 6\hat{\jmath} - 2\hat{k}$.

6. Are the following sets of vector linearly independent, and why? Give the simplest rigorous reason.
   - $3\hat{\imath} + 2\hat{\jmath}$ and $\hat{\imath} - \hat{\jmath}$.
   - $2\hat{\imath}$, $3\hat{\jmath}$, $5\hat{\imath} - 12\hat{k}$, $\hat{\imath} + \hat{\jmath} + \hat{k}$.
   - $(1,0,0,0), (0,1,1,0), (-4,6,6,0)$

   [2, 7.6].

7. Is the set S of vectors of the form $(x, y, 2x, 3y)$ in R^4 a vector space? Why? If so, give the dimension of the vector space and a basis. Also answer the same questions for the vector set (x, x^2) and the vector set $(x, y, 2 + x, 3 + y)$.

   Note: If for a vector in R^n you define multiplication by a scalar α as

$$\alpha\vec{v} \equiv \alpha(v_1, v_2, \ldots, v_n) \equiv (\alpha v_1, \alpha v_2, \ldots, \alpha v_n)$$

and addition as

$$\vec{v} + \vec{w} = (v_1, v_2, \ldots, v_n) + (w_1, w_2, \ldots, w_n) \equiv (v_1 + w_1, v_2 + w_2, \ldots, v_n + w_n)$$

then the needed "linear space" properties, such as

$$\alpha(\beta\vec{v}) = (\alpha\beta)\vec{v} \tag{1}$$
$$\alpha(\vec{v}+\vec{w}) = \alpha\vec{v}+\alpha\vec{w} \tag{2}$$
$$\vec{v}+\vec{w} = \vec{w}+\vec{v} \tag{3}$$
$$\vec{u}+(\vec{v}+\vec{w}) = (\vec{u}+\vec{v})+\vec{w} \tag{4}$$

will be certainly satisfied.

The problem is to show that the vector sets S are "complete." For a vector space, a multiple of a vector must still be in the same space. So must the sum of any two vectors still be in the space. Otherwise the space is not complete. In the first potential vector space S above, if you multiply a vector of the form $(x, y, 2x, 3y)$ by some constant, you get another vector. That other vector is only part of S if it can be written as $(x', y', 2x', 3y')$ for some values x' and y'. If it cannot be written in this form, the vector is not part of S, and so S is not a vector space. The same for adding two vectors in S. If you add two vectors $(x_1, y_1, 2x_1, 3y_1)$ and $(x_2, y_2, 2x_2, 3y_2)$, is the result of the form $(x_3, y_3, 2x_3, 3y_3)$ for some x_3 and y_3? Otherwise the sum of two vector in S is not in S and S is not a vector space. [2, 7.6].

5 Linear Algebra II

In this class,

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- *Solutions must be neat.*
- *Solutions must be exact and fully simplified.*
- *You must use the given symbols.*
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1. Given the matrices

$$A = \begin{pmatrix} -2 & 2 \\ 0 & 1 \\ 14 & 2 \\ 6 & 8 \end{pmatrix} \qquad B = \begin{pmatrix} 3 & 4 \\ 2 & 1 \\ 14 & 16 \\ 1 & 25 \end{pmatrix}$$

find, if they exist,

- $-5A + 3B$;
- A^{T};
- AB, BA, $A^{\mathrm{T}}B$, and BA^{T};
- the unit matrix or unit matrices that A can be pre-multiplied by, and demonstrating that this does not change A;
- the unit matrix or unit matrices that A can be post-multiplied by, and demonstrating that this does not change A;
- the zero matrix or matrices that can be added to A, stating what the result will be;
- the zero matrix or matrices that A can be pre-multiplied by, stating what the result will be;
- the zero matrix or matrices that A can be post-multiplied by, stating what the result will be.

Double-check you are using the correct matrices while doing all the above. Make sure you have answered everything. [2, 8.1].

2. Consider the system of equations

$$\begin{aligned} 4x_1 - x_2 + 4x_3 &= 1 \\ x_1 + x_2 - 5x_3 &= 0 \\ -2x_1 + x_2 + 7x_3 &= 4 \end{aligned}$$

Solve this system, as written (not in matrix notation), *using the class procedure* (See the revised notes on linear algebra[1].

In the first stage of the forward elimination, you will first need to do a partial pivoting to avoid fractions. In the second stage, partial pivoting does no good, so should not be done. Since you cannot take an integer multiple of the original

[1] `http://www.eng.fsu.edu/~dommelen/linalg/index.html`

equation greater than 1 in this stage (since it would mess up the L matrix asked later) you will need to live with fractions in the final equation.

Solve the resulting equations *using the class procedure*.

Next do the same but using augmented matrix notation.

Next find the L and U matrices of the LU decomposition and multiply the result. Do they give back the original matrix A of the system. If not, what do they give back? So, what is the right hand vector $\vec{b}^{\text{pp}}$ to use when solving $L\vec{y} = \vec{b}^{\text{pp}}$? Solve this system and check that you get indeed the correct right hand side vector to use in solving the system $U\vec{x} = \vec{y}$. Double-check you are using the correct matrix and right hand side while doing all the above. Make sure you have answered everything. [2, 8.1].

6 Linear Algebra III

In this class,

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1. Given

$$A = \begin{pmatrix} 6 & 4 & 2 & 2 \\ 9 & 6 & 0 & 0 \\ 3 & 2 & 4 & 4 \end{pmatrix}$$

Do all of the next things *using the class procedures*: (a) Reduce to echelon form. *Avoid fractions in both echelon matrix and multipliers, but use only legal partial pivoting to achieve that. Do not use non-unit multiples of the rows being changed. Check your result carefully.* (b) Using class procedures, find the null space of A. State its dimension. (c) Calling the echelon form U, find L so that $LU = A^{\rm pp}$ is like A, but with permuted rows. (Note that the second pivoting will permute the elements in the first column of L; the first column should be 1,2,3.) (d) Use matrices L and U to quickly find the solution space, if any, for $A\vec{x} = \vec{b}$ if $\vec{b}^{\rm T} = (0,1,0)$. (e) Repeat for $\vec{b}^{\rm T} = (0,-3,3)$. (f) Find the rank of A. (g) Explain why the sum of the rank of A and the dimension of the null space equals the number of columns of A. (h) What is the dimension of the row space of A? Find a fully simplified basis for it. Write the expression for the row space in terms of that basis. (i) Repeat the previous question for the column space. No, do not use U here, that is wrong as the column space gets destroyed going from A to U. (j) Neatly draw the two basis vectors of the column space in a three dimensional coordinate system and so illustrate a triangular piece of the row space plane.

2. Here are some quick ones. As always, explain all answers fully.

   1. Is an upper triangular matrix always in echelon form? Why?
   2. Is a nonsingular upper triangular square matrix always in echelon form? Why?
   3. What is the null space of an $m \times n$ zero matrix? What is its dimension? What is the rank of the matrix?
   4. What is the null space of a unit matrix? What is its dimension? What is the rank of the matrix?
   5. What is the dimension of the null space of an $1 \times n$ nonzero matrix (i.e. a nonzero row vector)? Give the null space for matrix $(0,0,3,0,0,6,0)$. What is the rank of the matrix?

6. Can a system $A_{m\times n}\vec{x} = \vec{0}$ where $m > n$ (more equations than unknowns) have a nontrivial solution?
7. Must there always be a solution to $A_{m\times n}\vec{x} = \vec{b}$ where $n > m$ (more unknowns than equations)?
8. Why does a system $A_{m\times n}\vec{x} = \vec{0}$ with $m < n$ *always* have a nontrivial solution? So what can you say about the dimension of the null space?

3. Given

$$A = \begin{pmatrix} 4 & 3 & -5 & 6 \\ 1 & -5 & 15 & 2 \\ 0 & -5 & 1 & 7 \\ 8 & 9 & 0 & 15 \end{pmatrix}$$

Find the determinant of this matrix using minors. Minimize the algebra in doing so. No Gaussian elimination steps, including partial pivoting, allowed.

4. Reconsider the matrix:

$$A = \begin{pmatrix} 4 & 3 & -5 & 6 \\ 1 & -5 & 15 & 2 \\ 0 & -5 & 1 & 7 \\ 8 & 9 & 0 & 15 \end{pmatrix}$$

Find the determinant of this matrix using Gaussian elimination. Compare to your earlier result.

7 Linear Algebra IV

In this class,

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1. For the matrix of the last question of the previous homework. find one row or column (state which row or column you took) of the inverse matrix using minors. Select the row or column so that you can reuse your earlier results, if correct, or else the posted solution, to cut down on the work.

2. For the matrix

$$A = \begin{pmatrix} 11 & 0 & -5 \\ 0 & 1 & 0 \\ 4 & -7 & 9 \end{pmatrix}$$

find the inverse both using minors and using Gaussian elimination. Make sure the result is the same.

3. For the matrix

$$A = \begin{pmatrix} 6 & -2 \\ -3 & 4 \end{pmatrix}$$

find the eigenvalues and eigenvectors. Take λ_1 to be the larger of the eigenvalues and λ_2 the smaller one. Is the matrix defective? Singular? What is the rank?

4. Take the two eigenvectors of the previous problem to be $(2, 1-\sqrt{7})$ and $(2, 1+\sqrt{7})$. (The eigenvectors that you found may have been different by some scalar factor.) Suppose you use the given two eigenvectors now as the basis of a new coordinate system. How do the coordinates v_1, v_2 of a vector $\vec{v}$ in the old coordinate system relate to the ones v'_1, v'_2 in the new coordinate system and vice versa? The instructor said that matrix $A' = P^{-1}AP$ in the new coordinates is a diagonal matrix. Verify that by direct matrix multiplication. Are the eigenvalues on the main diagonal? In the same order as the corresponding eigenvectors in P?

5. For the matrix

$$A = \begin{pmatrix} -2 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

find the eigenvalues and eigenvectors. Is the matrix defective? Singular? What is the rank?

6. Here are some quick ones. For each answer, *explain why.*

1. If a matrix is singular, how does that reflect in its eigenvalues?
2. Is it possible for an $n \times n$ matrix with all n eigenvalues different to be defective?
3. Is a square null-matrix defective? Singular?
4. Is a unit matrix defective? Singular?
5. Is a square matrix with all coefficients 1 singular? Defective?
6. Is a square matrix with all coefficients 0 except $a_{12} = 1$ singular? Defective? How many independent eigenvectors are there? To answer this, find the eigenvalue(s) and dimension of their null spaces for this simple triangular matrix.
7. Is a square matrix with all coefficients 0 except $a_{i\,i+1} = 1$ for $i = 1, 2, \ldots, n-1$ singular? Defective? How many independent eigenvectors are there?

8 Linear Algebra V

In this class,

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1. An anti-symmetric matrix is a matrix for which $A^{\mathrm{T}} = -A$. Are the eigenvalues of an antisymmetric real matrix real too? To check, write down the simplest nontrivial anti-symmetric 2×2 matrix you can think of (which may not be symmetric) and see. In fact, the eigenvalues of an antisymmetric matrix are always purely imaginary, i.e. proportional to $i = \sqrt{-1}$. The (complex) eigenvectors are orthogonal, as long as you remember that in the first vector of a dot product, you must take complex conjugate, i.e. replace every i by $-i$. Verify this for your antisymmetric matrix.

2. Analyze and accurately draw the quadratic curve

$$-2x^2 + xy + 3y^2 = 5$$

using matrix diagonalization. Show the exact, as well as the approximate values for all angles. Repeat for the curve,

$$1x^2 + xy + 6y^2 = 5$$

Note: if you add say 3 times the unit matrix to a matrix A, then the eigenvectors of A do not change. It only causes the eigenvalues to increase by 3, as you can readily verify from the definition of eigenvector. Use this to your advantage.

3. Given

$$A = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix}$$

Without doing any mathematics, what can you say immediately about the eigenvalues and eigenvectors of this matrix? Now find the equation for the eigenvalues. It is a cubic one. However, one eigenvalue is immediately obvious from looking at A. What eigenvalue λ_i, that makes $A - \lambda_i I$ singular, is immediately obvious from looking at A? Explain. Factor out the corresponding factor $(\lambda - \lambda_i)$ from the cubic, then find the roots of the remaining quadratic. Number the single eigenvalue λ_1, and the double one λ_2 and λ_3. The found two basis vectors of the null space of $A - \lambda_2 I$, call them $\vec{e}_2^{\,*}$ and $\vec{e}_3^{\,*}$, will not be orthogonal to each other. To make them orthogonal, you must eliminate the component that $\vec{e}_3^{\,*}$ has in the direction of $\vec{e}_2^{\,*}$. In particular, if $\vec{e}_2$ is the *unit* vector in the direction of $\vec{e}_2^{\,*}$, then $\vec{e}_3^{\,*} \cdot \vec{e}_2$ is the scalar component of $\vec{e}_3^{\,*}$ in the direction of $\vec{e}_2^{\,*}$. Multiply by the unit vector $\vec{e}_2$ to get the vector component, and substract it from $\vec{e}_3^{\,*}$:

$$\vec{e}_3^{\,**} = \vec{e}_3^{\,*} - (\vec{e}_3^{\,*} \cdot \vec{e}_2)\vec{e}_2$$

(This trick of making vectors orthogonal by substracting away the components in the wrong directions is called "Gram-Schmidt" orthogonalization.) Now make this vector of length 1. Then describe the transformation of basis that turns matrix A into a diagonal one. What is the transformation matrix P from old to new and what is its inverse? What is the diagonal matrix A'? Do your eigenvectors form a right or left-handed coordinate system?

4. Consider the quadratic surface given by

$$2x^2 + 2y^2 + 2z^2 + 2xy + 2yz + 2zx = 4$$

Write this in vector-matrix notation and identiy the matrix A. Compare with the matrix of the previous question. Now determine the shape of the quadratic surface. If it is a spheroid, state whether it is oblate or prolate. If it is a hyperboloid, state whether it is one of revolution or not, and whether it is of one or two sheets.

Repeat for the surface

$$2xy + 2yz + 2zx = 4$$

(Note that the new matrix is as if you subtracted 2 times the unit matrix from the previous matrix. As already noted in an earlier homework, this does nothing to the eigenvectors, but subtracts 2 from each eigenvalue.)

9 Ordinary Differential Equations I

In this class,

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1. For the ODE

$$y' = x(1-y)$$

sketch a dense direction field as a fully covering set of tiny line segments. A few hundred line segments will do, if you choose their locations well. Or use the Matlab quiver(u,v,scale) function or similar to draw the segments. (If $dy/dx = \tan\alpha$, you could take $u = \cos\alpha$ and $v = \sin\alpha$) Based on that, draw various solution curves accurately. Discuss maxima and minima, symmetry, asymptotes, and inflection points of the solutions. *Do NOT solve the equation algebraically. Use only the direction field to derive the solution properties. Cheating reduces credit!*

2. Solve

$$2y\frac{\mathrm{d}y}{\mathrm{d}x} = e^x e^{-y^2} \qquad y(4) = -2$$

Now solve the same ODE, but with initial condition that $y = 0$ at $x = 1$. Accurately draw these solutions.

3. Solve, using the class procedure (variation of parameter),

$$\sin(2x)\frac{\mathrm{d}y}{\mathrm{d}x} = 2\sin(x) - 2y\sin^2(x)$$

Draw a few representative solution curves.

4. Solve, using the class procedure,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{y}{x+\sqrt{xy}}$$

Draw a few representative solution curves. (Hint: solve for x as a function of y.) Also find the solution curve that satisfies the initial condition $y(0) = 1$ and draw it in your graph in a different color.

5. Solve, using the class procedure,

$$y' + \frac{2}{x}y = -x^9y^5$$

Draw a few representative solution curves. Also find the solution curve that satisfies the initial condition $y(-1) = 2$ and draw it in your graph in a different color. Where is its vertical asymptote?

6. Solve using class procedures

$$y'' + 2y' - 3y = 0 \qquad y(0) = 6 \qquad y'(0) = -2$$

7. Solve using class procedures

$$2\dddot{x} - 10\ddot{x} + 16\dot{x} - 8x = 0 \qquad x(0) = \dot{x}(0) = 0 \qquad \ddot{x}(0) = 2$$

8. Solve using class procedures

$$y'' + 2y' + 3y = 0 \qquad y(0) = 6 \qquad y'(0) = 2$$

10 Ordinary Differential Equations II

In this class,

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1. Solve using variation of parameters,

$$2y'' - 4y' - 6y = 4\sin^2(x)$$

2. Solve using undetermined coefficients:

$$y'' - 2y' + y = 3x + 25\sin(3x) + 2e^x \qquad y(0) = 1 \quad y'(0) = 2$$

3. Find the Laplace transform $\widehat{u}$ of

$$u = 1 - 4t + 2t^2 e^{-3t}$$

You may *only* use the brief Laplace transform table handed out in class. Everything else must be derived. Do *not* use convolution.

4. Solve

$$y' - 9y = t \qquad y(0) = 5$$

That would of course be quick using undetermined coefficients, or solving as a first order linear equation. Unfortunately, you must use Laplace transforms. You may *only* use the brief Laplace transform table handed out in class. Everything else must be derived. Do *not* use convolution. In solving the system of 3 equations in 3 unknowns of the partial fraction expansion, you may mess around; this is no longer linear algebra. However, you *must* substitute your solution into the original ODE and ICs and go back to fix any problem there may be.

5. Solve

$$y'' + 9y = t^2 \qquad y(0) = y'(0) = 0$$

That would of course be quick using undetermined coefficients. Unfortunately, you must use Laplace transforms. You may *only* use the brief Laplace transform table handed out in class. Everything else must be derived. Do *not* use convolution. In solving the system of 5 equations in 5 unknowns of the partial fraction expansion, you may mess around; this is no longer linear algebra. However, you *must* substitute your solution into the original ODE and ICs and go back to fix any problem there may be.

11 Ordinary Differential Equations III

In this class,

- *Questions must be answered in order asked.*
- *Solutions must be neat.*
- *Solutions must be exact and fully simplified.*
- *You must use the given symbols.*
- *You must show all reasoning.*
- *Copying is never allowed, even when working together.*

1. Resonant forcing of an undamped spring-mass system over some time period T that spans a large number of periods can introduce large-amplitude vibrations. To study the problem, consider the example

$$m\ddot{x} + kx = F(t) \qquad x(0) = \dot{x}(0) = 0$$

where the mass, spring constant, and applied force are given by

$$m = 1 \qquad k = 4 \qquad F(t) = \cos(2t) \text{ if } t < T \qquad F(t) = 0 \text{ if } t > T$$

Solve using the Laplace transform method. (Note: from S8 and S11 you can see that

$$\sin(\omega t) - \omega t \cos(\omega t) \Longleftrightarrow \frac{2\omega^3}{(s^2 + \omega^2)^2}$$

Call it result S15.) Clean up your answer. I find that beyond time $t = T$, the amplitude stays constant at

$$\tfrac{1}{4}T\sqrt{1 + 2\cos(2T)\frac{\sin(2T)}{2T} + \left(\frac{\sin(2T)}{2T}\right)^2}$$

which is approximately proportional to T for large T. Do your results agree?

2. The generic linearly damped spring-mass system that is initially at rest but receives a kick with momentum I_0 at a time T is described by

$$m\ddot{x} + c\dot{x} + kx = I_0\delta(t - T) \qquad x(0) = \dot{x}(0) = 0$$

Here the mass m, damping constant c, and spring constant k are given positive constants. As seen in a previous question, the natural frequency of the free undamped system is $\omega = \sqrt{k/m}$. It is also useful to define a nondimensional damping constant $\zeta \equiv c/2m\omega$ which is called the damping ratio. Check that in those terms, the ODE can be written as

$$\ddot{x} + 2\zeta\omega\dot{x} + \omega^2 x = \frac{I_0}{m}\delta(t - T)$$

Assuming that $\omega = 5$ rad/s and the damping ratio $\zeta = 4/5$, find x using Laplace transformation. (You must complete the square as explained, for example, in the revised notes on ordinary differential equations[2]. Complex roots in partial fractions are

[2] `http://www.eng.fsu.edu/~dommelen/odes/web-pages/cmpltsqr.html`

not allowed.) Note that after the kick, there is no further force and the system vibrates as a free system. Plot your solution accurately versus time and so show graphically that the mass keeps vibrating between negative and positive values although the amplitude of vibration after the kick decreases with time. More generally, it can be seen that if the damping ratio is less than one, the mass keeps vibrating. For damping ratio greater than 1, the amplitude changes sign at most once.

3. The generic linearly damped spring-mass system experiencing an external force with frequency ω can be written as

$$m_1\ddot{x}_1 + c_1\dot{x}_1 + k_1x_1 = F_1\cos(\widetilde{\omega}t)$$

Here F_1 is a constant. As seen in an earlier question, if $\widetilde{\omega}$ is close to the natural frequency of the system and damping is small, mass m_1 may experience severe vibration. If mass m_1 is, say, really a building and the force is really an earthquake, that may be very bad news. But suppose you hang a second mass m_2 from the first using a spring with constant k_2. Then the equation above becomes

$$m_1\ddot{x}_1 + c_1\dot{x}_1 + k_1x_1 = F_1\cos(\widetilde{\omega}t) + k_2(x_2 - x_1)$$

while the second mass satisfies the equation

$$m_2\ddot{x}_2 = -k_2(x_2 - x_1)$$

Find the Laplace transforms $\widehat{x}_1$ and $\widehat{x}_2$. To keep it simple, assume that before the quake,

$$x_1(0) = 0 \quad \dot{x}_1(0) = v_{10} \quad x_2(0) = 0 \quad \dot{x}_2(0) = v_{20}$$

You do not have to find x_1 and x_2; you can answer the next questions from what you know about partial fractions. To do so, show first that

$$\widehat{x}_1 = F_1\frac{s(m_2s^2 + k_2)}{Q(s^2 + \widetilde{\omega}^2)} + \frac{m_1v_{10}(m_2s^2 + k_2) + m_2v_{20}k_2}{Q}$$

where Q is the quadratic

$$Q = (m_1s^2 + c_1s + k_1 + k_2)(m_2s^2 + k_2) - k_2^2$$

Cramers rule works nicely in this case.

Now you need to find the qualitative form of the partial fraction expansion of $\widehat{x}_1$. Now the 4 roots of the quadratic Q in the bottom would be difficult to find.But look for a second at the free solution (i.e. with $F_1 = 0$). Based on your physical arguments in the previous question, you should be able to describe the qualitative nature of the four roots if the damping is low. Knowing about the nature of these four roots, you can now ignore the second term in $\widehat{x}_1$ and look at the first term with F_1 nonzero. The terms will correspond to two decaying modes of vibration and one term where m_1 vibrates with frequency $\widetilde{\omega}$. This term will have a large amplitude for small damping. However, if you look a bit closer, you see that if you choose the ratio k_2/m_2 to be $\widetilde{\omega}^2$, the third term disappears. Then the building returns to rest after a transition period, despite the ongoing vibrating force on it! The effect of the force has been eliminated!

You may be astonished by that, since only the ratio of k_2 to m_2 is specified. So you could eliminate the vibration in your building m_1 by suspending a single grain of sand m_2 from it using a very weak spring! (Actually, if you do this, and the natural frequency of the building is close to $\widetilde{\omega}$, and damping is small, then the coefficients of the decaying modes will be very large. So the building will still experience large transient vibration.)

4. Solve the system

$$\begin{aligned} x_1' &= 2x_1 + x_2 - 2x_3 \\ x_2' &= 3x_1 - 2x_2 \\ x_3' &= 3x_1 + x_2 - 3x_3 \end{aligned}$$

Find the general solution to this system in vector form and in terms of a fundamental matrix. Then find the vector of integration constants assuming that $\vec{x}(0) = (1, 7, 3)^{\mathrm{T}}$ and write $\vec{x}(t)$ for that case.

12 Ordinary Differential Equations IV

In this class,

- *Questions must be answered in order asked.*
- *Solutions must be neat.*
- *Solutions must be exact and fully simplified.*
- *You must use the given symbols.*
- *You must show all reasoning.*
- *Copying is never allowed, even when working together.*

1. Given the system

$$\dot{\vec{x}} = A\vec{x} \qquad A = \begin{pmatrix} 0 & 5 \\ -1 & -2 \end{pmatrix}$$

Find the general solution to this system in vector form and in terms of a fundamental matrix. Complex solutions not allowed.

2. Solve the system and initial condition

$$\dot{\vec{x}} = A\vec{x} \quad \vec{x}(0) = \vec{x}_0 \qquad A = \begin{pmatrix} 1 & 5 & 0 \\ 0 & 1 & 0 \\ 4 & 8 & 1 \end{pmatrix} \quad \vec{x}_0 = \begin{pmatrix} 9 \\ 1 \\ 1 \end{pmatrix}$$

Give a fundamental matrix. Clean up the final $\vec{x}$.

3. Solve the inhomogeneous system and initial condition

$$\dot{\vec{x}} = A\vec{x} + \vec{g} \quad \vec{x}(0) = \vec{x}_0 \qquad A = \begin{pmatrix} 2 & -3 & 1 \\ 0 & 2 & 4 \\ 0 & 0 & 1 \end{pmatrix} \quad \vec{g} = \begin{pmatrix} 10 \\ 6 \\ -1 \end{pmatrix} e^{2t} \quad \vec{x}_0 = \begin{pmatrix} 5 \\ 11 \\ -2 \end{pmatrix}$$

Use variation of parameters. Clean up the final $\vec{x}$.

4. Consider the autonomous system

$$x' = x + 3y - x^2 \sin y \qquad y' = 2x + y - xy^2$$

Analyze this system analytically:

(a) Find the critical points. One critical point is easy. Four more critical points can be found numerically. To help you a bit, their y-values are ± 1.1107 and ± 1.6074.

(b) Find the matrix of derivatives of vector $\vec{F}$ at each of the five critical point. (By symmetry around the origin, there are only three matrices that are different.)

5. For each of the three different matrices of the previous question, solve the linearized system. (For the simple point, your solution should be exact.) Then draw the phase plane, with $-3 \leq x \leq 3$ and $-2 \leq y \leq 2$ and locate the five stationary points. At each stationary points, draw its eigenvectors, or $\vec{u}$ and $\vec{v}$ if complex, or also a $\vec{f}$ if defective, as little vectors, but with the correct angles (and in case of $\vec{e}$ and $\vec{f}$, relative lengths).

13 Ordinary Differential Equations V

In this class,

- *Questions must be answered in order asked.*
- *Solutions must be neat.*
- *Solutions must be exact and fully simplified.*
- *You must use the given symbols.*
- *You must show all reasoning.*
- *Copying is never allowed, even when working together.*

1. Solve the system

$$\vec{x}\,' = A\vec{x} \qquad A = \begin{pmatrix} 3 & -5 \\ 8 & -3 \end{pmatrix}$$

Neatly and accurately sketch a comprehensive set of solution curves in the x_1, x_2 plane. Include the eigenvectors, and/or $\vec{f}$, $\vec{u}$, and $\vec{v}$ in the graph if applicable. Get the slopes right.

2. Solve the system

$$\dot{\vec{x}} = A\vec{x} \qquad A = \begin{pmatrix} -6 & -7 \\ 7 & -20 \end{pmatrix}$$

Neatly and accurately sketch a comprehensive set of solution curves in the x_1, x_2 plane. Include the eigenvectors, and/or $\vec{f}$, $\vec{u}$, and $\vec{v}$ in the graph if applicable. Get the slopes right.

3. Continuing the last questions of the previous homework for the autonomous system

$$x' = x + 3y - x^2 \sin y \qquad y' = 2x + y - xy^2$$

First analyze this system analytically:

(a) For each of the five stationary poitnt, list the type of point and its stability.

(b) In your phase plane with the eigenvectors, and $\vec{f}$ if appropriate, or $\vec{u}$ and $\vec{v}$, neatly draw the solution lines near each stationary point. Make sure to clearly show the key features, related to the directions of the vectors.

4. Continuing the previous quastion, create a numerical graph of the complete set of solutions lines in the entire plane. To so so, take $-5 \le x \le 5$ and $-5 \le y \le 5$.

Suitable programs to do this can be found on the web. Some I saw previously:

`http://www.math.uu.nl/people/beukers/phase/newphase.html`

`http://www.scottsarra.org/applets/dirField1/dirField1.html`

`http://www.math.rutgers.edu/courses/ODE/sherod/phase-local.html`

See here for Matlab software.[3] (You will need to convert to an ODE by taking the ratio of the equations, and then the software might crash when it divides by zero if it hits a critical point.)

You will need to use a screen grabber to make a copy that you can print. Typically you press Alt+PrintScreen or Shift-PrintScreen to get a printable copy of the active Window. In windows you might have to paste the copied graph into, say, M.S. Paint.

Additional note: It has been brought to my attention that the above phase plane plotters may no longer work because the College of Engineering no longer installs Java on the web browsers. Therefore I have created a little Matlab program that does essentially the same thing. It can be found at:

`http://www.eng.fsu.edu/~dommelen/courses/aim/odesols/phaseplane.m`

This program is set up to do the Van der Pol oscillator. To make it work with the system you are solving, you will need to make a few minor changes to this program, as explained in the comments at the start of the file. You will also need to create a function file myfunc.m. Pattern it after the Van der Pol example:

`http://www.eng.fsu.edu/~dommelen/courses/aim/odesols/vanderpol.m`

If you use MS Notepad to create this file, do not forget to save as "All Files", not "Text Files".

If you do not have Matlab, the free Octave clone will do just fine. You can use "File/Save As" in the plot window to save the plot for printing.

5. Compare the full numerical solution to the solutions you got near the stationary point. Do they really agree? Note: for nodes the direction of one eigenvector is readily deduced from the full solution. For the direction of the other eigenvector, look for a solution curve that goes straight through the critical point, without bending to meet the direction of the other eigenvector.

   State for each critical point whether critical point analysis *must* give the right solution near the point. In other words, could the real lines be qualitatively different from what you drew?

[3] `http://math.rice.edu/~dfield/index.html`

References

[1] F. Ayres and E. Mendelson. *Calculus*. Schaum's Outline Series. McGraw-Hill, 5th edition, 2009.

[2] Dennis G. Zill and Warren S. Wright. *Advanced Engineering Mathematics*. Jones and Bartlett Learning, 5th edition, 2014.