Analysis in Mechanical Engineering
EML 5060
Homework

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Contents

1 Calculus I

1. Plot $y^3 = 6x^2 - x^3$ following class procedures[a b c]. [1, Maximum and Minimum Values, Curve Sketching]
2. Plot $y = (x+1)/x^2$ following class procedures[a b]. [1, Maximum and Minimum Values, Curve Sketching]
3. Plot $y = \ln(e^{x+1} - 1)$ following class procedures[a b]. [1, Maximum and Minimum Values, Curve Sketching]

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[a] Include a plot

[b] Discuss intercepts, extents, symmetries, asymptotes, asymptotic behavior for $x \to \pm\infty$, local/global maxima/minima, concavity, inflection points, poles, cusps, corners, and other singularities

[c] Note that $\lim_{x\to\infty} = \lim_{u\downarrow 0}$ if $u = 1/x$. Alternatively, use the Taylor series approximation $(1+\varepsilon)^p \sim 1 + p\varepsilon$ for $\varepsilon \to 0$

2 Calculus II

1. A company charges orders as follows
   - Orders of 50,000 items or less are charged at \$30 per thousand.
   - For orders above 50,000, the charge per thousand is reduced by $37\frac{1}{2}$¢ for each thousand above 50,000. (This reduced charge per thousand is applied to the *complete* order.)

   Plot the receipts of this company and analyze the graph. What order size maximizes the receipts of the company? Give the reasons why the financial management of this company is clearly in incompetent hands.[a]. [1, Maximum and Minimum Values]
2. For a conical tent of given volume, find the ratio of the height h to the radius of the base r that requires the least amount of material. Note: verify first that the surface area of a cone is $\pi r\sqrt{r^2+h^2}$.[b]. [1, Maximum and Minimum Values]
3. Find the MacLaurin series for $1/(1+x^5)$. Hint: you may not want to crunch this out. Explain why not. Use a suitable trick instead. [1, Taylor and Maclaurin Series]
4. Write out the Taylor series for $\cos x$ around $\pi/3$ using the exact values of $\cos(\pi/3)$ and $\sin(\pi/3)$. Now find the largest distance r from $\pi/3$ so that the error in the three-term Taylor series is no more than 0.000 05 when $|x-\frac{1}{3}\pi|<r$. Find r to two significant digits accurate without using trial and error. [1, Taylor and Maclaurin Series]
5. Find $\lim_{x\downarrow 0}\ln(\cot x)/e^{\csc^2 x}$. [1, L'Hôpital's Rule]
6. Find $\lim_{x\to\pi/2}(\sec^3 x-\tan^3 x)$. [1, L'Hôpital's Rule]

[a] Include a plot

[b] Include a plot and picture

3 Calculus III

1. Variable z is given in terms of the measurable variables f and g as

$$\frac{1}{z} = \frac{1}{f} + \frac{1}{g}$$

The values of f and g and their uncertainties are:

$$f = 4 \pm 0.01 \qquad g = 8 \pm 0.02$$

What are the maximum relative and absolute errors in the computed z? Are you stunned by value of the relative error in z? Explain why not. [1, Total Differential]

2. The position of a particle in two-dimensions is given by

$$x = 2 + 3t \qquad y = t^2 + 4$$

What is the rate at which the distance r from the origin increases at time $t = 1$? [1, Total Differential]

3. Find I_x for the area between the curves

$$y = x \qquad y = 4x - x^2$$

Exact answers only, please. Since the integrand y^2 does not depend on x, it would seem logical to integrate x first. Comment on that. [1, Centroids and Moments of Inertia]

4. Find the volume of the region bounded by

$$z = 0 \qquad x^2 + y^2 = 4x \qquad x^2 + y^2 = 4z$$

Use cylindrical coordinates r, θ (or ϕ if you want), and z. What variable is obviously the one to integrate first? For the second integration, discuss each possibility and explain which is the best choice. Would it have been easier to use Cartesian coordinates x, y and z instead of cylindrical ones? [1, Triple Integrals]

4 Linear Algebra I

1. Show the vectors

$$\vec{F} = \sqrt{2}\hat{\imath} + \hat{\jmath} - 6\hat{k} \qquad \vec{G} = 8\hat{\imath} + 2\hat{k}$$

as neatly and accurately as possible in a 3D left-handed coordinate system with the x-axis to the right and the z axis upwards in the plane of the paper. Now, using a different color, construct graphically, as accurately as possible, $2\vec{F}$, $3\vec{G}$, $\vec{F}+\vec{G}$, $\vec{F}-\vec{G}$ and also demonstrate them graphically in a neat plot, as accurately as possible. Compute these quantities also algebraically. Does that look about right in the graph? Compute $||\vec{F}||$ and $||\vec{G}||$. After evaluating the results in your calculator, do they agree with what you measure in the graph (ignoring that $\vec{F}$ is sticking a bit out of the paper)? Evaluate the angle between vectors $\vec{F}$ and $\vec{G}$ using a dot product. Does that angle seem about right? How would the above work out for the vectors

$$\vec{v} = (\sqrt{2}, 1, -6) \qquad \vec{w} = (8, 0, 2)$$

Double-check you are using the correct vectors while doing all the above. Make sure you have answered everything. As always, give *exact, cleaned up*, answers. As always, explain all reasoning! [2, chapter 6].

2. A flat mirror passes through the points

$$\vec{r}_A = (-2, 1, 6) \quad \vec{r}_B = (2, 1, -7) \quad \vec{r}_C = (4, 2, 1)$$

Find a vector $\vec{N}$, with integer components, that is normal to the plane of the mirror using the appropriate vector combinations and product. Find a simple scalar equation for the plane of the mirror. Check that A, B, and C satisfy it. Now assume a laser beam moves in the positive $\vec{e} = \hat{\imath}$ direction before it is reflected by the mirror. Find the unit vector $\vec{e}\,'$ in the direction of the reflected beam. (Note: Snell's law of reflection may be formulated as follows: reflection leaves the component of $\vec{e}$ in the direction parallel to the mirror the same. However, it inverts the component normal to the mirror. So to get $\vec{e}\,'$, first consider the expression for the component of $\vec{e}$ in the direction normal to the mirror. You know that if $\vec{n}$ is a unit vector normal to the mirror, then this component is $\vec{e}\cdot\vec{n}$. The *vector* component is the scalar one multiplied by the unit vector $\vec{n}$. To invert this vector compont subtract it *twice* from $\vec{e}$. Once to zero the component and a second time to add the negative component. Therefore you see that:

$$\vec{e}\,' = \vec{e} - 2(\vec{e}\cdot\vec{n})\vec{n}$$

Do not compute $\vec{n}$ explicitly, as this would bring in a nasty square root. Just substitute $\vec{n} = \vec{N}/|\vec{N}|$ in the above expression.) You must use the procedures that require the *least* amount of algebra to answer all questions. Double-check you are using the correct vectors while doing all the above. Make sure you have answered everything. [2, chapter 6].

3. Given

$$\vec{r}_A = (-2, 1, 6) \quad \vec{r}_B = (2, 1, -7) \quad \vec{r}_C = (4, 2, 1) \quad \vec{r}_D = (2, 2, 2)$$

Using appropriate vector products, find the area of the parallelogram with sides AB and AC, and the volume of the parallelepiped with sides AB, AC, and AD. Also find the angle between AD and the parallelogram. You must use the procedures that require the *least* amount of algebra to answer all questions. (To find the angle between a line and a plane, first find the angle between the line and a line normal to the plane, in the range from 0 to $\pi/2$ and then take the complement of that.) Double-check you are using the correct vectors while doing all the above. Make sure you have answered everything. [2, chapter 6].

4. Are the following sets of vector linearly independent, and why? Give the simplest rigorous reason.
   - $3\hat{\imath} + 2\hat{\jmath}$ and $\hat{\imath} - \hat{\jmath}$.
   - $2\hat{\imath}$, $3\hat{\jmath}$, $5\hat{\imath} - 12\hat{k}$, $\hat{\imath} + \hat{\jmath} + \hat{k}$.
   - $(1, 0, 0, 0), (0, 1, 1, 0), (-4, 6, 6, 0)$

   [2, chapter 6].

5. Is the set S of vectors of the form $(x, y, 2x, 3y)$ in R^4 a vector space? Why? If so, give the dimension of the vector space and a basis. Also answer the same questions for the vector set (x, x^2) and the vector set $(x, y, 2 + x, 3 + y)$. [2, chapter 6].

   Note: If for a vector in R^n you define multiplication by a scalar α as

$$\alpha\vec{v} \equiv \alpha(v_1, v_2, \ldots, v_n) \equiv (\alpha v_1, \alpha v_2, \ldots, \alpha v_n)$$

and addition as

$$\vec{v} + \vec{w} = (v_1, v_2, \ldots, v_n) + (w_1, w_2, \ldots, w_n) \equiv (v_1 + w_1, v_2 + w_2, \ldots, v_n + w_n)$$

then the needed "linear space" properties, such as

$$\alpha(\beta\vec{v}) = (\alpha\beta)\vec{v} \tag{1}$$
$$\alpha(\vec{v} + \vec{w}) = \alpha\vec{v} + \alpha\vec{w} \tag{2}$$
$$\vec{v} + \vec{w} = \vec{w} + \vec{v} \tag{3}$$
$$\vec{u} + (\vec{v} + \vec{w}) = (\vec{u} + \vec{v}) + \vec{w} \tag{4}$$

will be certainly satisfied. However, S is a *subset* of R^4 in which the vectors can all be written as $(x, y, 2x, 3y)$ for some values of x and y. It is not obvious whether such a subset is *complete* under multiplication be a scalar or vector addition. In particular, if you take a scalar multiple of a vector $(x, y, 2x, 3y)$, is the result still of the form $(x', y', 2x', 3y')$ (with in general $x' \neq x$ and $y' \neq y$)? Or can the result no longer necessarily be written in a form $(x', y', 2x', 3y')$? In that case the result is no longer in S, so S is not complete under multiplication by a scalar. At least some multiples of vectors in S are outside S. The same for adding two vectors in S: is the result necessarily still in S?

5 Linear Algebra II

1. Given the matrices

$$A = \begin{pmatrix} -2 & 2 \\ 0 & 1 \\ 14 & 2 \\ 6 & 8 \end{pmatrix} \qquad B = \begin{pmatrix} 3 & 4 \\ 2 & 1 \\ 14 & 16 \\ 1 & 25 \end{pmatrix}$$

   find, if they exist,
   - $-5A + 3B$;
   - A^{T};
   - AB, BA, $A^{\mathrm{T}}B$, and BA^{T};
   - the unit matrix or unit matrices that A can be pre-multiplied by, demonstrating that this does not change A;
   - the unit matrix or unit matrices that A can be post-multiplied by, demonstrating that this does not change A;
   - the zero matrix or matrices that can be added to A, stating what the result will be;
   - the zero matrix or matrices that A can be pre-multiplied by, stating what the result will be;
   - the zero matrix or matrices that A can be post-multiplied by, stating what the result will be.

   Double-check you are using the correct matrices while doing all the above. Make sure you have answered everything. [2, chapter 7].

2. Consider the system of equations

$$\begin{aligned} 4x_1 - x_2 + 4x_3 &= 1 \\ x_1 + x_2 - 5x_3 &= 0 \\ -2x_1 + x_2 + 7x_3 &= 4 \end{aligned}$$

   Solve this system, as written, *using the class procedure*. In the first stage of the forward elimination, you will first need to do a partial pivoting to avoid fractions. In the second stage, partial pivoting does no good, so should not be done. Since you cannot take an integer multiple of the original equation greater than 1 in this stage (since it would mess up the L matrix asked later) you will need to live with fractions in the final equation. Solve the resulting equations *using the class procedure*. Next do the same but using augmented matrix notation. Next find the L and U matrices of the LU decomposition and multiply the result. Do they give back the original matrix A of the system. If not, what do they give back? So, what is the right hand vector $\vec{b}^{\mathrm{pp}}$ to use when solving $L\vec{y} = \vec{b}^{\mathrm{pp}}$? Solve this system and check that you get indeed the correct right hand side vector to use in solving the system $U\vec{x} = \vec{y}$. Double-check you are using the correct matrix and right hand side while doing all the above. Make sure you have answered everything. [2, chapter 7].

6 Linear Algebra III

1. Given

$$A = \begin{pmatrix} 8 & 2 & 1 & 0 \\ 0 & 1 & 1 & 3 \\ 4 & 0 & 0 & -3 \end{pmatrix}$$

   Reduce to row canonical form using the class procedure (i.e. first to echelon avoiding fractions). At the same time find the matrix Ω so that $A_{\rm R} = \Omega A$. Is $\Omega = A^{-1}$? If not, why not? Is $A_{\rm R} = I$? What is the rank of A? What is its nullspace? What is the dimension of the null space? What is the solution space if the right hand side vector is $(1, 2, 3)$? Do you get the same solution space from the echelon and row canonical forms? Do you really need to include this right hand side in the augmented matrix or is there a simpler way?

2. Here are some quick ones. As always, explain all answers fully.
    1. Is an upper triangular matrix always in echelon form?
    2. Is a nonsingular upper triangular matrix always in echelon form?
    3. What is the null space of an $m \times n$ zero matrix? What is its dimension? What is the rank of the matrix?
    4. What is the null space of a unit matrix? What is the rank of the matrix?
    5. What is the dimension of the null space of an $1 \times n$ nonzero matrix (i.e. a nonzero row vector)? Give it for matrix $(0, 0, 3, 0, 0, 6, 0)$. What is the rank of the matrix?
    6. Can a system $A_{m\times n}\vec{x} = \vec{0}$ where $m > n$ (more equations than unknowns) have a nontrivial solution?
    7. Must there always be a solution to $A_{m\times n}\vec{x} = \vec{b}$ where $n > m$ (more unknowns than equations)?
    8. Prove that a system $A_{m\times n}\vec{x} = \vec{0}$ with $m < n$ *always* has a nontrivial solution. So what can you say about the dimension of the null space?

7 Linear Algebra IV

1. Given
$$A = \begin{pmatrix} -4 & -2 & 1 & 6 \\ 0 & 4 & -4 & 3 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$
First find the rank, dimension of the row space, and dimension of the column space. Then find *fully* simplified bases of the row and column spaces.

2. Given
$$A = \begin{pmatrix} 4 & 3 & -5 & 6 \\ 1 & -5 & 15 & 2 \\ 0 & -5 & 1 & 7 \\ 8 & 9 & 0 & 15 \end{pmatrix}$$
Find the determinant of this matrix both using minors and Gaussian elimination. Find the first row of the inverse matrix using minors.

8 Linear Algebra V

1. Here are some quick ones:
   1. If a matrix is singular, how does that reflect in its eigenvalues?
   2. Is it possible for an $n \times n$ matrix with all n eigenvalues different to be defective?
   3. Is a square null-matrix defective? Singular?
   4. Is a unit matrix defective? Singular?
   5. Is a square matrix with all coefficients 1 singular? Defective?
   6. Is a square matrix with all coefficients 0 except $a_{12} = 1$ singular? Defective? What are the eigenvalues and eigenvectors?
   7. Is a square matrix with all coefficients 0 except $a_{ii+1} = 1$ for $i = 1, 2, \ldots, n-1$ singular? Defective? What are the eigenvalues and eigenvectors?
   8. An anti-symmetric matrix is a matrix for which $A^{\mathrm{T}} = -A$. Are the eigenvalues of an antisymmetric matrix real too? To check, write down a nontrivial anti-symmetric 2×2 matrix and see.

2. Analyze and accurately draw the quadratic curve

$$-2x^2 + xy + 3y^2 = 5$$

   using matrix diagonalization. Show the exact, as well as the approximate values for all angles. Repeat for the curve,

$$1x^2 + xy + 6y^2 = 5$$

   Note: if you add say 3 times the unit matrix to a matrix A, then the eigenvectors of A do not change. It only causes the eigenvalues to increase by 3, as you can readily verify from the definition of eigenvector.

3. Given

$$A = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix}$$

   Without doing any mathematics, what can you say immediately about the eigenvalues and eigenvectors of this matrix? Now find the equation for the eigenvalues. It is a cubic one. However, one eigenvalue is immediately obvious from looking at A. What eigenvalue λ_i, that makes $A - \lambda_i I$ singular, is immediately obvious from looking at A? Factor out the corresponding factor $(\lambda - \lambda_i)$ from the cubic, then find the roots of the remaining quadratic. Number the single eigenvalue λ_1, and the double one λ_2 and λ_3. The found two basis vectors of the null space of $A - \lambda_2 I$, call them $\vec{e}_2^{\,*}$ and $\vec{e}_3^{\,*}$, will not be orthogonal to each other. To make them orthogonal, simply eliminate the component that $\vec{e}_3^{\,*}$ has in the direction of $\vec{e}_2^{\,*}$. In particular, if $\vec{e}_2$ is the *unit* vector in the direction of $\vec{e}_2^*$, then $\vec{e}_3^* \cdot \vec{e}_2$ is the scalar component of $\vec{e}_3^*$ in the direction of $\vec{e}_2^*$. Multiply by the unit vector $\vec{e}_2$ to get the vector component, and substract it from $\vec{e}_3^*$:

$$\vec{e}_3^{**} = \vec{e}_3^* - (\vec{e}_3^* \cdot \vec{e}_2)\vec{e}_2$$

(This trick of making vectors orthogonal by substracting away the components in the wrong directions is called "Gram-Schmidt" orthogonalization.) Now describe the transformation of basis that turns matrix A into a diagonal one. What is the transformation matrix P from old to new and what is its inverse? What is the diagonal matrix A?

9 Ordinary Differential Equations I

1. For the ODE
$$y' = x(1-y)$$
sketch a dense direction field as a fully covering set of tiny line segments. Based on that, draw various solution curves. Discuss maxima and minima, symmetry, asymptotes, and inflection points of the solutions. *Do not solve the equation algebraically. Use only the direction field to derive the solution properties.*

2. Solve
$$2y\frac{\mathrm{d}y}{\mathrm{d}x} = e^x e^{-y^2} \qquad y(4) = -2$$
Now solve the same ODE, but with initial condition that $y = 0$ at $x = 1$. Accurately draw these solutions.

3. Solve, using the class procedure (variation of parameter),
$$\sin(2x)\frac{\mathrm{d}y}{\mathrm{d}x} = 2\sin(x) - 2y\sin^2(x)$$

4. Solve
$$2y'' - 4y' + 20y = 0$$
As always, don't forget to clean up.

5. Solve
$$2y'' - 4y' - 6y = 4\sin^2(x)$$
Use variation of parameters.

10 Ordinary Differential Equations II

1. Solve using undetermined coefficients:

$$y'' - 2y' + y = 3x + 25\sin(3x) + 2e^x \qquad y(0) = 1 \quad y'(0) = 2$$

2. Find the Laplace transform $\widehat{u}$ of

$$u = 1 - 4t + 2t^2 e^{-3t}$$

You may *only* use the brief Laplace transform table handed out in class. Everything else must be derived. Do *not* use convolution.

3. Solve

$$y'' + 9y = t^2 \qquad y(0) = y'(0) = 0$$

That would of course be quick using undetermined coefficients. Unfortunately, you must use Laplace transforms. You may *only* use the brief Laplace transform table handed out in class. Everything else must be derived. Do *not* use convolution. In solving the system of 5 equations in 5 unknowns of the partial fraction expansion, you may mess around; this is no longer linear algebra. However, you *must* substitute your solution into the original ODE and ICs and go back to fix any problem there may be.

4. Resonant forcing of an undamped spring-mass system over some time period T that spans a large number of periods can introduce large-amplitude vibrations. To study the problem, consider the example

$$m\ddot{x} + kx = F(t) \qquad x(0) = \dot{x}(0) = 0$$

where the mass, spring constant, and applied force are given by

$$m = 1 \qquad k = 4 \qquad F(t) = \cos(2t) \text{ if } t < T \qquad F(t) = 0 \text{ if } t > T$$

Solve using the Laplace transform method. Note: from S8 and S11 you can see that

$$\sin(\omega t) - \omega t\cos(\omega t) \Longleftrightarrow \frac{2\omega^3}{(s^2+\omega^2)^2}$$

Clean up your answer. I find that beyond time $t = T$, the amplitude stays constant at

$$\tfrac{1}{4}T\sqrt{1 + 2\cos(2T)\frac{\sin(2T)}{2T} + \left(\frac{\sin(2T)}{2T}\right)^2}$$

which is approximately proportional to T for large T. Do your results agree?

11 Ordinary Differential Equations III

1. The generic undamped spring-mass system with external forcing is

$$m\ddot{x} + kx = F(t) \qquad x(0) = x_0 \quad \dot{x}(0) = v_0$$

where the mass m and spring constant k are given positive constants, $F(t)$ is the given external force, and the initial displacement x_0 and velocity v_0 are given constants. Give the solution using Laplace transformation, as always restricting use of convolution to the bare minimum. Write the solution in the form

$$A(t)\sin(\omega t) + B(t)\cos(\omega t)$$

Identify the natural frequency ω.

Next check that for a resonant force $F = F_0 \cos(\tilde{\omega} t)$ with F_0 a constant and $\tilde{\omega} = \omega$, the integrand in the A integral is always positive, so A will grow without bound in time. However, the integrand in the B integral will periodically change sign, so that B stays finite. Address the question why if I apply a *cosine* forcing, it is the magnitude of the *sine* that keeps increasing instead of the cosine. Does that make physical sense? For nonresonant forcing, $\tilde{\omega} \neq \omega$, both integrands will periodically change sign, so that both A and B stay finite.

To see these things more precisely, find $\widehat{x}$ for the nonresonant case $\tilde{\omega} \neq \omega$, assuming $x_0 = v_0 = 0$. Its partial fraction expansion will be

$$\widehat{x} = \frac{Cs + D}{s^2 + \tilde{\omega}^2} + \frac{Es + G}{s^2 + \omega^2}$$

To quickly find C and D, you can use a trick. Multiple $\widehat{x}$ as found (not the partial fraction expansion) by $s^2 + \tilde{\omega}^2$, simplify, and evaluate at $s = i\tilde{\omega}$ with $i = \sqrt{-1}$. That will give you the value of $i\tilde{\omega}C + D$. Also evaluate at $s = -i\tilde{\omega}$ to get $-i\tilde{\omega}C + D$. The sum and difference of the equations directly give D and C. Show that you get a $\ldots\cos(\tilde{\omega} t)$ term (why now a cosine?) with an amplitude that will be large if $\tilde{\omega} \approx \omega$. Also find the solution for $\tilde{\omega} = \omega$ exactly, and show that it is indeed a growing sine.

2. The generic linearly damped spring-mass system that is initially at rest but receives a kick with momentum I_0 at a time T is described by

$$m\ddot{x} + c\dot{x} + kx = I_0\delta(t - T) \qquad x(0) = \dot{x}(0) = 0$$

Here the mass m, damping constant c, and spring constant k are given positive constants. As seen in the previous question, the natural frequency of the free undamped system $\omega = \sqrt{k/m}$. It is also useful to define a nondimensional damping constant $\zeta \equiv c/2m\omega$ which is called the damping ratio. Check that in those terms, the ODE can be written as

$$\ddot{x} + 2\zeta\omega\dot{x} + \omega^2 x = \frac{I_0}{m}\delta(t - T)$$

Assuming that $\omega = 5$ rad/s and the damping ratio 4/5, find x using Laplace transformation. (You will need to complete the square as explained on the handout.)

Note that after the kick, there is no further force and the system vibrates as a free system. Show graphically from your result that the mass keeps vibrating between negative and positive values though the amplitude of vibration after the kick decreases with time. More generally, it can be seen that if the damping ratio is less than one, the mass keeps vibrating. For damping ratio greater than 1, the amplitude changes sign at most once.

To understand what is going on in vibrations in more general terms, note that the solution of any homogeneous second order constant coefficient equation is always of the form

$$\frac{As + B}{(s - s_1)(s - s_2)}$$

Ignoring the degenerate cases like $s_1 = s_2$, there are two possibilities. The first is that s_1 and s_2 are distinct real numbers. Show from partial fractions that such a solution will give rise to two exponentials in time. The second main possibility is that $s_1 = s_r + is_i$ and $s_2 = s_r - is_i$ are a complex conjugate pair. In that case, check that the above expression multiplies out as

$$\frac{As + B}{(s - s_r)^2 + s_i^2}$$

and that that gives rise to an exponential multiplied by a sine or cosine. Those things are true for any homogeneous 2nd order CC ODE. However, give a clear and solid physical reason that for a freely vibrating damped spring-mass system, you can ignore the possibility of growing exponentials. Also give a solid physical reason that for the undamped spring-mass system, the solution must be pure sines and cosines. In other words, show that simple exponentials, decaying, constant, or growing are not an option if undamped. And that sines and cosines times a growing or decaying exponential are not an option either. Then for an *slightly* damped system, you should get an oscillating solution times a slowly decaying exponential. (I do not see a physical reason why you could not have an oscillating solution for very high damping constant. But the mathematical reason is clear: for high enough damping constant, the discriminant of the quadratic in s cannot be negative. So oscillation is then not possible. There can be only one sign change, and only if the exponentials have coefficients of opposite sign.) Note also that it is the real part of the roots (s_1 and s_2 if real, otherwise s_r) that determines whether there is exponential growth. That is the basis for the root-locus method, where you look where the real part of roots are to determine the stability of your system.

3. The generic linearly damped spring-mass system experiencing an external force with frequency ω can be written as

$$m_1\ddot{x}_1 + c_1\dot{x}_1 + k_1 x_1 = F_1 \cos(\widetilde{\omega} t)$$

Here F_1 is a constant. As seen two questions back, if $\widetilde{\omega}$ is close to the natural frequency of the system and damping is small, mass m_1 may experience severe vibration. But suppose you hang a second mass m_2 from the first using a spring with constant k_2. Then the equation above becomes

$$m_1\ddot{x}_1 + c_1\dot{x}_1 + k_1x_1 = F_1\cos(\widetilde{\omega}t) + k_2(x_2 - x_1)$$

while the second mass satisfies the equation

$$m_2\ddot{x}_2 = -k_2(x_2 - x_1)$$

To keep it simple, assume initial conditions

$$x_1(0) = 0 \quad \dot{x}_1(0) = v_{10} \quad x_2(0) = 0 \quad \dot{x}_2(0) = v_{20}$$

Find the Laplace transforms $\widehat{x}_1$ and $\widehat{x}_2$. You do not have to find x_1 and x_2; you can answer the next questions from what you know about partial fractions.

First, your solution should have a quartic in the bottom for which the roots would be difficult to find. But look for a second at the free solution (i.e. with $F_1 = 0$). Based on your physical arguments in the previous question, you should be able to describe the qualitative nature of the four roots if the damping is low. Then consider the partial fraction nature of the solution for F_1 nonzero (but you can further ignore v_{10} and v_{20}). The terms will correspond to two decaying modes of vibration and one term where m_1 vibrates with frequency $\widetilde{\omega}$. However, if you look a bit closer, you see that if you choose the ratio k_2/m_2 to be $\widetilde{\omega}^2$, the third term disappears. Then the building returns to rest after a transition period, despite the ongoing vibrating force on it! The effect of the force has been eliminated!

You may be astonished by that, since only the ratio of k_2 to m_2 is specified. So you could eliminate the vibration in your building m_1 by suspending a single grain of sand m_2 from it using a very weak spring! (Actually, if you do this, and the natural frequency of the building is close to $\widetilde{\omega}$, and damping is small, then the coefficients of the decaying modes will be very large. That can easily be seen using the same trick as used two questions back.)

4. Solve the system

$$\begin{aligned} x_1' &= 2x_1 + x_2 - 2x_3 \\ x_2' &= 3x_1 - 2x_2 \\ x_3' &= 3x_1 + x_2 - 3x_3 \end{aligned}$$

Find the general solution to this system in vector form and in terms of a fundamental matrix. Then find the vector of integration constants assuming that $x(0) = (1, 7, 3)^{\mathrm{T}}$ and write $\vec{x}$ for that case.

5. Given the system

$$\dot{\vec{x}} = A\vec{x} \qquad A = \begin{pmatrix} 0 & 5 \\ -1 & -2 \end{pmatrix}$$

Find the general solution to this system in vector form and in terms of a fundamental matrix.

12 Ordinary Differential Equations IV

1. Solve the system

$$\dot{\vec{x}} = A\vec{x} \quad \vec{x}(0) = \vec{x}_0 \qquad A = \begin{pmatrix} 1 & 5 & 0 \\ 0 & 1 & 0 \\ 4 & 8 & 1 \end{pmatrix} \quad \vec{x}_0 = \begin{pmatrix} 9 \\ 1 \\ 1 \end{pmatrix}$$

   Use the nonmatrix exponential method. Give a fundamental matrix.

2. Solve the previous question, but this time use the matrix exponential method. Find e^{At}, then find the solution $\vec{x}$ as $e^{At}\vec{x}_0$ and check that it is the same as before.

3. Solve the system

$$\dot{\vec{x}} = A\vec{x} + \vec{g} \quad \vec{x}(0) = \vec{x}_0 \qquad A = \begin{pmatrix} 2 & -3 & 1 \\ 0 & 2 & 4 \\ 0 & 0 & 1 \end{pmatrix} \quad \vec{g} = \begin{pmatrix} 10 \\ 6 \\ -1 \end{pmatrix} e^{2t} \quad \vec{x}_0 = \begin{pmatrix} 5 \\ 11 \\ -2 \end{pmatrix}$$

   Use the nonmatrix exponential method. Give a fundamental matrix.

13 Ordinary Differential Equations V

1. Solve the system

$$\vec{x}\,' = A\vec{x} \qquad A = \begin{pmatrix} 3 & -5 \\ 8 & -3 \end{pmatrix}$$

   Neatly and accurately sketch a comprehensive set of solution curves in the x_1, x_2 plane. Include the eigenvectors, $\vec{f}$, $\vec{u}$, and $\vec{v}$ in the graph if applicable. Get the slopes right.

2. Solve the system

$$\dot{\vec{x}} = A\vec{x} \qquad A = \begin{pmatrix} -6 & -7 \\ 7 & -20 \end{pmatrix}$$

   Neatly and accurately sketch a comprehensive set of solution curves in the x_1, x_2 plane. Include the eigenvectors, $\vec{f}$, $\vec{u}$, and $\vec{v}$ in the graph if applicable. Get the slopes right.

3. (30 pt).

   Please answer in order asked (even if you do the numerics first to guard against errors).

   Consider the autonomous system

$$x' = x + 3y - x^2 \sin y \qquad y' = 2x + y - xy^2$$

   First analyze this system analytically:

   (a) Find the critical points. One critical point is easy. Four more critical points can be found numerically. To help you a bit, their y-values are ± 1.1107 and ± 1.6074.

   (b) Find the matrix of derivatives of vector $\vec{F}$ at each of the five critical point. (Actually, you can use symmetry around the origin and only find three.)

   (c) Use it to analyze each critical point. List type of point and its stability.

   (d) Also find the relevant eigenvectors or $\vec{u}$ and $\vec{v}$ if complex, and a $\vec{f}$ if defective.

   (e) Sketch the solution lines in the immediate vicinity of each critical point in a single x, y plane. Take $-5 \leq x \leq 5$ and $-5 \leq y \leq 5$. Try to get it as accurate as possible. To do so, use the directions of the eigenvectors, $\vec{f}$, or $\vec{u}$ and $\vec{v}$.

   (f) State whether critical point analysis *must* give the right solution near the point. Could the real lines be qualitatively different from what you drew?

   Next draw the complete solution plane as a dense and complete set of solution lines using a numerical solution method.

   Suitable programs to do this can be found on the web. Some I saw previously:

   `http://www.math.uu.nl/people/beukers/phase/newphase.html`

   `http://www.scottsarra.org/applets/dirField1/dirField1.html`

   `http://www.math.rutgers.edu/courses/ODE/sherod/phase-local.html`

See here for Matlab software.[1] (You will need to convert to an ODE by taking the ratio of the equations, and then the software might crash when it divides by zero if it hits a critical point.)

You will need to use a screen grabber to make a copy that you can print. Typically you press Alt+PrintScreen or Shift-PrintScreen to get a printable copy of the active Window.

Comment on how well your predictions came out.

[1] `http://math.rice.edu/~dfield/index.html`

References

[1] F. Ayres and E. Mendelson. *Calculus*. Schaum's Outline Series. McGraw-Hill, 5th edition, 2009.

[2] P.V. O'Neil. *Advanced Engineering Mathematics*. Thomson-Engineering, 6th edition, 2007.