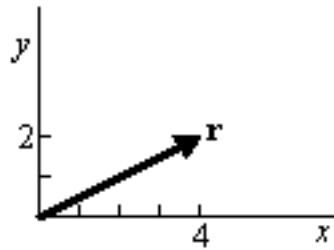# Introduction

Vectors:

- Statics and Dynamics

- Physics

- Geometry

- Computer solutions

- ...

Properties:

- The vector above as a list of numbers:

$$\vec{r} = \vec{x} = \mathbf{x} = \begin{pmatrix} 4 \\ 2 \end{pmatrix} \text{ or } \vec{r}^T = \vec{x}^T = \mathbf{x}^T = (4, 2)$$
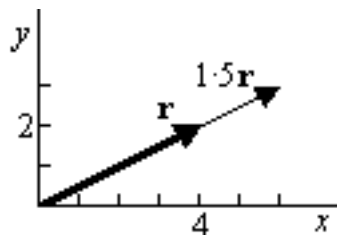
  If the list is written vertically, the vector is called a column vector; if it is written horizontally, it is a row vector. An n-dimensional row vector is equivalent to a $1 \times n$ size matrix; an n-dimensional column vector to a $n \times 1$ matrix,

- Components of the vector above: $r_1 = r_x = x_1 = x = 4, r_2 = r_y = x_2 = y = 2$.
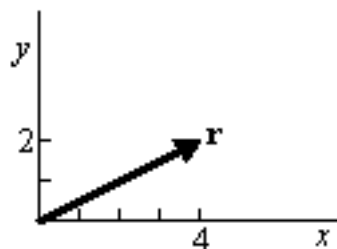
- Addition of vectors:
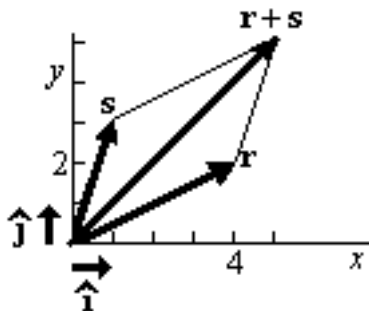
  $(4, 2) + (1, 3) = (4 + 1, 2 + 3) = (5, 5)$.

- Multiplication of a vector by a scalar:

$1.5(4, 2) = (1.5 \ 4, 1.5 \ 2) = (6, 3).$

- Length or norm of a vector:

  - Definition:
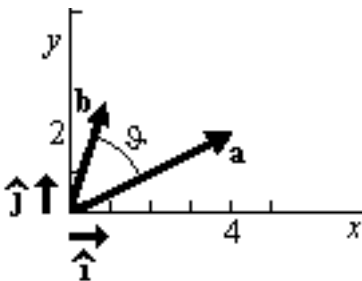 $$||\vec{a}|| = |\vec{a}| = a = \sqrt{a_x^2 + a_y^2 + a_z^2 + \ldots}$$

  - Unit vectors: Unit vectors have length one.

  - Distance: The distance between two points $\vec{r}_1$ and $\vec{r}_2$ is by definition $||\vec{r}_2 - \vec{r}_1||$:
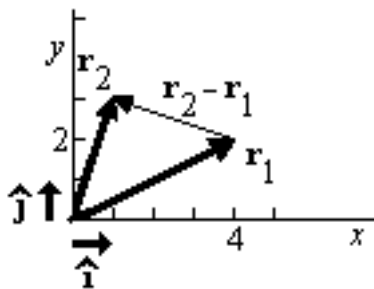
- Dot (scalar) product:

  - Definition:
  $$\vec{a} \cdot \vec{b} = a_x b_x + a_y b_y + a_z b_z + \ldots = ||\vec{a}|| \ ||\vec{b}|| \cos\vartheta$$

- Orthogonality: If the dot product is zero, the vectors are by definition orthogonal to each other.
- Length: $||\vec{a}|| = \sqrt{\vec{a} \cdot \vec{a}}$.

• Projection:
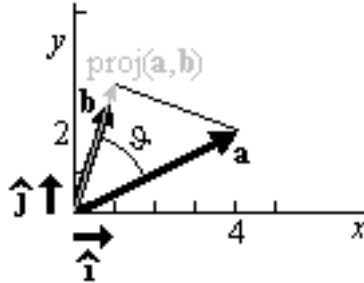
The magnitude of the (orthogonal) component (or coordinate) of $\vec{a}$ in the direction of $\vec{b}$ is:

$$a_b = a\cos(\vartheta) = \vec{a} \cdot \hat{b} = \frac{\vec{a} \cdot \vec{b}}{||\vec{b}||}$$

The projection of $\vec{a}$ onto $\vec{b}$ is

$$\text{proj}(\vec{a}, \vec{b}) = \vec{a}_b = a_b \hat{b} = \frac{\vec{a} \cdot \vec{b}}{||\vec{b}||} \frac{\vec{b}}{||\vec{b}||}$$

1.41

1 1.41, §1 Asked

Given: The vectors
$$\vec{u} = (1, -2, 4) \qquad \vec{v} = (3, 5, 1)$$

Asked: various.

2 1.41, §2 Solution

$$\vec{u} = (1, -2, 4) \qquad \vec{v} = (3, 5, 1)$$

Sum:
$$3\vec{u} - 2\vec{v} = (3 \cdot 1 - 2 \cdot 3, 3 \cdot (-2) - 2 \cdot 5, 3 \cdot 4 - 2 \cdot 1) = (-3, -16, 10)$$

Dot product:
$$\vec{u} \cdot \vec{v} = 1 \cdot 3 + (-2) \cdot 5 + 4 \cdot 1 = -3$$

Norm or length:
$$||\vec{u}|| = \sqrt{1^2 + (-2)^2 + 4^2} = \sqrt{21} \qquad ||\vec{v}|| = \sqrt{3^2 + 5^2 + 1^2} = \sqrt{35}$$

Angle between vectors:
$$\cos(\vartheta) = \vec{u} \cdot \vec{v}/(||\vec{u}|| \, ||\vec{v}||) = -3/\sqrt{21}\sqrt{35} = -3/(7\sqrt{15})$$

Distance between the end points:
$$d(\vec{u}, \vec{v}) = ||\vec{u} - \vec{v}|| = ||(-2, -7, 3)|| = \sqrt{4 + 49 + 9} = \sqrt{62}$$

Projection:
$$\text{proj}(\vec{u}, \vec{v}) = (\vec{u} \cdot \vec{v}/||\vec{v}||)(\vec{v}/||\vec{v}||) = (-3/35)(3, 5, 1) = (-9/35, -3/7, -3/35)$$

Basis vectors

Base vectors:

- Writing an example vector as a combination of the base vectors $\hat{i}$ and $\hat{j}$:
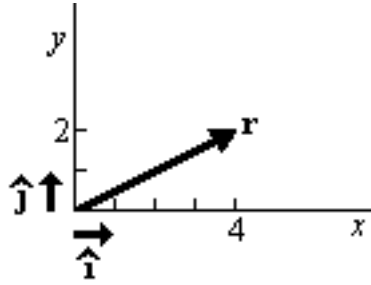
$$\vec{r} = 4\hat{i} + 2\hat{j}$$

- Addition:
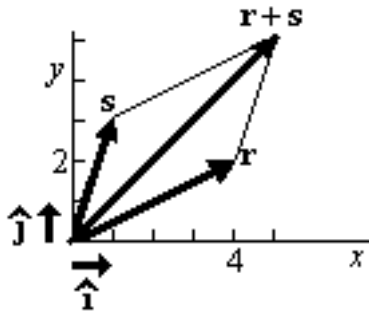
$$4\hat{i} + 2\hat{j} + 1\hat{i} + 3\hat{j} = 5\hat{i} + 5\hat{j}.$$

- Multiplication by a scalar:
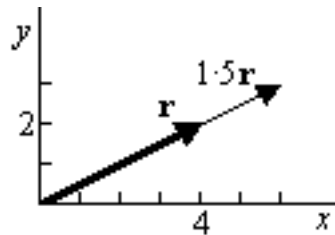
$$1.5(4\hat{i} + 2\hat{j}) = 6\hat{i} + 3\hat{j}.$$

- Dot (scalar) product:
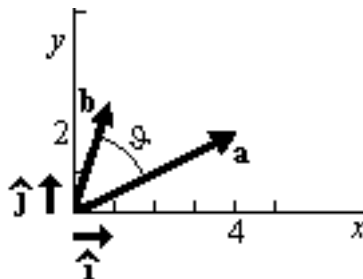
$$(a_x\hat{\imath} + a_y\hat{\jmath} + a_z\hat{k} + \ldots) \cdot (b_x\hat{\imath} + b_y\hat{\jmath} + b_z\hat{k} + \ldots) \cdot = a_xb_x + a_yb_y + a_zb_z + \ldots$$

since $\hat{\imath} \cdot \hat{\imath} = 1$, $\hat{\jmath} \cdot \hat{\jmath} = 1$, $\hat{k} \cdot \hat{k} = 1$, and $\hat{\imath} \cdot \hat{\jmath} = 0$, $\hat{\imath} \cdot \hat{k} = 0$, $\hat{\jmath} \cdot \hat{k} = 1$.

1.48(a)

1   1.48(a), §1 Asked

Given: The vectors
$$\vec{v} = (2,5) \qquad \vec{u}_1 = (1,2) \qquad \vec{u}_2 = (3,5)$$

Asked: Write $\vec{v}$ as a linear combination $a\vec{u}_1 + b\vec{u}_2$, i.e., find a and b so that $\vec{v} = a\vec{u}_1 + b\vec{u}_2$

2   1.48(a), §2 Solution

$$\vec{v} = (2,5) \qquad \vec{u}_1 = (1,2) \qquad \vec{u}_2 = (3,5)$$

Write $\vec{v}$ as a linear combination $a\vec{u}_1 + b\vec{u}_2$, i.e., find a and b so that $\vec{v} = a\vec{u}_1 + b\vec{u}_2$

$$\begin{pmatrix} 2 \\ 5 \end{pmatrix} = a \begin{pmatrix} 1 \\ 2 \end{pmatrix} + b \begin{pmatrix} 3 \\ 5 \end{pmatrix} = \begin{pmatrix} 1a + 3b \\ 2a + 5b \end{pmatrix}$$

$$\begin{array}{rcrcll} a & + & 3b & = & 2 & (1) \\ 2a & + & 5b & = & 5 & (2) \end{array}$$

Eliminate a from equation (2) by substracting 2 times (1):

$$\begin{array}{rcrcll} a & + & 3b & = & 2 & (1) \\ 0 & - & b & = & 1 & (2') = (2) - 2(1) \end{array}$$

Solve from the bottom up, (2') giving that $b = -1$ and then (1) giving that $a = 5$.

1.54

1 1.54, §1 Asked

Given: The vectors

$$\vec{u} = 3\hat{\imath} - 4\hat{\jmath} + 2\hat{k} \qquad \vec{v} = 2\hat{\imath} + 5\hat{\jmath} - 3\hat{k}$$

Asked: Various

2 1.54, §2 Solution

Given:

$$\vec{u} = 3\hat{\imath} - 4\hat{\jmath} + 2\hat{k} \qquad \vec{v} = 2\hat{\imath} + 5\hat{\jmath} - 3\hat{k}$$

Sum:

$$2\vec{u} - 3\vec{v} = 6\hat{\imath} - 8\hat{\jmath} + 4\hat{k} - 6\hat{\imath} - 15\hat{\jmath} + 9\hat{k} = -23\hat{\jmath} + 13\hat{k}$$

Dot product:

$$\vec{u} \cdot \vec{v} = (3\hat{\imath} - 4\hat{\jmath} + 2\hat{k}) \cdot (2\hat{\imath} + 5\hat{\jmath} - 3\hat{k})$$

Use the fact that $\hat{\imath} \cdot \hat{\imath} = \hat{\jmath} \cdot \hat{\jmath} = \hat{k} \cdot \hat{k} = 1$ and $\hat{\imath} \cdot \hat{\jmath} = \hat{\jmath} \cdot \hat{k} = \hat{k} \cdot \hat{\imath} = 0$:

$$\vec{u} \cdot \vec{v} = 3 \; 2 + 0 + 0 - 0 - 4 \; 5 + 0 - 0 + 0 - 2 \; 3 = -20$$

Norm or length:

$$||\vec{u}|| = \sqrt{\vec{u} \cdot \vec{u}} = \sqrt{(3\hat{\imath} - 4\hat{\jmath} + 2\hat{k}) \cdot (3\hat{\imath} - 4\hat{\jmath} + 2\hat{k})}$$

Multiply out as before:

$$||\vec{u}|| = \sqrt{3^2 + (-4)^2 + 2^2} = \sqrt{29}$$

Hyperplanes

A hyperplane in R^n (n-dimensional space) is the collection of points satisfying a single scalar linear equation:

$$a_1 x_1 + a_2 x_2 + \ldots + a_n x_n = d$$

i.e.

$$\vec{n} \cdot \vec{r} = d \qquad \vec{n} = (a_1, a_2, \ldots, a_n)$$
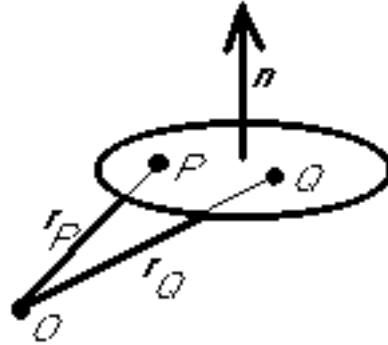
3D: A plane:

$$ax + by + cz = d$$

2D: A line:

$$ax + by = d$$

1.55(a)

1   1.55(a), §1 Asked

Given: The point P with $\vec{r}_P = (1, 2, -3)$ and the vector $\vec{N} = 3\hat{\imath} - 4\hat{\jmath} + 5\hat{k}$.

Asked: The equation for the plane through P and normal to $\vec{N}$.

2   1.55(a), §2 Solution

$$\vec{r}_P = (1, 2, -3) \qquad \vec{N} = 3\hat{\imath} - 4\hat{\jmath} + 5\hat{k}$$

In general

$$\vec{r} \cdot \vec{N} = \vec{r}_P \cdot \vec{N}$$

where $\vec{r} = (x, y, z) = x\hat{\imath} + y\hat{\jmath} + z\hat{k}$.

Plug in the numbers and dot out:

$$3x - 4y + 5z = 1\,3 - 2\,4 - 3\,5 = -20$$

Lines

Line through point P parallel to vector $\vec{s}$:
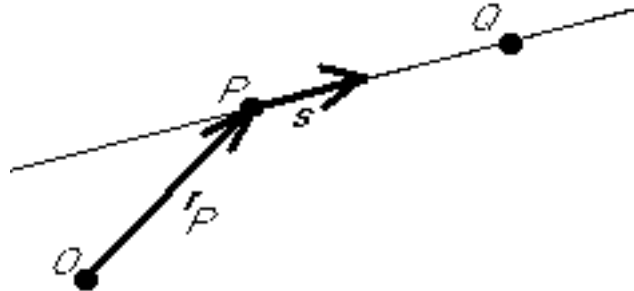
$$\vec{r} = \vec{r}_P + \lambda\vec{s}$$

This applies to any number of dimensions.

1.56(b)

1   1.56(b), §1 Asked

Given: The plane $2x - 3y + 7z = 4$ and the point P with coordinates $(x, y, z) = (1, -5, 7)$.

Asked: The parametric equation for the line ℓ through P and normal to the plane.

2   1.56(b), §2 Solution

Plane $2x - 3y + 7z = 4$ and the point $(1, -5, 7)$.

In general, the equation for the line through P is

$$\vec{r} = \vec{r}_P + \lambda \vec{s}$$

where $\vec{s}$ is any nonzero vector in the direction of the line.

The line is given to be normal to the plane, so the direction of the line is the direction of a normal vector to the plane, which can be picked out of the equation:

$$\vec{r} = (x, y, z) = (1, -5, 7) + \lambda(2, -3, 7) = (1 + 2\lambda, -5 - 3\lambda, 7 + 7\lambda)$$

Curves

Curves in R^n: $\vec{r} = \vec{r}(t)$ with t the parameter. The unit tangent to the curve is $\vec{T} = \dot{\vec{r}}/||\dot{\vec{r}}||$.

1.57

1 1.57, §1 Asked

Given:

$$\vec{r} = t^3\hat{\imath} - t^2\hat{\jmath} + (2t - 3)\hat{k}$$

for $0 \le t \le 5$.

Asked: (a) Find the point P on the curve curresponding to $t = 2$. (b) Find the initial point Q and the terminal point Q'. (c) Find the unit tangent vector $\vec{T}$ to the curve when $t = 2$.

2 1.57, §2 Solution

$$\vec{r} = t^3\hat{\imath} - t^2\hat{\jmath} + (2t - 3)\hat{k}$$

for $0 \le t \le 5$.

Point P: When $t = 2$, $\vec{r} = (8, -4, 1)$.

At end point Q, $t = 0$, $\vec{r} = (0, 0, -3)$; at end point Q', $t = 5$, $\vec{r} = (125, -25, 7)$.
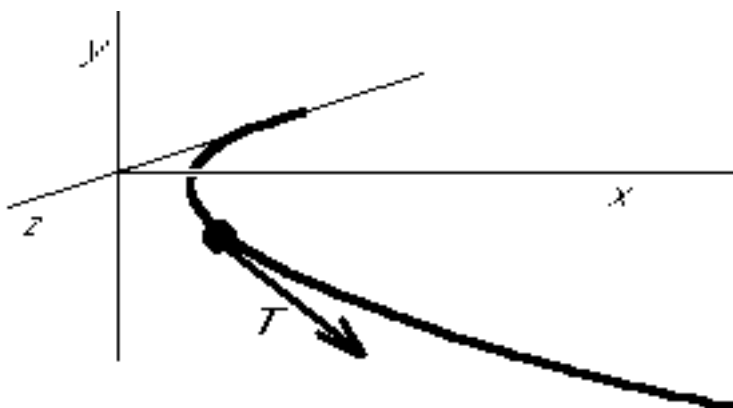
Vector $\vec{T}$ is proportional to

$$\frac{d\vec{r}}{dt} = \begin{pmatrix} 3t^2 \\ -2t \\ 2 \end{pmatrix}.$$

Then

$$\vec{T} = \frac{\mathrm{d}\vec{r}}{\mathrm{d}t} \bigg/ \left| \frac{\mathrm{d}\vec{r}}{\mathrm{d}t} \right| = \begin{pmatrix} 3t^2 \\ -2t \\ 2 \end{pmatrix} \bigg/ \sqrt{9t^4 + 4t^2 + 4}.$$

At $t = 2$

$$\vec{T} = \begin{pmatrix} 12 \\ -4 \\ 2 \end{pmatrix} \bigg/ \sqrt{144 + 16 + 4} = \begin{pmatrix} 6/\sqrt{41} \\ -2/\sqrt{41} \\ 1/\sqrt{41} \end{pmatrix}$$

Tangential planes

Tangent planes to a surface $F(x, y, z) = 0$ at a point P on the surface:

$$\vec{r} \cdot \vec{N} = \vec{r}_P \cdot \vec{N}$$

where N can be taken as the gradient ∇F of F:

$$N = (\partial F/\partial x, \partial F/\partial y, \partial F/\partial z)$$

1.59(b)

1 1.59(b), §1 Asked

Given: The hyperboloid of one sheet

$$x^2 + 3y^2 - 5z^2 = 160$$

and the point P with position vector (3,-2,1) on that hyperboloid.

Asked: A normal vector $\vec{N}$ to the surface at P and the tangent plane at P.

2 1.59(b), §2 Solution

$$x^2 + 3y^2 - 5z^2 = 160 \qquad P = (3, -2, 1)$$

Correct problem:

$$x^2 + 3y^2 - 5z^2 = 16 \qquad P = (3, -2, 1)$$

Bring equation of surface in *standard form* (zero right hand side):

$$x^2 + 3y^2 - 5z^2 - 16 \equiv F(x, y, z) = 0$$

A normal vector to a surface in standard form is given by the gradient of F:

$$\nabla F \equiv \begin{pmatrix} \frac{\partial F}{\partial x} \\ \frac{\partial F}{\partial y} \\ \frac{\partial F}{\partial z} \end{pmatrix} = \begin{pmatrix} 2x \\ 6y \\ -10z \end{pmatrix}$$

At P, $(x, y, z) = (3, -2, 1)$, so:

$$\vec{N} = \nabla F|_P = \begin{pmatrix} 6 \\ -12 \\ -10 \end{pmatrix}$$

Tangent plane:

$$\vec{N} \cdot \vec{r} = \vec{N} \cdot \vec{r}_P$$

or

$$6x - 12y - 10z = 6\ 3 - 12\,(-2) - 10\ 1 = 32$$

Can divide by 2 to simplify:

$$3x - 6y - 5z = 16$$

Cross products

Cross (or vector) product $\vec{a} \times \vec{b}$ (in 3D only):
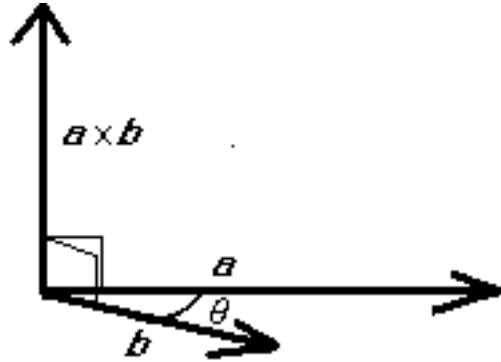
Magnitude:

$$||\vec{a} \times \vec{b}|| = ||\vec{a}||\,||\vec{b}|| \sin \vartheta$$

Direction: $\vec{a} \times \vec{b}$ is normal to both $\vec{a}$ and $\vec{b}$.

$$\vec{a} \times \vec{b} \equiv \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \hat{\imath}(a_2 b_3 - a_3 b_2) + \hat{\jmath}(a_3 b_1 - a_1 b_3) + \hat{k}(a_1 b_2 - a_2 b_1)$$

Reminder: Evaluating small determinants:

$$|a| = a \qquad \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = aei + bfg + cdh - afh - bdi - ceg$$

1.64(a)

1   1.64(a), §1 Asked

Given: The vectors
$$\vec{v} = (1, 2, 3) \qquad \vec{w} = (1, -1, 2)$$

Asked: A unit vector $\vec{u}$ normal to these two.

2   1.64(a), §2 Solution

$$\vec{v} = (1, 2, 3) \qquad \vec{w} = (1, -1, 2)$$

If we cross $\vec{v}$ and $\vec{w}$, we get a vector normal to them. If we divide that cross product by its length it will become a unit vector:

$$\vec{v} \times \vec{w} \equiv \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 1 & 2 & 3 \\ 1 & -1 & 2 \end{vmatrix} = \hat{\imath}(2 \cdot 2 - 3\,(-1)) + \hat{\jmath}(3 \cdot 1 - 1 \cdot 2) + \hat{k}(1\,(-1) - 2 \cdot 1) = (7, 1, -3)$$

$$\vec{u} = \frac{\vec{v} \times \vec{w}}{||\vec{v} \times \vec{w}||} = \frac{(7, 1, -3)}{\sqrt{7^2 + 1^2 + (-3)^2}} = (7/\sqrt{59}, 1/\sqrt{59}, -3/\sqrt{59})$$

Introduction

1   General

The most usual representation of systems on computers and elsewhere is using matrices. Finite element problems, dynamics, fluid mechanics, ..., are almost always matrix problems for the computer.

A matrix A is a table of numbers:

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} & \ldots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \ldots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \ldots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \ldots & a_{mn} \end{pmatrix}$$

An $m \times n$ matrix consists of n column vectors (the columns), or equivalently of m row vectors (the rows).

Conversely, a column vector is equivalent to a matrix with only one column and a row vector is a matrix with only one row.

Square matrices are matrices with the same number of rows as columns.

Index notation:

$$A = \{a_{ij}\} \qquad (i = 1, \ldots, m; j = 1, \ldots, n)$$

where $\{\cdot\}$ indicates "the collection of values" or "set of values".

2   Scalar multiplication

Multiplying a matrix by a scalar (i.e. a number) means multiplying each coefficient by that scalar:

$$kA = \begin{pmatrix} ka_{11} & ka_{12} & ka_{13} & \ldots & ka_{1n} \\ ka_{21} & ka_{22} & ka_{23} & \ldots & ka_{2n} \\ ka_{31} & ka_{32} & ka_{33} & \ldots & ka_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ ka_{m1} & ka_{m2} & ka_{m3} & \ldots & ka_{mn} \end{pmatrix}$$

Just like for vectors.

Scalar multiplication in index notation:

$$B = kA \qquad \Longrightarrow \qquad b_{ij} = ka_{ij} \text{ for all } i \text{ and } j$$

3 Addition

Summation of two matrices adds corresponding coefficients:

$$A + B = \begin{pmatrix} a_{11} + b_{11} & a_{12} + b_{12} & a_{13} + b_{13} & \dots & a_{1n} + b_{1n} \\ a_{21} + b_{21} & a_{22} + b_{22} & a_{23} + b_{23} & \dots & a_{2n} + b_{2n} \\ a_{31} + b_{31} & a_{32} + b_{32} & a_{33} + b_{33} & \dots & a_{3n} + b_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{m1} + b_{m1} & a_{m2} + b_{m2} & a_{m3} + b_{m3} & \dots & a_{mn} + b_{mn} \end{pmatrix}$$

(just like for vectors.) The matrices *must* be of the same size.

Summation in index notation:

$$C = A + B \qquad \Longrightarrow \qquad c_{ij} = a_{ij} + b_{ij} \text{ for all } i \text{ and } j$$

4 Zero matrices

Zero matrices have *all* coefficients zero. Adding a zero matrix to a matrix does not change the matrix.

2.37(b)

1 2.37(b), §1 Asked

Given:

$$A = \begin{pmatrix} 1 & 2 \\ 3 & -4 \end{pmatrix} \qquad B = \begin{pmatrix} 5 & 0 \\ -6 & 7 \end{pmatrix}$$

Asked:

$$2A + 3B$$

(3, not 32).

2 2.37(b), §2 Solution

$$2\begin{pmatrix} 1 & 2 \\ 3 & -4 \end{pmatrix} + 3\begin{pmatrix} 5 & 0 \\ -6 & 7 \end{pmatrix} = \begin{pmatrix} 2 & 4 \\ 6 & -8 \end{pmatrix} + \begin{pmatrix} 15 & 0 \\ -18 & 21 \end{pmatrix} = \begin{pmatrix} 17 & 4 \\ -12 & 13 \end{pmatrix}$$

Matrix multiplication

1 General

Matrix multiplication is defined in terms of the *row-column* product:

$$C = AB = \begin{pmatrix} a_{11} & a_{12} & a_{13} & \ldots & a_{1p} \\ \vdots & \vdots & \vdots & \ldots & \vdots \\ \vdots & \vdots & \vdots & \ldots & \vdots \\ a_{i1} & a_{i2} & a_{i3} & \ldots & a_{ip} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \ldots & a_{mp} \end{pmatrix} \begin{pmatrix} b_{11} & \ldots & b_{1j} & \ldots & \ldots & b_{1n} \\ b_{21} & \ldots & b_{2j} & \ldots & \ldots & b_{2n} \\ b_{31} & \ldots & b_{3j} & \ldots & \ldots & b_{3n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ b_{p1} & \ldots & b_{pj} & \ldots & \ldots & b_{pn} \end{pmatrix}$$

$$C = \begin{pmatrix} c_{11} & \ldots & \ldots & \ldots & c_{1n} \\ \vdots & \vdots & \vdots & \ldots & \vdots \\ \vdots & \vdots & \vdots & \ldots & \vdots \\ \ldots & \ldots & c_{ij} & \ldots & \ldots \\ \vdots & \vdots & \vdots & \ldots & \vdots \\ c_{m1} & \ldots & \ldots & \ldots & c_{mn} \end{pmatrix}$$

where

$$c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \ldots + a_{ip}b_{pj}$$

In other words, c_{ij} is the dot product of the i-th *row*-vector of A times the j-th *column*-vector of B:

$$AB = \begin{pmatrix} \vec{a}_1^T \\ \vec{a}_2^T \\ \vec{a}_3^T \\ \vdots \\ \vec{a}_m^T \end{pmatrix} \begin{pmatrix} \vec{b}_1 & \vec{b}_2 & \vec{b}_3 & \ldots & \vec{b}_n \end{pmatrix} = \begin{pmatrix} \vec{a}_1^T \cdot \vec{b}_1 & \vec{a}_1^T \cdot \vec{b}_2 & \vec{a}_1^T \cdot \vec{b}_3 & \ldots & \vec{a}_1^T \cdot \vec{b}_n \\ \vec{a}_2^T \cdot \vec{b}_1 & \vec{a}_2^T \cdot \vec{b}_2 & \vec{a}_2^T \cdot \vec{b}_3 & \ldots & \vec{a}_2^T \cdot \vec{b}_n \\ \vec{a}_3^T \cdot \vec{b}_1 & \vec{a}_3^T \cdot \vec{b}_2 & \vec{a}_3^T \cdot \vec{b}_3 & \ldots & \vec{a}_3^T \cdot \vec{b}_n \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \vec{a}_m^T \cdot \vec{b}_1 & \vec{a}_m^T \cdot \vec{b}_2 & \vec{a}_m^T \cdot \vec{b}_3 & \ldots & \vec{a}_m^T \cdot \vec{b}_n \end{pmatrix}$$

The dots in the above product can be omitted since the matrix product of a row vector times a column vector is by definition the same as the dot product of those vectors.

Multiplication in index notation:

$$C = AB \quad \implies \quad c_{ij} = \sum_k a_{ik}b_{kj} \text{ for all } i \text{ and } j$$

The summation is over *neighboring indices*.

For matrices to be multiplied, the second dimension of A must be the same as the first dimension of B.

Matrix multiplication does not ordinarily commute:

$$AB \neq BA$$

2   Unit matrix

The unit (or identity) matrix I is like the number 1 for numbers: multiplying by I does not change a matrix.

Form of the unit matrix:

$$I = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix}$$

Note, blocks of zeros are often omitted, (or written as a humongous zero,) so

$$I = \begin{pmatrix} 1 & & & & \\ & 1 & & & \\ & & 1 & & \\ & & & \ddots & \\ & & & & 1 \end{pmatrix}$$

Index notation

$$I_{ij} = \delta_{ij} \qquad (= 1 \text{ if } i = j; \quad = 0 \text{ if } i \neq j)$$

The tensor δ_{ij} is called the Kronecker delta.

2.38(b)

1   2.38(b), §1 Asked

Given:

$$A = \begin{pmatrix} 1 & 2 \\ 3 & -4 \end{pmatrix} \qquad B = \begin{pmatrix} 5 & 0 \\ -6 & 7 \end{pmatrix} \qquad C = \begin{pmatrix} 1 & -3 & 4 \\ 2 & 6 & -5 \end{pmatrix}$$

Asked:

$$BC \text{ and } A(BC)$$

2   2.38(b), §2 Solution

$$BC = \begin{pmatrix} 5 & 0 \\ -6 & 7 \end{pmatrix} \begin{pmatrix} 1 & -3 & 4 \\ 2 & 6 & -5 \end{pmatrix}$$

$$= \begin{pmatrix} 5\cdot 1 + 0\cdot 2 & 5\cdot -3 + 0\cdot 6 & 5\cdot 4 + 0\cdot -5 \\ -6\cdot 1 + 7\cdot 2 & -6\cdot -3 + 7\cdot 6 & -6\cdot 4 + 7\cdot -5 \end{pmatrix}$$

$$= \begin{pmatrix} 5 & -15 & 20 \\ 8 & 60 & -59 \end{pmatrix}$$

$$A(BC) = \begin{pmatrix} 1 & 2 \\ 3 & -4 \end{pmatrix} \begin{pmatrix} 5 & -15 & 20 \\ 8 & 60 & -59 \end{pmatrix}$$

$$= \begin{pmatrix} 21 & 105 & -98 \\ -17 & -285 & 296 \end{pmatrix}$$

Transpose matrices

1 General

Transposing a matrix turns the columns into rows and vice-versa

$$
A = \begin{pmatrix}
a_{11} & a_{12} & \ldots & a_{1n} \\
a_{21} & a_{22} & \ldots & a_{2n} \\
a_{31} & a_{32} & \ldots & a_{3n} \\
\vdots & \vdots & \ddots & \vdots \\
a_{m1} & a_{m2} & \ldots & a_{mn}
\end{pmatrix}
\quad
A^T = \begin{pmatrix}
a_{11} & a_{21} & a_{31} & \ldots & a_{m1} \\
a_{12} & a_{22} & a_{32} & \ldots & a_{m2} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
a_{1n} & a_{2n} & a_{3n} & \ldots & a_{mn}
\end{pmatrix}
$$

Similarly, transposing turns a column vector into a row vector and vice-versa.

Another way of thinking about it is that the elements are flipped over around the *"main diagonal"*, which runs from top left to bottom right:

$$
\begin{pmatrix}
\boxed{a_{11}} & a_{12} & a_{13} & \ldots & a_{1n} \\
a_{21} & \boxed{a_{22}} & a_{23} & \ldots & a_{2n} \\
a_{31} & a_{32} & \boxed{a_{33}} & \ldots & a_{3n} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
a_{n1} & a_{n2} & a_{n3} & \ldots & \boxed{a_{nn}}
\end{pmatrix}
$$

(The sum of the elements on the main diagonal is called the *trace* of the matrix.)

Note that $\left(A^T \right)^T = A$.

Transpose in index notation:
$$
a_{ij}^T = a_{ji} \text{ for all } i \text{ and } j
$$

Note that in index notation, the main diagonal consists of the elements where $i = j$. These stay put during transposing.

Transposing matrix products:
$$
(AB)^T = B^T A^T
$$

For complex matrices, the normal generalization of transpose is "Hermitian conjugate", where you take the complex conjugate of each complex number, *in addition* to interchanging rows and columns: $A^H \equiv \bar{A}^T$, or $a_{ij}^H = \bar{a}_{ji}$.

2 Special matrices

Symmetric matrices satisfy

$$S^T = S$$

Symmetric matrices are very common in engineering. For example, most statics deals with symmetric matrices, as does solid body dynamics, and a lot of the simpler fluid flows.

Complex matrices for which $A^H = A$ are called "Hermitian matrices." The complex Fourier transform is really a Hermitian matrix.

Skew-symmetric matrices satisfy

$$K^T = -K$$

Skew-symmetric matrices determine the velocity field in solid body motion, and other fields involving cross products.

Diagonal matrices have only nonzero elements on the main diagonal:

$$D = \begin{pmatrix} d_{11} & 0 & 0 & \dots & 0 \\ 0 & d_{22} & 0 & \dots & 0 \\ 0 & 0 & d_{33} & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & d_{nn} \end{pmatrix}$$

An example is the unit matrix. In index notation, a matrix is diagonal iff $d_{ij} = 0$ if $i \neq j$.

Upper triangular matrices have only nonzero elements on and above the main diagonal:

$$U = \begin{pmatrix} u_{11} & u_{12} & u_{13} & \dots & u_{1n} \\ 0 & u_{22} & u_{23} & \dots & u_{2n} \\ 0 & 0 & u_{33} & \dots & u_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & u_{nn} \end{pmatrix}$$

In index notation, $u_{ij} = 0$ if $j < i$.

Lower triangular matrices:

$$L = \begin{pmatrix} l_{11} & 0 & 0 & \dots & 0 \\ l_{21} & l_{22} & 0 & \dots & 0 \\ l_{31} & l_{32} & l_{33} & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ l_{n1} & l_{n2} & l_{n3} & \dots & l_{nn} \end{pmatrix}$$

In index notation, $l_{ij} = 0$ if $j > i$.

The transpose of an upper triangular matrix is a lower triangular one and vice-versa.

2.40(b)

1   2.40(b), §1 Asked

Given:

$$B = \begin{pmatrix} 5 & 0 \\ -6 & 7 \end{pmatrix}$$

Asked:

$$B^T$$

2   2.40(b), §2 Solution

$$B = \begin{pmatrix} 5 & 0 \\ -6 & 7 \end{pmatrix} \qquad \implies \qquad B^T = \begin{pmatrix} 5 & -6 \\ 0 & 7 \end{pmatrix}$$

Inverse Matrices

1 General

Inverse matrices are like inverses for numbers:

$$AA^{-1} = A^{-1}A = I$$

Note that $\left(A^{-1}\right)^{-1} = A$.

The inverse only exists when the determinant of the matrix, $|A|$, is nonzero.

To get the inverses of *small* matrices, you could use the procedure of taking "minors". In index notation:

$$a_{ij}^{-1\,T} = (-1)^{i+j}|A_{ij}|/|A|$$

where A_{ij} is the matrix A after you remove the column and row of element a_{ij}. See the example problems.

Inverting products:

$$(AB)^{-1} = B^{-1}A^{-1}$$

Transposing and inversing commute:

$$\left(A^T\right)^{-1} = \left(A^{-1}\right)^T$$

2 Orthonormal matrices

Orthonormal (orthogonal) matrices are matrices in which the columns vectors form an orthonormal set (each column vector has length one and is orthogonal to all the other colum vectors).

For square orthonormal matrices, the inverse is simply the transpose,

$$Q^{-1} = Q^T$$

This can be seen from:

$$Q^T Q = \begin{pmatrix} \vec{q}_1^{\,T} \\ \vec{q}_2^{\,T} \\ \vec{q}_3^{\,T} \\ \vdots \\ \vec{q}_n^{\,T} \end{pmatrix} \begin{pmatrix} \vec{q}_1 & \vec{q}_2 & \vec{q}_3 & \cdots & \vec{q}_n \end{pmatrix}$$

$$= \begin{pmatrix} \vec{q}_1^T \vec{q}_1 & \vec{q}_1^T \vec{q}_2 & \vec{q}_1^T \vec{q}_3 & \cdots & \vec{q}_1^T \vec{q}_n \\ \vec{q}_2^T \vec{q}_1 & \vec{q}_2^T \vec{q}_2 & \vec{q}_2^T \vec{q}_3 & \cdots & \vec{q}_2^T \vec{q}_n \\ \vec{q}_3^T \vec{q}_1 & \vec{q}_3^T \vec{q}_2 & \vec{q}_3^T \vec{q}_3 & \cdots & \vec{q}_3^T \vec{q}_n \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \vec{q}_n^T \vec{q}_1 & \vec{q}_n^T \vec{q}_2 & \vec{q}_n^T \vec{q}_3 & \cdots & \vec{q}_n^T \vec{q}_n \end{pmatrix} = I$$

It can be seen, from inverting the order of the factors, that the rows of a square orthonormal matrices are an orthonormal set too.

Complex orthogonal matrices are called "unitary".

2.53(b)

1   2.53(b), §1 Asked

Asked: Find the inverse of

$$\begin{pmatrix} 2 & 3 \\ 4 & 5 \end{pmatrix}$$

2   2.53(b), §2 Solution

Use minors:

$$a_{ij}^{-1\,T} = (-1)^{i+j}|A_{ij}|/|A|$$

$$\begin{pmatrix} 2 & 3 \\ 4 & 5 \end{pmatrix}^{-1} = \frac{1}{\begin{vmatrix} 2 & 3 \\ 4 & 5 \end{vmatrix}}\begin{pmatrix} 5 & -4 \\ -3 & 2 \end{pmatrix}^{T}$$

$$= \frac{1}{-2}\begin{pmatrix} 5 & -3 \\ -4 & 2 \end{pmatrix} = \begin{pmatrix} -5/2 & 3/2 \\ 2 & -1 \end{pmatrix}$$

2.54(a)

1   2.54(a), §1 Asked

Asked: Find the inverse of

$$\begin{pmatrix} 1 & 1 & 2 \\ 1 & 2 & 5 \\ 1 & 3 & 7 \end{pmatrix}$$

2   2.54(a), §2 Solution

Use minors:

$$\begin{pmatrix} 1 & 1 & 2 \\ 1 & 2 & 5 \\ 1 & 3 & 7 \end{pmatrix}^{-1} =$$

$$\frac{1}{\begin{vmatrix} 1 & 1 & 2 \\ 1 & 2 & 5 \\ 1 & 3 & 7 \end{vmatrix}} \begin{pmatrix} \begin{vmatrix} 2 & 5 \\ 3 & 7 \end{vmatrix} & -\begin{vmatrix} 1 & 5 \\ 1 & 7 \end{vmatrix} & \begin{vmatrix} 1 & 2 \\ 1 & 3 \end{vmatrix} \\ -\begin{vmatrix} 1 & 2 \\ 3 & 7 \end{vmatrix} & \begin{vmatrix} 1 & 2 \\ 1 & 7 \end{vmatrix} & -\begin{vmatrix} 1 & 1 \\ 1 & 3 \end{vmatrix} \\ \begin{vmatrix} 1 & 2 \\ 2 & 5 \end{vmatrix} & -\begin{vmatrix} 1 & 2 \\ 1 & 5 \end{vmatrix} & \begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix} \end{pmatrix}^{T}$$

and since the determinant in the bottom is -1,

$$\begin{pmatrix} 1 & 1 & -1 \\ 2 & -5 & 3 \\ -1 & 2 & -1 \end{pmatrix}$$

A quicker way to find determinants of large matrices will be given in chapter 3.
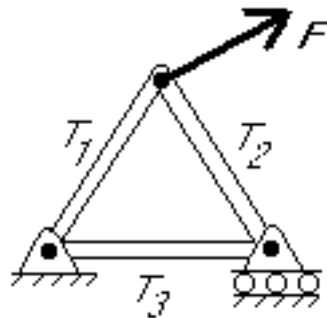
Introduction

Determinate linear systems:

- Trusses;

- FEM codes;

- Finite difference codes;

- Economics;

- Design optimization;

- CAD/CAM;

- ...

For a unique solution (under all conditions):

- The number of equations must be the number of unknowns (square matrix A);

- The matrix must be nonsingular (the determinant $|A|$ must be nonzero).
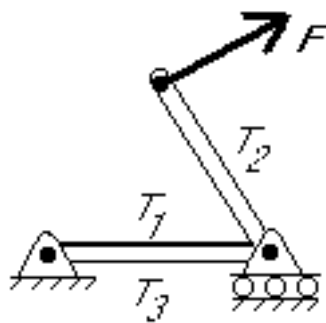
Cramer's rule is useless for anything but very small systems. The general purpose method is Gaussian elimination (LU-decomposition.)

3.51(a)

1 3.51(a), §1 Asked

Asked: Solve

$$
\begin{aligned}
2x + 3y &= 1 \qquad (1)\\
5x + 7y &= 3 \qquad (2)
\end{aligned}
$$

2 3.51(a), §2 Graphically
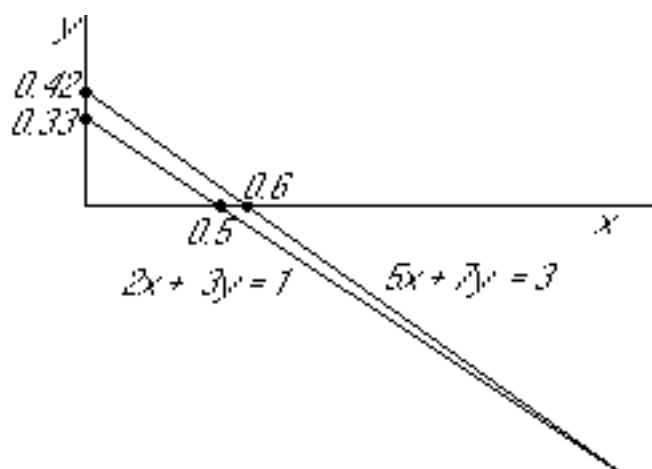
One unique solution point $(x, y) = (2, -1)$

3 3.51(a), §3 Elimination

Gaussian elimination:

$$
\begin{aligned}
2x + 3y &= 1 \qquad (1)\\
5x + 7y &= 3 \qquad (2)
\end{aligned}
$$

A. Forward Elimination:

Use (1) to eliminate x from (2):

$$
\begin{aligned}
2x + 3y &= 1 \qquad (1)\\
- y &= 1 \qquad (2') = 2(2) - 5(1)
\end{aligned}
$$

Note: you always must keep at least some of the original equation.

B. Back Substitution:

Solve (2') to find $y = -1$. Then use that value in (1) to find $x = 2$.

4   3.51(a), §4 Matrix Form

$$
\begin{array}{rcrcll}
2x & + & 3y & = & 1 & \quad (1) \\
5x & + & 7y & = & 3 & \quad (2)
\end{array}
$$

This can be written as

$$
\begin{pmatrix} 2 & 3 \\ 5 & 7 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \end{pmatrix} \qquad \begin{array}{l}(1)\\(2)\end{array}
$$

or $A\vec{x} = \vec{b}$ where

$$
A = \begin{pmatrix} 2 & 3 \\ 5 & 7 \end{pmatrix} \qquad \vec{x} = \begin{pmatrix} x \\ y \end{pmatrix} \qquad \vec{b} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}
$$

More concisely, only write the augmented matrix:

$$
\left(\begin{array}{cc|c} 2 & 3 & 1 \\ 5 & 7 & 3 \end{array} \right) \qquad \begin{array}{l}(1)\\(2)\end{array}
$$

After elimination:

$$
\left(\begin{array}{cc|c} 2 & 3 & 1 \\ 0 & -1 & 1 \end{array} \right) \qquad \begin{array}{l}(1)\\(2') = 2(2) - 5(1)\end{array}
$$

5   3.51(a), §5 Determinant

$$
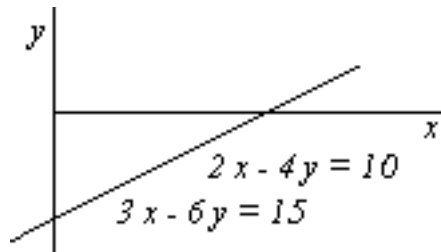|A| = \begin{vmatrix} 2 & 3 \\ 5 & 7 \end{vmatrix} = 2\,7 - 5\,3 = -1
$$

3.51(d)

1 3.51(d), §1 Asked

Asked: Solve

$$
\begin{array}{rcrcll}
2x & - & 4y & = & 10 & \quad (1) \\
3x & - & 6y & = & 15 & \quad (2)
\end{array}
$$

2 3.51(d), §2 Graphically

One complete line of solution points $y = -2.5 + 0.5x$

3 3.51(d), §3 Elimination

Gaussian elimination:

$$
\begin{array}{rcrcll}
2x & - & 4y & = & 10 & \quad (1) \\
3x & - & 6y & = & 15 & \quad (2)
\end{array}
$$

A. Forward Elimination:

Use (1) to eliminate x from (2):

$$
\begin{array}{rcrcll}
2x & - & 4y & = & 10 & \quad (1) \\
 & & 0 & = & 0 & \quad (2') = 2(2) - 3(1)
\end{array}
$$

The second equation is trivial.

B. Back Substitution:

Solve (1) to find $x = 5 + 2y$. y can be anything, but for each possible y there is only one corresponding x.

4 3.51(d), §4 Matrix Form

$$
\begin{array}{rrrclc}
2x & - & 4y & = & 10 & \quad (1) \\
3x & - & 6y & = & 15 & \quad (2)
\end{array}
$$

Rewritten:

$$
\left(\begin{array}{cc|c}
2 & -4 & 10 \\
3 & -6 & 15
\end{array} \right)
\qquad
\begin{array}{l}
(1) \\
(2)
\end{array}
$$

After elimination:

$$
\left(\begin{array}{cc|c}
2 & -4 & 10 \\
0 & 0 & 0
\end{array} \right)
\qquad
\begin{array}{l}
(1) \\
(2') = 2(2) - 3(1)
\end{array}
$$

5 3.51(d), §5 Determinant

$$
|A| = \begin{vmatrix} 2 & -4 \\ 3 & -6 \end{vmatrix} = 2\,(-6) - 4\,(-3) = 0
$$

Echelon form

When solving equations, we reduce the matrix to echelon form. It is in echelon form when the first nonzero element, if any, in each row is to the right of the first nonzero element of the previous row

$$
\begin{pmatrix}
0 & \boxed{P} & \cdot & \cdot & & \cdot & \cdot & \cdot \\
0 & 0 & 0 & \boxed{P} & & \cdot & \cdot & \cdot \\
0 & 0 & 0 & 0 & & \boxed{P} & \cdot & \cdot \\
0 & 0 & 0 & 0 & & 0 & 0 & 0
\end{pmatrix}
$$

These first nonzero elements will be called "pivots."

Example echelon form

$$
\begin{pmatrix}
\boxed{1} & 2 & -1 & 3 \\
0 & 0 & \boxed{6} & -3 \\
0 & 0 & 0 & 0
\end{pmatrix}
\qquad
\begin{matrix}
(1) \\
(2') \\
(3'')
\end{matrix}
$$

Example nonechelon form

$$
\begin{pmatrix}
\boxed{1} & 2 & -1 & 3 \\
0 & 0 & \boxed{6} & -3 \\
0 & \boxed{1} & 0 & 0
\end{pmatrix}
\qquad
\begin{matrix}
(1) \\
(2') \\
(3'')
\end{matrix}
$$

Another nonechelon form

$$
\begin{pmatrix}
\boxed{1} & 2 & -1 & 3 \\
0 & 0 & 0 & 0 \\
0 & \boxed{1} & 0 & 0
\end{pmatrix}
\qquad
\begin{matrix}
(1) \\
(2') \\
(3'')
\end{matrix}
$$

You *must* reduce your systems *completely* to echelon form. You may not delete any rows as the book says.

1.53(c)

1 1.53(c), §1 Asked

Asked: Solve:

$$\left(\begin{array}{ccc|c} 1 & 2 & 3 & 3 \\ 2 & 3 & 8 & 4 \\ 5 & 8 & 19 & 11 \end{array}\right) \qquad \begin{array}{l} (1) \\ (2) \\ (3) \end{array}$$

2 1.53(c), §2 Elimination

$$\left(\begin{array}{ccc|c} 1 & 2 & 3 & 3 \\ 2 & 3 & 8 & 4 \\ 5 & 8 & 19 & 11 \end{array}\right) \qquad \begin{array}{l} (1) \\ (2) \\ (3) \end{array}$$

Forward elimination:

$$\left(\begin{array}{ccc|c} 1 & 2 & 3 & 3 \\ 0 & -1 & 2 & -2 \\ 0 & -2 & 4 & -4 \end{array}\right) \qquad \begin{array}{l} (1) \\ (2') = (2) - 2(1) \\ (3') = (3) - 5(1) \end{array}$$

$$\left(\begin{array}{ccc|c} 1 & 2 & 3 & 3 \\ 0 & -1 & 2 & -2 \\ 0 & 0 & 0 & 0 \end{array}\right) \qquad \begin{array}{l} (1) \\ (2') \\ (3'') = (3') - 2(2') \end{array}$$

Echelon form. *You must bring it completely to this form.*

3 1.53(c), §3 Solution

$$\left(\begin{array}{ccc|c} 1 & 2 & 3 & 3 \\ 0 & -1 & 2 & -2 \\ 0 & 0 & 0 & 0 \end{array}\right) \qquad \begin{array}{l} (1) \\ (2') \\ (3'') = (3') - 4(2') \end{array}$$

Back substitution: From $(3'')$ nothing, then from $(2')$, $y = 2 + 2z$, this in (1) to give $x = 3 - 2(2 + 2z) - 3z = -1 - 7z$.

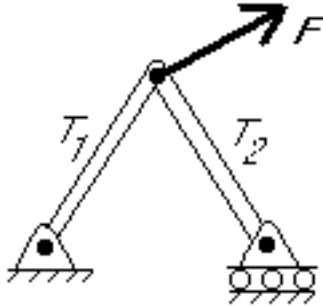# 4 1.53(c), §4 Determinant

$$\begin{vmatrix} 1 & 2 & 3 \\ 2 & 3 & 8 \\ 5 & 8 & 19 \end{vmatrix} = 1\cdot3\cdot19 + 2\cdot8\cdot5 + 3\cdot2\cdot8 - 1\cdot8\cdot8 - 2\cdot2\cdot19 - 3\cdot3\cdot5 = 0$$
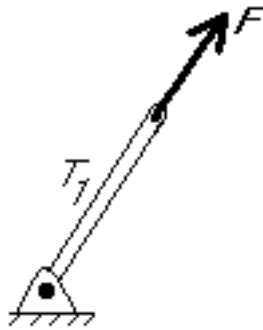
Gaussian elimination

1 Rectangular systems

- Misdesigned systems;

- Systems with limited forces;

- Statically indeterminate systems;
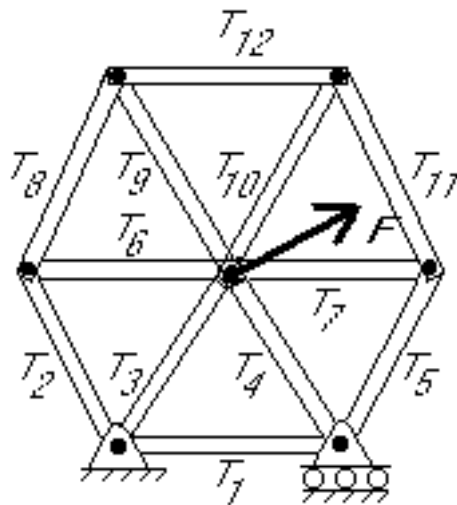
- Measured data;

- Compression;

- Redistribution;

- ...

Matrix shapes:

$$
\begin{pmatrix} \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}
\quad
\begin{pmatrix} \cdot & \cdot \\ \cdot & \cdot \\ \cdot & \cdot \end{pmatrix}
$$

There may be no solution, a unique solution, or infinitely many solutions, depending on circumstances.

2   Partial Pivoting

$$
\left(\begin{array}{cccc|c}
\boxed{1} & 2 & 3 & 4 & 5 \\
2 & 4 & 6 & 9 & 10 \\
1 & 2 & 4 & 5 & 6 \\
1 & 2 & 4 & 6 & 7
\end{array} \right)
\qquad
\begin{array}{c}
(1) \\ (2) \\ (3) \\ (4)
\end{array}
$$

$$
\left(\begin{array}{cccc|c}
\boxed{1} & 2 & 3 & 4 & 5 \\
0 & |\overline{0} & \overline{0} & \overline{1} & \overline{0} \\
0 & |0 & \boxed{1} & 1 & 1 \\
0 & |0 & 1 & 2 & 2
\end{array} \right)
\qquad
\begin{array}{c}
(1) \\ (2') \\ (3') \\ (4')
\end{array}
$$

I must interchange $(2')$ and $(3')$ before proceeding.

$$
\left(\begin{array}{cccc|c}
\boxed{1} & 2 & 3 & 4 & 5 \\
0 & 0 & \boxed{1} & 1 & 1 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 1 & 2 & 2
\end{array} \right)
\qquad
\begin{array}{c}
(1) \\ (3') \\ (2') \\ (4')
\end{array}
$$

$$
\left(\begin{array}{cccc|c}
\boxed{1} & 2 & 3 & 4 & 5 \\
0 & 0 & \boxed{1} & 1 & 1 \\
0 & 0 & 0 & |\overline{1} & \overline{0} \\
0 & 0 & 0 & |1 & 1
\end{array} \right)
\qquad
\begin{array}{c}
(1) \\ (3') \\ (2') \\ (4') - (3')
\end{array}
$$

For best accuracy: always use the pivot with the largest absolute magnitude that you can.

$$\left(\begin{array}{cccc|c} 1 & 2 & 3 & 4 & 5 \\ \boxed{2} & 4 & 6 & 9 & 10 \\ 1 & 2 & 4 & 5 & 6 \\ 1 & 2 & 4 & 6 & 7 \end{array}\right) \qquad \begin{array}{c} (1) \\ (2) \\ (3) \\ (4) \end{array}$$

$$\left(\begin{array}{cccc|c} \boxed{2} & 4 & 6 & 9 & 10 \\ 1 & 2 & 3 & 4 & 5 \\ 1 & 2 & 4 & 5 & 6 \\ 1 & 2 & 4 & 6 & 7 \end{array}\right) \qquad \begin{array}{c} (2) \\ (1) \\ (3) \\ (4) \end{array}$$

$$\left(\begin{array}{cccc|c} \boxed{2} & 4 & 6 & 9 & 10 \\ 0 & 0 & 0 & -\frac{1}{2} & 0 \\ 0 & 0 & \boxed{1} & \frac{1}{2} & 1 \\ 0 & 0 & 1 & 1\frac{1}{2} & 2 \end{array}\right) \qquad \begin{array}{c} (2) \\ (1') \\ (3') \\ (4') \end{array}$$

$$\left(\begin{array}{cccc|c} \boxed{2} & 4 & 6 & 9 & 10 \\ 0 & 0 & \boxed{1} & \frac{1}{2} & 1 \\ 0 & 0 & 0 & -\frac{1}{2} & 0 \\ 0 & 0 & 1 & 1\frac{1}{2} & 2 \end{array}\right) \qquad \begin{array}{c} (2) \\ (3') \\ (1') \\ (4') \end{array}$$

$$\left(\begin{array}{cccc|c} \boxed{2} & 4 & 6 & 9 & 10 \\ 0 & 0 & \boxed{1} & \frac{1}{2} & 1 \\ 0 & 0 & 0 & -\frac{1}{2} & 0 \\ 0 & 0 & 0 & \boxed{1} & 1 \end{array}\right) \qquad \begin{array}{c} (2) \\ (3') \\ (1') \\ (4'') \end{array}$$

$$\left(\begin{array}{cccc|c} \boxed{2} & 4 & 6 & 9 & 10 \\ 0 & 0 & \boxed{1} & \frac{1}{2} & 1 \\ 0 & 0 & 0 & \boxed{1} & 1 \\ 0 & 0 & 0 & -\frac{1}{2} & 0 \end{array}\right) \qquad \begin{array}{c} (2) \\ (3') \\ (4'') \\ (1') \end{array}$$

$$\left(\begin{array}{cccc|c} \boxed{2} & 4 & 6 & 9 & 10 \\ 0 & 0 & \boxed{1} & \frac{1}{2} & 1 \\ 0 & 0 & 0 & \boxed{1} & 1 \\ 0 & 0 & 0 & 0 & \frac{1}{2} \end{array}\right) \qquad \begin{array}{c} (2) \\ (3') \\ (4'') \\ (1'') \end{array}$$

3 Reduction Algorithm

The following algorithm corrects the one in section 3.6 in the book:

1. Start assuming that the pivot will be the coefficient of the first unknown in the first equation.

2. If that coefficient is zero, look below the coefficient for one that is nonzero and swap equations to replace the zero coefficient with the nonzero one.

3. If there are no nonzero coefficients below either, go to the next unknown, i.e. move one place to the right in the matrix and repeat the previous step. (If there are no more unknowns, you are done.)

4. With the obtained nonzero pivot, create zeros below it.

5. Go process the submatrix consisting of the remaining equations below the pivot and the remaining unknowns beyond the pivot in the same way.

What is wrong with the book: Step 1 in the book procedure is impossible if the first row is zero. *Deleting equations or unknowns is a no-no with your instructor.* And step 4 may not be possible since there may not be any equations left after step 3 following the book.

3.54

1 3.54(a), §1 Asked

Asked: Solve:

$$\left(\begin{array}{cc|c} 1 & -2 & 5 \\ 2 & 3 & 3 \\ 3 & 2 & 7 \end{array}\right) \qquad \begin{array}{c} (1) \\ (2) \\ (3) \end{array}$$

2 3.54(a), §2 Solution

$$\left(\begin{array}{cc|c} \boxed{1} & -2 & 5 \\ 2 & 3 & 3 \\ 3 & 2 & 7 \end{array}\right) \qquad \begin{array}{c} (1) \\ (2) \\ (3) \end{array}$$

Forward elimination:

$$\left(\begin{array}{cc|c} \boxed{1} & -2 & 5 \\ 0 & \boxed{7} & -7 \\ 0 & 8 & -8 \end{array}\right) \qquad \begin{array}{l} (1) \\ (2') = (2) - 2(1) \\ (3') = (3) - 3(1) \end{array}$$

$$\left(\begin{array}{cc|c} \boxed{2} & 3 & 3 \\ 0 & \boxed{-7} & 7 \\ 0 & 0 & 0 \end{array}\right) \qquad \begin{array}{l} (1) \\ (2') \\ (3'') = 7(3') - 8(2') \end{array}$$

Echelon form. *You must bring it completely to this form.*

Back substitution:

From $(3'')$, nothing; from $(2')$, $y = -1$; from (1), $x = 3$.

A unique solution.

3   3.54(b), §3 Asked

Asked: Solve:

$$\left(\begin{array}{cccc|c} 1 & 2 & -3 & 2 & 2 \\ 2 & 5 & -8 & 6 & 5 \\ 3 & 4 & -5 & 2 & 4 \end{array}\right) \qquad \begin{array}{c} (1) \\ (2) \\ (3) \end{array}$$

4   3.54(b), §4 Solution

$$\left(\begin{array}{cccc|c} \boxed{1} & 2 & -3 & 2 & 2 \\ 2 & 5 & -8 & 6 & 5 \\ 3 & 4 & -5 & 2 & 4 \end{array}\right) \qquad \begin{array}{c} (1) \\ (2) \\ (3) \end{array}$$

Forward elimination:

$$\left(\begin{array}{cccc|c} \boxed{1} & 2 & -3 & 2 & 2 \\ 0 & \boxed{1} & -2 & 2 & 1 \\ 0 & -2 & 4 & -4 & -2 \end{array}\right) \qquad \begin{array}{l} (1) \\ (2') = (2) - 2(1) \\ (3') = (3) - 3(1) \end{array}$$

$$\left(\begin{array}{cccc|c} \boxed{1} & 2 & -3 & 2 & 2 \\ 0 & \boxed{1} & -2 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right) \qquad \begin{array}{l} (1) \\ (2') \\ (3'') = (3') + 2(2') \end{array}$$

Echelon form. *You must bring it completely to this form.*

Back substitution:

From $(3'')$, nothing; from $(2')$, $y = 1 + 2z - 2t$; from (1), $x = 2 - 2(1 + 2z - 2t) + 3z - 2t = -z + 2t$.

Solution space is 2D.

5   3.54(c), §5 Asked

Asked: Solve (corrected):

$$\left(\begin{array}{cccc|c} 1 & 2 & 4 & -5 & 3 \\ 3 & -1 & 5 & 2 & 4 \\ 5 & -4 & 6 & 9 & 2 \end{array}\right) \qquad \begin{array}{c} (1) \\ (2) \\ (3) \end{array}$$

6   3.54(c), §6 Solution

$$
\left(\begin{array}{cccc|c}
\boxed{1} & 2 & 4 & -5 & 3 \\
3 & -1 & 5 & 2 & 4 \\
5 & -4 & 6 & 9 & 2
\end{array}\right)
\qquad
\begin{array}{l}
(1) \\
(2) \\
(3)
\end{array}
$$

Forward elimination:

$$
\left(\begin{array}{cccc|c}
\boxed{1} & 2 & 4 & -5 & 3 \\
0 & \boxed{-7} & -7 & 17 & -5 \\
0 & -14 & -14 & 34 & -13
\end{array}\right)
\qquad
\begin{array}{l}
(1) \\
(2') = (2) - 3(1) \\
(3') = (3) - 5(1)
\end{array}
$$

$$
\left(\begin{array}{cccc|c}
\boxed{1} & 2 & -1 & 3 & 3 \\
0 & \boxed{-7} & -7 & 17 & -5 \\
0 & 0 & 0 & 0 & -3
\end{array}\right)
\qquad
\begin{array}{l}
(1) \\
(2') \\
(3'') = (3') - 2(2')
\end{array}
$$

Echelon form. *You must bring it completely to this form.*

Back substitution:

Equation $(3'')$ cannot be satisfied: there is no solution.

Bases

A basis $\vec{a}_1, \vec{a}_2, \ldots, \vec{a}_n$ to a space is a chosen set of vectors so that any vector $\vec{x}$ in the space can be *uniquely* expressed in terms of the basis vectors:

$$\vec{x} = c_1\vec{a}_1 + c_2\vec{a}_2 + \ldots + c_n\vec{a}_n$$

Example: $\hat{i}, \hat{j}, \hat{k}$ are a basis to coordinate space:

$$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$$

Basis vectors must be *independent*, which means you need all of them to express any arbitrary vector in the space. If some basis vector can be expressed in terms of the others, you do not need that vector and should throw it out.

For example if $\hat{i}$ would be $\hat{j} + \hat{k}$, then you would not need it, since $x\hat{i}$ could then be written as $x\hat{j} + x\hat{k}$. But there is no way to get $\hat{i}$ from a linear combination of $\hat{j}$ and $\hat{k}$

To check independence of supposed basis vector $\vec{a}_1, \vec{a}_2, \ldots, \vec{a}_n$, verify that

$$c_1\vec{a}_1 + c_2\vec{a}_2 + \ldots + c_n\vec{a}_n = 0$$

only when all coefficients $c_1, c_2, \ldots, c_n$ are zero.

Why this works: If, for example, c_1 would be nonzero, you can take $c_1\vec{a}_1$ to the other side and divide by $-c_1$.

3.57(a)

1   3.57(a), §1 Asked

Given:

$$\vec{u}_1 = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} \quad \vec{u}_2 = \begin{pmatrix} 1 \\ 4 \\ 2 \end{pmatrix} \quad \vec{u}_3 = \begin{pmatrix} 1 \\ -3 \\ 2 \end{pmatrix}$$

Asked: Express

$$\vec{v} = \begin{pmatrix} 4 \\ -9 \\ 2 \end{pmatrix}$$

in terms of $\vec{u}_1$, $\vec{u}_2$, and $\vec{u}_3$.

2   3.57(a), §2 Solution

We need c_1, c_2, and c_3 so that

$$\vec{v} = c_1 \vec{u}_1 + c_2 \vec{u}_2 + c_3 \vec{u}_3$$

In matrix form:

$$\begin{pmatrix} \vec{u}_1 & \vec{u}_2 & \vec{u}_3 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \vec{v}$$

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 4 \\ 2 & 4 & -3 & -9 \\ -1 & 2 & 2 & 2 \end{array} \right) \qquad \begin{pmatrix} (1) \\ (2) \\ (3) \end{pmatrix}$$

Forward elimination:

$$\left(\begin{array}{ccc|c} \boxed{1} & 1 & 1 & 4 \\ 0 & \boxed{2} & -5 & -17 \\ 0 & 3 & 3 & 6 \end{array} \right) \qquad \begin{pmatrix} (1) \\ (2') = (2) - 2(1) \\ (3') = (3) + (1) \end{pmatrix}$$

$$\left(\begin{array}{ccc|c} \boxed{1} & 1 & 1 & 4 \\ 0 & \boxed{2} & -5 & -17 \\ 0 & 0 & \boxed{21} & 63 \end{array} \right) \qquad \begin{pmatrix} (1) \\ (2') \\ (3'') = 2(3') - 3(2') \end{pmatrix}$$

Back substitution:

From $(3'')$, $c_3 = 3$; from $(2')$, $c_2 = -1$; from (1), $c_1 = 2$.

If the right hand side $\vec{v}$ would have been zero, the only possible values for c_1, c_2, and c_3 would be all zero. A set of vectors is dependent if you can create zero from them with some nonzero coefficients. (This allows you to express one of the set in terms of the others.)

Since you cannot do so with u_1, u_2 and u_3, they are independent vectors.

Also, since you can find a solution for any vector $\vec{v}$, you can express *any* vector in terms of u_1, u_2, and u_3. Vectors for which that is true are called a basis, in this case for three-dimensional vector space.

Row-canonical form

1   Row Canonical

Row canonical form:

- also zeros *above* the pivots;

- the pivots are normalized to one.

Example echelon form

$$\begin{pmatrix} \boxed{1} & 2 & -1 & 3 \\ 0 & 0 & \boxed{6} & -3 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad \begin{matrix} (1) \\ (2') \\ (3'') \end{matrix}$$

Example canonical form

$$\begin{pmatrix} \boxed{1} & 2 & 0 & 2\frac{1}{2} \\ 0 & 0 & \boxed{1} & -\frac{1}{2} \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad \begin{matrix} (1') = (1) + \frac{1}{6}(2') \\ (2'') = \frac{1}{6}(2') \\ (3'') \end{matrix}$$

Note that

$$\left(\begin{array}{cccc|c} \boxed{1} & 2 & 0 & 2\frac{1}{2} & 3\frac{1}{2} \\ 0 & 0 & \boxed{1} & -\frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 0 & 0 & 0 \end{array}\right) \qquad \begin{matrix} (1') \\ (2'') \\ (3'') \end{matrix}$$

directly solves to $x = 3\frac{1}{2} - 2y - 2\frac{1}{2}t$ and $z = \frac{1}{2} + \frac{1}{2}t$.

To obtain a row-canonical matrix, first reduce to echelon form. Next eliminate all nonzero elements above the pivots, starting from the last one, and divide each equation by its pivot.

2   Operations Counts

To solve a reasonably sized system of n equations in n unknowns, the amount of work the computer must do varies with algorithm. The operations the computer must do are mostly additions and multiplications, and the number that must be done is roughly:

- Cramer's rule: $n!$ operations (prohibitive);

- Gaussian Elimination/LU-decomposition: reduce to echelon form at $\frac{1}{3}n^3$ operations;

- Gauss-Jordan: reduce to row canonical form at $\frac{1}{2}n^3$ operations;

- Matrix inversion: n^3 operations.

Note that Gausian elimination can be much more efficient still for sparse matrices (i.e. matrices with a lot of zeros.) Always use the most specific algorithm for your matrix.

3.61(b)

1 3.61(b), §1 Asked

Given: A matrix

$$A = \begin{pmatrix} 1 & 2 & -1 & 2 & 1 \\ 2 & 4 & 1 & -2 & 3 \\ 3 & 6 & 3 & -7 & 7 \end{pmatrix} \quad \begin{matrix} (1) \\ (2) \\ (3) \end{matrix}$$

Asked: Reduce the matrix to echelon and row canonical forms.

2 3.61(b), §2 Solution

$$A = \begin{pmatrix} \boxed{1} & 2 & -1 & 2 & 1 \\ 2 & 4 & 1 & -2 & 3 \\ 3 & 6 & 3 & -7 & 7 \end{pmatrix} \quad \begin{matrix} (1) \\ (2) \\ (3) \end{matrix}$$

$$\begin{pmatrix} \boxed{1} & 2 & -1 & 2 & 1 \\ 0 & 0 & \boxed{3} & -6 & 1 \\ 0 & 0 & 6 & -13 & 4 \end{pmatrix} \quad \begin{matrix} (1) \\ (2') = (2) - 2(1) \\ (3') = (3) - 3(1) \end{matrix}$$

$$\begin{pmatrix} \boxed{1} & 2 & -1 & 2 & 1 \\ 0 & 0 & \boxed{3} & -6 & 1 \\ 0 & 0 & 0 & \boxed{-1} & 2 \end{pmatrix} \quad \begin{matrix} (1) \\ (2') \\ (3'') = (3') - 2(2') \end{matrix}$$

This is in echelon form.

$$\begin{pmatrix} \boxed{1} & 2 & -1 & 0 & 5 \\ 0 & 0 & \boxed{3} & 0 & -11 \\ 0 & 0 & 0 & \boxed{1} & -2 \end{pmatrix} \quad \begin{matrix} (1') = (1) + 2(3'') \\ (2'') = (2') - 6(3'') \\ (3''') = -(3'') \end{matrix}$$
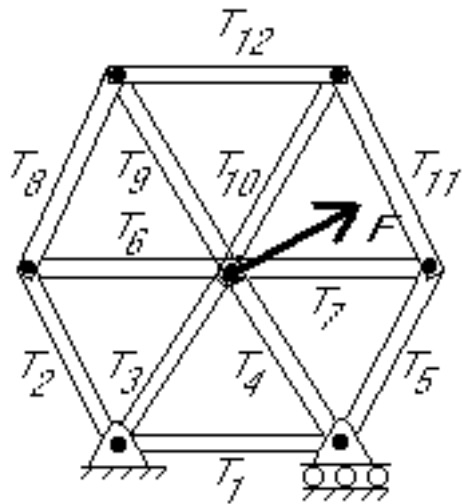
$$\begin{pmatrix} \boxed{1} & 2 & 0 & 0 & \frac{4}{3} \\ 0 & 0 & \boxed{1} & 0 & -\frac{11}{3} \\ 0 & 0 & 0 & \boxed{1} & -2 \end{pmatrix} \quad \begin{matrix} (1'') = (1') + \frac{1}{3}(2'') \\ (2''') = \frac{1}{3}(2'') \\ (3''') \end{matrix}$$

This is row canonical form.

Null spaces

1 Null spaces

The null space of a matrix A are all vectors $\vec{x}$ so that $A\vec{x} = 0$. If A is square and $|A|$ is nonzero, the null space is simply $\vec{x} = 0$ and has dimension 0.

Nontrivial null spaces may correspond to internal stresses in structures, connectivity problems, vibrational mode shape, buckling shapes, eigenvectors corresponding to a given eigenvalue, etcetera.

You typically want to describe the null spaces as simply as possible. Defining a basis for the null space allows you to do so.

2 2D Example

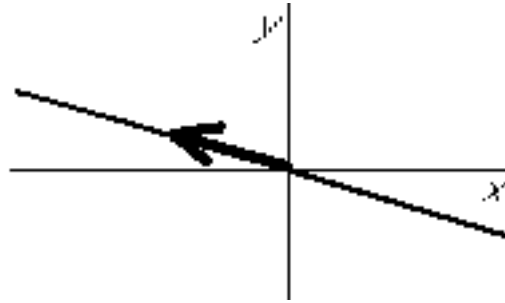$$\begin{pmatrix} 1 & 2 & \bigm| & 0 \\ 2 & 4 & \bigm| & 0 \end{pmatrix} \qquad \begin{matrix} (1) \\ (2) \end{matrix}$$

Forward elimination:

$$\begin{pmatrix} \boxed{1} & 2 \\ 0 & 0 \end{pmatrix} \qquad \begin{matrix} (1) \\ (2') \end{matrix}$$

Back substitution: $x = -2y$. So, the null space is a line through the origin:

Vector form:

$$\vec{r} = \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -2 \\ 1 \end{pmatrix} y$$

so that $(-2, 1)$ is one possible basis vector for this line. A line is a one-dimensional space, so it needs exactly one basis vector.

3   3D Example

$$\left(\boxed{1} \ \ -2 \ \ -3 \mid 0 \right) \qquad \begin{matrix} (1) \\ (2) \end{matrix}$$

Forward elimination is trivial.

Back substitution: $x = 2y + 3z$. The solution space is a plane through the origin:

Vector form:

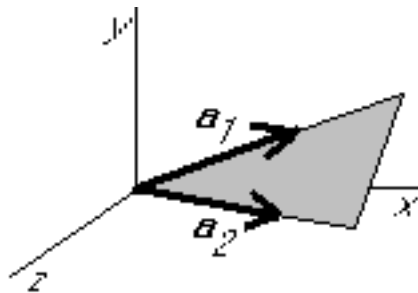$$\vec{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} y + \begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix} z$$

so that $(2, 1, 0)$ and $(3, 0, 1)$ are one possible set of two basis vector for this plane. A plane is a 2D space.

4   12D Example

Assuming there is no external force, i.e. $\vec{F} = 0$ in the truss below,
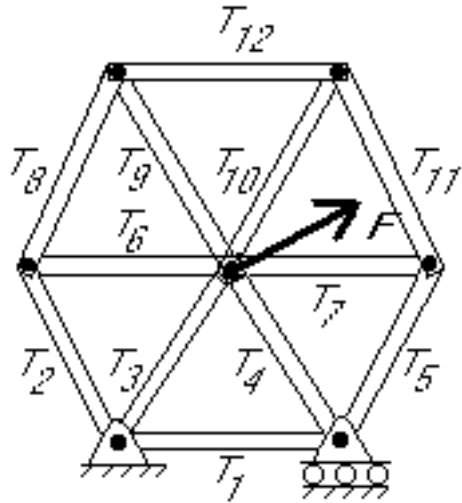
the solution of the homogeneous equilibrium equations is

$$
\begin{pmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \\ T_5 \\ T_6 \\ T_7 \\ T_8 \\ T_9 \\ T_{10} \\ T_{11} \\ T_{12} \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ -1 \\ -1 \\ 1 \\ -1 \\ -1 \\ 1 \\ -1 \\ -1 \\ 1 \\ 1 \end{pmatrix} T_{12}
$$

All bars in the outer ring have the same tension force T_{12}, while the spokes have an opposite compressive force.

1.53

1   3.60(a), §1 Asked

Asked: The dimension and a basis for the solution space of the following homogeneous system:

$$\left(\begin{array}{ccccc|c} 1 & 3 & 2 & -1 & -1 & 0 \\ 2 & 6 & 5 & 1 & -1 & 0 \\ 5 & 15 & 12 & 1 & -3 & 0 \end{array}\right) \qquad \begin{array}{c} (1) \\ (2) \\ (3) \end{array}$$

2   3.60(a), §2 Solution

We need the null space of the matrix:

$$\left(\begin{array}{ccccc} \boxed{1} & 3 & 2 & -1 & -1 \\ 2 & 6 & 5 & 1 & -1 \\ 5 & 15 & 12 & 1 & -3 \end{array}\right) \qquad \begin{array}{c} (1) \\ (2) \\ (3) \end{array}$$

Forward elimination:

$$\left(\begin{array}{ccccc} \boxed{1} & 3 & 2 & -1 & -1 \\ 0 & 0 & \boxed{1} & 3 & 1 \\ 0 & 0 & 2 & 6 & 2 \end{array}\right) \qquad \begin{array}{l} (1) \\ (2') = (2) - 2(1) \\ (3') = (3) - 5(1) \end{array}$$

$$\left(\begin{array}{ccccc} \boxed{1} & 3 & 2 & -1 & -1 \\ 0 & 0 & \boxed{1} & 3 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right) \qquad \begin{array}{l} (1) \\ (2') \\ (3'') = (3') - 2(2') \end{array}$$

Continue to row canonical:

$$\left(\begin{array}{ccccc} \boxed{1} & 3 & 0 & -7 & -3 \\ 0 & 0 & \boxed{1} & 3 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right) \qquad \begin{array}{l} (1') = (1) - 2(2') \\ (2') \\ (3'') \end{array}$$

Back substitution:

From $(3'')$, nothing; from $(2')$, $z = -3s - t$; from (1'), $x = -3y + 7s + 3t$. Variables y, s, and t cannot be determined: space is 3D.

Vector form;

$$
\begin{pmatrix} x \\ y \\ z \\ s \\ t \end{pmatrix} = y \begin{pmatrix} -3 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + s \begin{pmatrix} 7 \\ 0 \\ -3 \\ 1 \\ 0 \end{pmatrix} + t \begin{pmatrix} 3 \\ 0 \\ -1 \\ 0 \\ 1 \end{pmatrix}
$$

The three vectors in the right hand side form a basis for the solution space.

Inverse matrices

Inverse matrices computed with minors are very bad news for matrix sizes more than say 2 or 3.

Inverse matrices are usually bad news anyway. They waste operations, storage, and tend to decrease numerical accuracy. This is especially so for sparse matrices.

If you *really* need the inverse matrix, augment the matrix to be inverted with the unit matrix and reduce to row-canonical form. The inverse matrix will then be in the right hand side.

Partial pivoting may be used as needed.

3.67(c)

1 3.67(c), §1 Asked

Asked: Find the inverse of

$$\begin{pmatrix} 1 & 3 & -2 \\ 2 & 8 & -3 \\ 1 & 7 & 1 \end{pmatrix}$$

2 3.67(c), §2 Solution

Augment with the unit matrix:

$$\left(\begin{array}{ccc|ccc} 1 & 3 & -2 & 1 & 0 & 0 \\ 2 & 8 & -3 & 0 & 1 & 0 \\ 1 & 7 & 1 & 0 & 0 & 1 \end{array}\right) \qquad \begin{array}{l} (1) \\ (2) \\ (3) \end{array}$$

Reduce to row canonical:

$$\left(\begin{array}{ccc|ccc} 1 & 3 & -2 & 1 & 0 & 0 \\ 0 & 2 & 1 & -2 & 1 & 0 \\ 0 & 4 & 3 & -1 & 0 & 1 \end{array}\right) \qquad \begin{array}{l} (1) \\ (2') = (2) - 2(1) \\ (3') = (3) - (1) \end{array}$$

$$\left(\begin{array}{ccc|ccc} 2 & 0 & -7 & 8 & -3 & 0 \\ 0 & 2 & 1 & -2 & 1 & 0 \\ 0 & 0 & 1 & 3 & -2 & 1 \end{array}\right) \qquad \begin{array}{l} (1') = 2(1) - 3(2') \\ (2') \\ (3'') = (3') - 2(2') \end{array}$$

$$\left(\begin{array}{ccc|ccc} 2 & 0 & 0 & 29 & -17 & 7 \\ 0 & 2 & 0 & -5 & 3 & -1 \\ 0 & 0 & 1 & 3 & -2 & 1 \end{array}\right) \qquad \begin{array}{l} (1'') = (1') + 7(3'') \\ (2'') = (2') - (3'') \\ (3'') \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 14.5 & -8.5 & 3.5 \\ 0 & 1 & 0 & -2.5 & 1.5 & -0.5 \\ 0 & 0 & 1 & 3 & -2 & 1 \end{array}\right) \qquad \begin{array}{l} (1''') = (1'')/2 \\ (2''') = (2'')/2 \\ (3'') \end{array}$$

The inverse matrix is at the right. Verify that $AA^{-1} = I$.

Introduction

The *span* of a set of independent basis vectors $\vec{a}_1, \vec{a}_2, \ldots, \vec{a}_n$ is the set of all vectors $\vec{v}$ that can be described as linear combinations of the basis vectors:

$$\vec{v} = c_1\vec{a}_1 + c_2\vec{a}_2 + \ldots c_n\vec{a}_n$$

- One basis vector spans a line:
$$\vec{v} = c_1\vec{a}_1$$
  is a 1D straight line through the origin.

- Two basis vectors span a plane:
$$\vec{v} = c_1\vec{a}_1 + c_2\vec{a}_2$$
  is a 2D plane through the origin.

- Three basis vectors span a 3D space:
$$\vec{v} = c_1\vec{a}_1 + c_2\vec{a}_2 + c_3\vec{a}_3$$
  is a 3D space through the origin (in n-dimensional space.)

Remember the definition of independence: the basis vectors are independent if the only way to get zero is to take every c_i zero. It implies that you cannot express one basis vector in terms of the rest.

- Two vectors are linearly independent if they are not along the same line.

- Three vectors are linearly independent if they are not in the same plane.

The *rank* of a matrix is the number of independent rows, or columns. These are the same; see the example.

4.89(a)

1 4.89(a), §1 Asked

Asked: Are (1,2,-3,1), (3,7,1,-2), and (1,3,7,-4) independent?

2 4.89(a), §2 Solution

Most straightforward is to do Gaussian elimination with the vectors as rows:

$$\begin{pmatrix} \boxed{1} & 2 & -3 & 1 \\ 3 & 7 & 1 & -2 \\ 1 & 3 & 7 & -4 \end{pmatrix} \qquad \begin{matrix} \vec{u}_1 \\ \vec{u}_2 \\ \vec{u}_3 \end{matrix}$$

$$\begin{pmatrix} \boxed{1} & 2 & -3 & 1 \\ 0 & \boxed{1} & 10 & -5 \\ 0 & 1 & 10 & -5 \end{pmatrix} \qquad \begin{matrix} \vec{u}_1 \\ \vec{u}_2' = \vec{u}_2 - 3\vec{u}_1 \\ \vec{u}_3' = \vec{u}_3 - \vec{u}_1 \end{matrix}$$

$$\begin{pmatrix} \boxed{1} & 2 & -3 & 1 \\ 0 & \boxed{1} & 10 & -5 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad \begin{matrix} \vec{u}_1 \\ \vec{u}_2' \\ \vec{u}_3'' = \vec{u}_3' - \vec{u}_2' \end{matrix}$$

The vectors are linearly dependent, since the third vector is all zero. The rank of the matrix is 2: there are only two independent rows.

We can clean up a bit more by going to canonical:

$$\begin{pmatrix} 1 & 0 & -23 & 11 \\ 0 & 1 & 10 & -5 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad \begin{matrix} \vec{u}_1' \\ \vec{u}_2' \\ \vec{u}_3'' \end{matrix}$$

The space spanned is then the set of linear combinations of the two simplified vectors. The first vector is normal to the y-axis, the second is normal to the x-axis.

The alternate procedure uses the vectors as columns:

$$\vec{u}_1 c_1 + \vec{u}_2 c_2 + \vec{u}_3 c_3 = 0$$

should have no nontrivial solutions for linear independence. Note that this produces the transpose matrix from the one above:

$$\begin{pmatrix} \boxed{1} & 3 & 1 \\ 2 & 7 & 3 \\ -3 & 1 & 7 \\ 1 & -2 & -4 \end{pmatrix} \implies \begin{pmatrix} \boxed{1} & 3 & 1 \\ 0 & \boxed{1} & 1 \\ 0 & 10 & 10 \\ 0 & -5 & -5 \end{pmatrix} \implies \begin{pmatrix} \boxed{1} & 3 & 1 \\ 0 & \boxed{1} & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Since the solution of the system c_1, c_2, and c_3 can be nonzero, the vectors are linearly dependent.

The number of independent rows in this matrix, which is the number of independent columns in the first matrix, is again 2. So the rank is 2 whether I look at rows or columns.

Why is the rank the same whether I take the vectors as rows or columns? Well, since there are two independent vectors (let's take them as the first two,) I can take c_3 whatever I want and only find unique values for c_1 and c_2 given c_3. That means there must be two nonzero pivots. So the number of independent row vectors established in the first method must be the number of pivots in the second method. And the number of pivots is the number of independent row vectors in the second matrix.

4.104(b)

1   4.104(b), §1 Asked

Asked: Find the rank of the matrix

$$\begin{pmatrix} 1 & 2 & -3 & -2 \\ 1 & 3 & -2 & 0 \\ 3 & 8 & -7 & -2 \\ 2 & 1 & -9 & -10 \end{pmatrix}$$

2   4.104(b), §2 Solution

$$\begin{pmatrix} 1 & 2 & -3 & -2 \\ 1 & 3 & -2 & 0 \\ 3 & 8 & -7 & -2 \\ 2 & 1 & -9 & -10 \end{pmatrix} \quad \begin{matrix} (1) \\ (2) \\ (3) \\ (4) \end{matrix}$$

I expect the rank to be 4.

$$\begin{pmatrix} 1 & 2 & -3 & -2 \\ 0 & 1 & 1 & 2 \\ 0 & 2 & 2 & 4 \\ 0 & -3 & -3 & 6 \end{pmatrix} \quad \begin{matrix} (1) \\ (2') = (2) - (1) \\ (3') = (3) - 3(1) \\ (4') = (4) - 2(1) \end{matrix}$$

I already see it is not.

$$\begin{pmatrix} \boxed{1} & 2 & -3 & -2 \\ 0 & \boxed{1} & 1 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \begin{matrix} (1) \\ (2') \\ (3'') = (3') - 2(2') \\ (4'') = (4') - 3(2') \end{matrix}$$

True rank is 2. There are only two independent row vectors in the matrix. There are only two independent column vectors in the matrix.

Basis Changes

1 Simple example

Student request: change notations. Mine seem better than the book's, though. I think the books exposition (p207-210) is very confusing, partly by not using vector symbols to indicate vectors versus coordinates. I suggest you stick with my exposition.

To solve problems, it is often desirable or essential to change basis.

As an example, consider the vector of gravity $\vec{g}$. If I use a Cartesian coordinate system $\hat{i}, \hat{j}$ with the x-axis horizontal, the vector $\vec{g}$ will be along the negative y-axis. I will call this coordinate system, $(\hat{i}, \hat{j})$, the E-system.
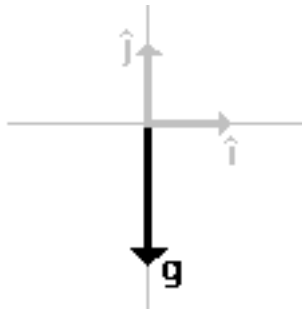
Using the E-system, I can write the vector $\vec{g}$ as:

$$\vec{g} = 0\hat{i} - g\hat{j} \quad \text{or} \quad \vec{g}\big|_E = \begin{pmatrix} 0 \\ -g \end{pmatrix}$$

In other words, the *coordinates* of vector $\vec{g}$ in the E-coordinate system are $g_1\big|_E = 0$ and $g_2\big|_E = -g$.

But if, say, the ground is under an angle θ with the horizontal, it might be much more convenient to use a coordinate system E^*, $(\hat{i}^*, \hat{j}^*)$, with the x-axis aligned with the ground:
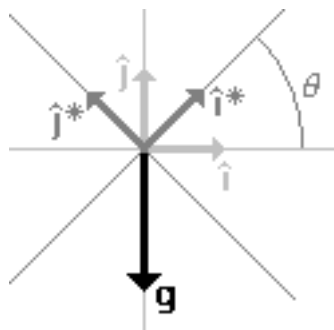
In this new coordinate system, the coordinates of $\vec{g}$ will be different. With a bit of trig, you see:

$$\vec{g} = -g\sin(\theta)\hat{i}^* - g\cos(\theta)\hat{j}^* \quad \text{or} \quad \vec{g}\big|_{E^*} = \begin{pmatrix} -g\sin(\theta) \\ -g\cos(\theta) \end{pmatrix}$$

The coordinates of vector $\vec{g}$ are now $g_1\big|_{E^*} = -g\sin(\theta)$ and $g_2\big|_{E^*} = -g\cos(\theta)$

What if I need to change the coordinates of a lot of vectors from one coordinate system to the other? Is there a systematic way of doing this? The answer is yes; the following formula applies:

$$\vec{v}\big|_E = P\vec{v}\big|_{E^*} \quad \text{with } P = \left(\hat{i}^*\big|_E \ \hat{j}^*\big|_E \right)$$

So the transformation of coordinates can be done by multiplying by a matrix P. This matrix consists of the basis vectors of the new coordinate system E^* expressed in terms of the old coordinate system E.
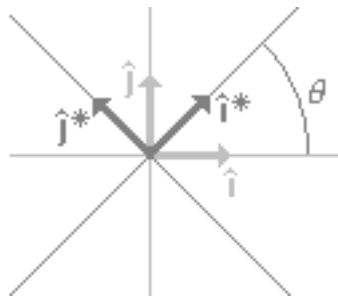
In particular,

$$\hat{i}^* = \cos(\theta)\hat{i} + \sin(\theta)\hat{j} \quad \text{so} \quad \hat{i}^*\big|_E = \begin{pmatrix} \cos(\theta) \\ \sin(\theta) \end{pmatrix}$$

$$\hat{j}^* = -\sin(\theta)\hat{i} + \cos(\theta)\hat{j} \quad \text{so} \quad \hat{j}^*\big|_E = \begin{pmatrix} -\sin(\theta) \\ \cos(\theta) \end{pmatrix}$$

and matrix P becomes:

$$P = \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix}$$

Let's test it: P times the coordinates of vector $\vec{g}$ in the E^*-system should give the coordinates in the E-system:

$$\begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix} \begin{pmatrix} -g\sin(\theta) \\ -g\cos(\theta) \end{pmatrix}$$

Multiplying out gives 0 and $-g$, which is exactly right.

Matrix P is called *the transformation matrix from E to E^**. Note however that it really transforms coordinates in the E^*-system to coordinates in the E-system. You just have to get used to that language: a transformation matrix from A to B transforms B coordinates into A coordinates. No, I do not know who thought of that first.

What if you really want to transform E coordinates into E^* coordinates? No big deal: just multiply by the inverse matrix P^{-1}.

2 General

The basis vectors do not have to be orthogonal, as in the example. In general, suppose I have a basis S, $\{\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_n\}$. Then any arbitrary vector $\vec{w}$ can be written as

$$\vec{w} = w_1\big|_S \vec{u}_1 + w_2\big|_S \vec{u}_2 + \ldots + w_n\big|_S \vec{u}_n$$

where $w_1|_S, w_2|_S, \ldots, w_n|_S$ are the coordinates of $\vec{w}$ in basis S. More briefly,

$$\vec{w}\big|_S = \begin{pmatrix} w_1\big|_S \\ w_2\big|_S \\ \vdots \\ w_n\big|_S \end{pmatrix}$$

Suppose I have another basis S', $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$. Then the same vector $\vec{w}$ can also be written as

$$\vec{w} = w_1\big|_{S'} \vec{v}_1 + w_2\big|_{S'} \vec{v}_2 + \ldots + w_n\big|_{S'} \vec{v}_n$$

or

$$\vec{w}\big|_{S'} = \begin{pmatrix} w_1\big|_{S'} \\ w_2\big|_{S'} \\ \vdots \\ w_n\big|_{S'} \end{pmatrix}$$

The relationship between the two sets of coordinates is always

$$\boxed{\vec{w}\big|_S = P \, \vec{w}\big|_{S'}}$$

where P is a matrix that is called the transformation matrix from S to S'. (Although it really works the opposite way.)

Matrix P takes the form:

$$\boxed{P = \left(\vec{v}_1\big|_S \; \vec{v}_2\big|_S \; \cdots \; \vec{v}_n\big|_S \right)}$$
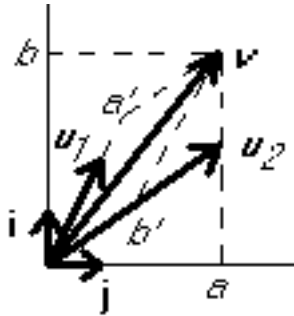
It contains the basis vectors of the S' system written in the S system. (That is why if I multiply with P, I get a vector in the S system.)

To get the transformation the other way, use the matrix P^{-1}.

6.47(a)

1   6.47(a), §1 Asked

Given: A new basis $S = \{\vec{u}_1, \vec{u}_2\}$. The new basis vectors can be expressed in terms of the original Cartesian basis $E = \{\hat{\imath}, \hat{\jmath}\}$ as $\vec{u}_1|_E = (1, 2)$ and $\vec{u}_2|_E = (3, 5)$.

Asked: Find (1) the change of basis matrix P from E to S; (2) the change of basis matrix Q from S to E; (3) the components (a', b') of an arbitrary vector $\vec{v}$ in the S basis if $\vec{v}$ has components (a, b) in the Cartesian coordinate system ($\vec{v} = a\hat{\imath} + b\hat{\jmath}$.)

2   6.47(a), §2 Solution

The basis vectors of the S-system are given in terms of the E-system, so, to get the transformation matrix, we simply put them as columns of the matrix:

$$P \equiv \left(\vec{u}_1\big|_E \ \vec{u}_2\big|_E \right) = \begin{pmatrix} 1 & 3 \\ 2 & 5 \end{pmatrix}$$

Let's check this, just to be sure. The arbitrary vector $\vec{v}$ can be expressed as

$$\vec{v} = a\hat{\imath} + b\hat{\jmath} = a'\vec{u}_1 + b'\vec{u}_2$$

so

$$\vec{v}\big|_E \equiv \begin{pmatrix} a \\ b \end{pmatrix} = a'\vec{u}_1\big|_E + b'\vec{u}_2\big|_E = \left(\vec{u}_1\big|_E \ \vec{u}_2\big|_E \right) \begin{pmatrix} a' \\ b' \end{pmatrix} = P\vec{v}\big|_S$$

Since

$$\vec{v}\big|_S = P^{-1}\vec{v}\big|_E$$

the transformation matrix Q from S to E is

$$Q = P^{-1} = \frac{1}{-1}\begin{pmatrix} 5 & -3 \\ -2 & 1 \end{pmatrix} = \begin{pmatrix} -5 & 3 \\ 2 & -1 \end{pmatrix}$$

So

$$\begin{pmatrix} a' \\ b' \end{pmatrix} = \begin{pmatrix} -5 & 3 \\ 2 & -1 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix}$$

So $a' = -5a + 3b$ and $b' = 2a - b$.

6.48

1 6.48, §1 Asked

Given: A basis $S = \{\vec{u}_1, \vec{u}_2\}$ and another basis $S' = \{\vec{v}_1, \vec{v}_2\}$. The basis vectors of S can be expressed in terms of the Cartesian basis $E = \{\hat{\imath}, \hat{\jmath}\}$ as $\vec{u}_1|_E = (1, 2)$ and $\vec{u}_2|_E = (2, 3)$ and those of S' as $\vec{v}_1|_E = (1, 3)$ and $\vec{v}_2|_E = (1, 4)$.

Asked: Find (a) the change of basis matrix P from S to S'; (b) the change of basis matrix Q from S' back to S.

2 6.48, §2 Solution

By definition, for any vector $\vec{w}$,

$$\vec{w}\big|_S = P\vec{w}\big|_{S'}$$

where P contains the basis vectors of the S'-system in terms of the S-system. Unfortunately, these basis vectors are given in terms of the E-system, not the S-system.

Trick: go over the E system:

$$\vec{w}\big|_{S'} \implies \vec{w}\big|_E \implies \vec{w}\big|_S$$

$$\vec{w}\big|_E = \left(\vec{v}_1\big|_E \, \vec{v}_2\big|_E\right)\vec{w}\big|_{S'} \qquad \vec{w}\big|_E = \left(\vec{u}_1\big|_E \, \vec{u}_2\big|_E\right)\vec{w}\big|_S$$

So

$$\vec{w}\big|_S = \left(\vec{u}_1\big|_E \, \vec{u}_2\big|_E\right)^{-1}\left(\vec{v}_1\big|_E \, \vec{v}_2\big|_E\right)\vec{w}\big|_{S'}$$

$$P = \begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix}^{-1} \begin{pmatrix} 1 & 1 \\ 3 & 4 \end{pmatrix} = \begin{pmatrix} -3 & 2 \\ 2 & -1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 3 & 4 \end{pmatrix} = \begin{pmatrix} 3 & 5 \\ -1 & -2 \end{pmatrix}$$

$$Q = P^{-1} = \begin{pmatrix} 2 & 5 \\ -1 & -3 \end{pmatrix}$$

Transforming Matrices

We saw that a transformation matrix P from an old basis S to new basis S' transforms between $\vec{v}$ $(= \vec{v}\big|_S)$ and $\vec{v}'$ $(= \vec{v}\big|_{S'})$ as:

$$\boxed{\vec{v} = P\vec{v}' \text{ or } \vec{v}' = P^{-1}\vec{v}}$$

A square matrix A transforms similarly, but has in addition the inverse of the transformation matrix at the far right:

$$\boxed{A = PA'P^{-1} \text{ or } A' = P^{-1}AP}$$

The need for two transformation matrices comes from the fact that a matrix provides a transformation of vectors. Given an "original vector" $\vec{x}$, multiplying by matrix A produces an "image vector" $\vec{y} = A\vec{x}$. When we change coordinates, one transformation matrix is needed to transform $\vec{x}$, the other to transform $\vec{y}$:
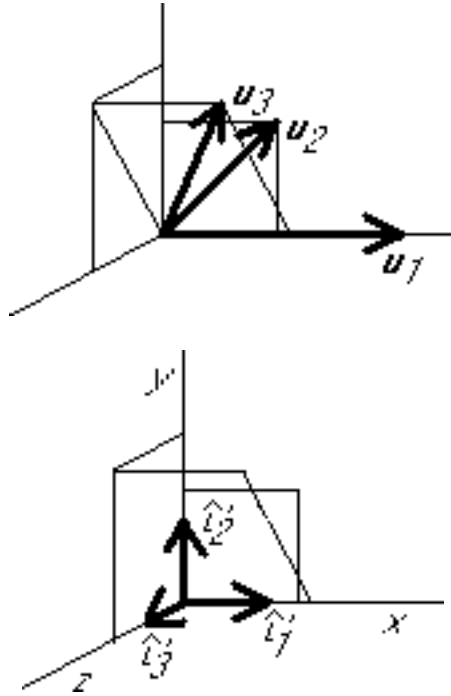
$$\vec{y}' = P^{-1}\vec{y} = P^{-1}(A\vec{x}) = P^{-1}AP\vec{x}'$$

So the matrix that transforms $\vec{x}'$ into $\vec{y}'$ is $P^{-1}AP$.

Gram-Schmidt

Description:

Gram-Schmidt orthogonalization is a way of converting a given arbitrary basis $\{\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_n\}$ into an equivalent orthonormal basis:

This often leads to better accuracy (e.g. in least square problems) and/or simplifications.

Modified Gram-Schmidt Procedure

Given a set of linearly independent vectors, $\vec{u}_1, \vec{u}_2, \ldots$, turn them into an equivalent orthonormal set $\hat{\imath}'_1, \hat{\imath}'_2, \ldots$ as follows:
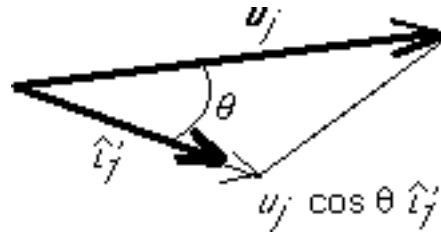
Step 1:

1. Normalize the first vector $\vec{u}_1$. That will be your $\hat{\imath}'_1$

$$\hat{\imath}'_1 = \frac{\vec{u}_1}{||\vec{u}_1||}$$

2. For the remaining vectors $\vec{u}_2, \vec{u}_3, \ldots$, eliminate their component in the direction of $\hat{\imath}'_1$ using the following formula:

$$\vec{u}_j^* = \vec{u}_j - \hat{\imath}'_1 \left(\hat{\imath}'^H_1 \vec{u}_j \right)$$

Note that $\hat{\imath}_1'^H \vec{u}_j = ||\hat{\imath}_1'||\,||\vec{u}_j||\cos\theta = ||\vec{u}_j||\cos\theta$ is the component of $\vec{u}_j$ in the direction of $\hat{\imath}_1'$:

Also $\hat{\imath}_1'\hat{\imath}_1'^H\vec{u}_j = \mathrm{proj}(\hat{\imath}_1', \vec{u}_j)$. The matrix $\hat{\imath}_1'\hat{\imath}_1'^H$ is called the projection operator onto $\hat{\imath}_1'$.

Ignore $\hat{\imath}_1'$ in the remaining process.

Step 2:

1. Normalize the second vector $\vec{u}_2^*$. That will be your $\hat{\imath}_2'$

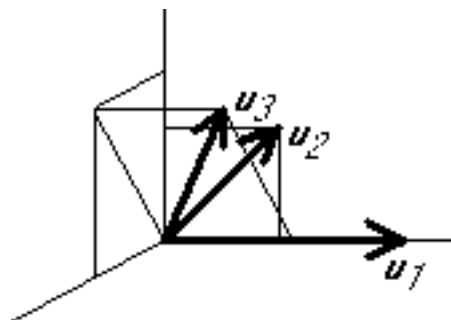$$\hat{\imath}_2' = \frac{\vec{u}_2^*}{||\vec{u}_2^*||}$$

2. For the remaining vectors $\vec{u}_3, \vec{u}_4, \ldots$, eliminate their component in the direction of $\hat{\imath}_2'$ using the following formula:

$$\vec{u}_j^{**} = \vec{u}_j^* - \hat{\imath}_2'\left(\hat{\imath}_2'^H\vec{u}_j^*\right)$$
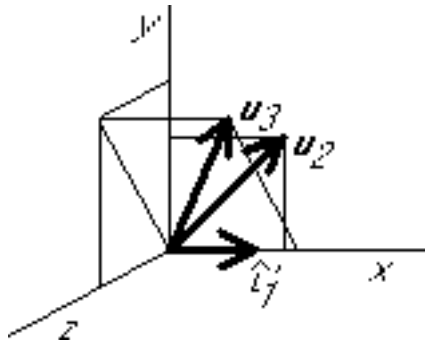
Ignore $\hat{\imath}_2'$ in the remaining process.

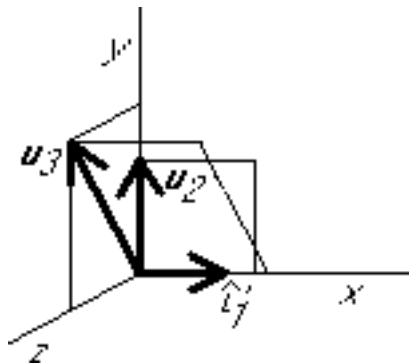Repeat the process along the same lines until you run out of vectors.
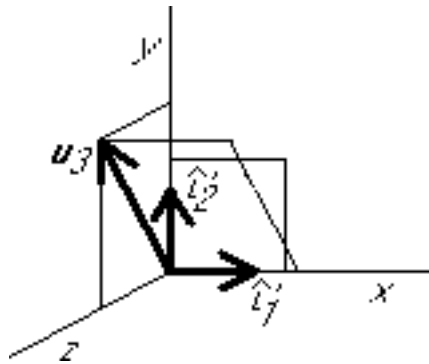
Graphical example:
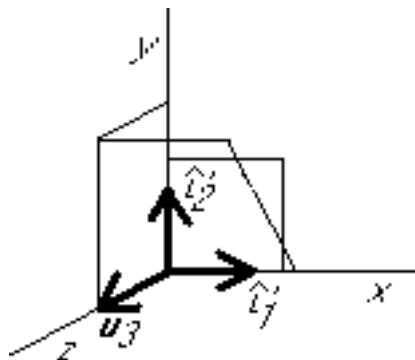
Normalize $\vec{u}_1$:

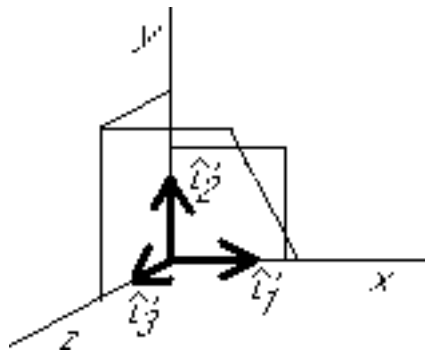Eliminate the components in the $\vec{u}_1$ direction from the rest:

Normalize $\vec{u}_2$:

Eliminate the components in the $\vec{u}_2$ direction from the rest:

Normalize $\vec{u}_3$:

7.21

1 7.21, §1 Asked

Given: The basis vectors

$$v_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \qquad v_2 = \begin{pmatrix} 1 \\ 1 \\ 2 \\ 4 \end{pmatrix} \qquad v_3 = \begin{pmatrix} 1 \\ 2 \\ -4 \\ -3 \end{pmatrix}$$

Asked: Find (a) an orthogonal basis for the space spanned by $\vec{v}_1$, $\vec{v}_2$, and $\vec{v}_3$; (b) an orthonormal basis for the space spanned by $\vec{v}_1$, $\vec{v}_2$, and $\vec{v}_3$.

2 7.21, §2 Solution

Since an orthonormal basis *is* orthogonal, I only need do (b).

$$v_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \qquad v_2 = \begin{pmatrix} 1 \\ 1 \\ 2 \\ 4 \end{pmatrix} \qquad v_3 = \begin{pmatrix} 1 \\ 2 \\ -4 \\ -3 \end{pmatrix}$$

Normalize v_1:

$$\hat{\imath}_1' = \frac{\vec{v}_1}{||\vec{v}_1||} = \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \end{pmatrix}$$

Get rid of the $\hat{\imath}_1'$-components:

$$\vec{v}_2^* = \vec{v}_2 - \hat{\imath}_1'\hat{\imath}_1'^H \vec{v}_2 = \begin{pmatrix} 1 \\ 1 \\ 2 \\ 4 \end{pmatrix} - \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \end{pmatrix} \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 2 \\ 4 \end{pmatrix}$$

$$= \begin{pmatrix} 1 \\ 1 \\ 2 \\ 4 \end{pmatrix} - \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \end{pmatrix} 4 = \begin{pmatrix} -1 \\ -1 \\ 0 \\ 2 \end{pmatrix}$$

$$\vec{v}_3^{\,*} = \vec{v}_3 - \hat{i}_1'\hat{i}_1'^{H}\vec{v}_3 = \begin{pmatrix} 1 \\ 2 \\ -4 \\ -3 \end{pmatrix} - \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \end{pmatrix} \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ -4 \\ -3 \end{pmatrix}$$

$$= \begin{pmatrix} 1 \\ 2 \\ -4 \\ -3 \end{pmatrix} - \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \end{pmatrix} (-2) = \begin{pmatrix} 2 \\ 3 \\ -3 \\ -2 \end{pmatrix}$$

Normalize v_2^*:

$$\hat{i}_2' = \frac{\vec{v}_2^{\,*}}{||\vec{v}_2^{\,*}||} = \begin{pmatrix} -\frac{1}{\sqrt{6}} \\ -\frac{1}{\sqrt{6}} \\ 0 \\ \frac{2}{\sqrt{6}} \end{pmatrix}$$

Get rid of the $\hat{i}_2'$-components:

$$\vec{v}_3^{\,**} = \vec{v}_3^{\,*} - \hat{i}_2'\hat{i}_2'^{H}\vec{v}_3^{\,*} = \begin{pmatrix} 2 \\ 3 \\ -3 \\ -2 \end{pmatrix} - \begin{pmatrix} -\frac{1}{\sqrt{6}} \\ -\frac{1}{\sqrt{6}} \\ 0 \\ \frac{2}{\sqrt{6}} \end{pmatrix} \begin{pmatrix} -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & 0 & \frac{2}{\sqrt{6}} \end{pmatrix} \begin{pmatrix} 2 \\ 3 \\ -3 \\ -2 \end{pmatrix}$$

$$= \begin{pmatrix} 2 \\ 3 \\ -3 \\ -2 \end{pmatrix} - \begin{pmatrix} -\frac{1}{\sqrt{6}} \\ -\frac{1}{\sqrt{6}} \\ 0 \\ \frac{2}{\sqrt{6}} \end{pmatrix} \frac{-9}{\sqrt{6}} = \begin{pmatrix} \frac{1}{2} \\ \frac{3}{2} \\ -3 \\ 1 \end{pmatrix}$$

Normalize v_3^{**}:

$$\hat{i}_3' = \frac{\vec{v}_3^{\,**}}{||\vec{v}_2^{\,**}||} = \begin{pmatrix} \frac{1}{5\sqrt{2}} \\ \frac{3}{5\sqrt{2}} \\ -\frac{6}{5\sqrt{2}} \\ \frac{2}{5\sqrt{2}} \end{pmatrix}$$

Introduction

Determinants are most of the time *not* very useful:

- The system $0.1x_1 = 0.3$, $0.1x_2 = 0.3$, ..., $0.1x_n = 0.3$ is pefectly well solvable, but $|A|$ will underflow on typical computers for values of n as low as 40.

- Direct evaluation of a determinant of an $n \times n$ matrix takes $n!$ multiplications. The big bang was about $5\,10^{17}$ seconds ago; evaluating a 70×70 determinant takes 10^{100} multiplications. (And allows an interesting possible accumulation of numerical errors.)

Small determinants may be convenient.

8.41(a)

1 8.41(a), §1 Asked

Asked:

$$\begin{vmatrix} 1 & 2 & 2 & 3 \\ 1 & 0 & -2 & 0 \\ 3 & -1 & 1 & -2 \\ 4 & -3 & 0 & 2 \end{vmatrix}$$

2 8.41(a), §2 Direct

Put in a checkerboard sign pattern (starting with +):

$$|A| = \begin{vmatrix} 1^+ & 2^- & 2^+ & 3^- \\ 1^- & 0^+ & -2^- & 0^+ \\ 3^+ & -1^- & 1^+ & -2^- \\ 4^- & -3^+ & 0^- & 2^+ \end{vmatrix}$$

Select a row (or a column) and expand in signs, coefficients, and minors. Here the second row may be best:

$$|A| = -(1)\begin{vmatrix} 2 & 2 & 3 \\ -1 & 1 & -2 \\ -3 & 0 & 2 \end{vmatrix} - (-2)\begin{vmatrix} 1 & 2 & 3 \\ 3 & -1 & -2 \\ 4 & -3 & 2 \end{vmatrix}$$

Repeat for each of the smaller determinants until the determinants are small enough to be directly written out, eg,

$$|a| = a \qquad \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = aei + bfg + cdh - ceg - afh - bdi$$

3   8.41(a), §3 Elimination

$$|A| = \begin{vmatrix} 1 & 2 & 2 & 3 \\ 1 & 0 & -2 & 0 \\ 3 & -1 & 1 & -2 \\ 4 & -3 & 0 & 2 \end{vmatrix} \quad \begin{array}{l} (1) \\ (2) \\ (3) \\ (4) \end{array}$$

Interchange rows:

$$-|A| = \begin{vmatrix} 1 & 0 & -2 & 0 \\ 1 & 2 & 2 & 3 \\ 3 & -1 & 1 & -2 \\ 4 & -3 & 0 & 2 \end{vmatrix} \quad \begin{array}{l} (1') = (2) \\ (2') = (1) \\ (3) \\ (4) \end{array}$$

Substract multiples of the first equation from the rest:

$$-|A| = \begin{vmatrix} 1 & 0 & -2 & 0 \\ 0 & 2 & 4 & 3 \\ 0 & -1 & 7 & -2 \\ 0 & -3 & 8 & 2 \end{vmatrix} \quad \begin{array}{l} (1') \\ (2'') = (2') - (1') \\ (3') = (3) - 3(1') \\ (4') = (4) - 4(1') \end{array}$$

Interchange the second and third equations:

$$|A| = \begin{vmatrix} 1 & 0 & -2 & 0 \\ 0 & -1 & 7 & -2 \\ 0 & 2 & 4 & 3 \\ 0 & -3 & 8 & 2 \end{vmatrix} \quad \begin{array}{l} (1') \\ (2''') = (3') \\ (3'') = (2'') \\ (4') = (4) - 4(1') \end{array}$$

Substract multiples of the second equation from the rest:

$$|A| = \begin{vmatrix} 1 & 0 & -2 & 0 \\ 0 & -1 & 7 & -2 \\ 0 & 0 & 18 & -1 \\ 0 & 0 & -13 & 8 \end{vmatrix} \quad \begin{array}{l} (1') \\ (2''') \\ (3''') = (3'') + 2(2''') \\ (4'') = (4') - 3(2''') \end{array}$$

Replace the fourth equation by a combination of the fourth and third:

$$18|A| = \begin{vmatrix} 1 & 0 & -2 & 0 \\ 0 & -1 & 7 & -2 \\ 0 & 0 & 18 & -1 \\ 0 & 0 & 0 & 131 \end{vmatrix} \quad \begin{array}{l} (1') \\ (2''') \\ (3''') \\ (4''') = 18(4'') + 13(3''') \end{array}$$

The determinant of a *triangular* matrix is the product of the elements on the main diagonal:

$$18|A| = (1)(-1)(18)(131) \quad \Longrightarrow \quad |A| = -131$$

Introduction

Eigenvalues:

- buckling;
- modes of vibration;
- dynamical systems;
- principal axes;
- boundary layer instability;
- heat conduction;
- acoustics;
- electrical circuits;
- stability of numerical methods;
- exam questions;
- ...

Definition

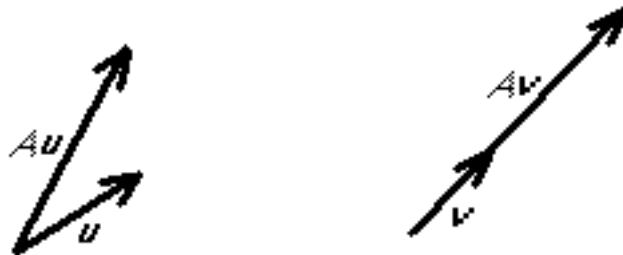A nonzero vector $\vec{v}$ is an eigenvector of a matrix A if $A\vec{v}$ is a multiple of $\vec{v}$:

$$A\vec{v} = \lambda\vec{v}$$

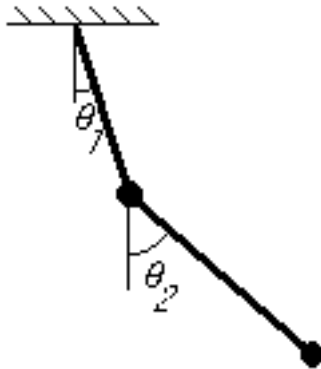The number λ is called the corresponding eigenvalue.

Graphically, if $\vec{v}$ is an eigenvector of A, then the vector $A\vec{v}$ is in the same (or exactly opposite direction) as $\vec{v}$:

An eigenvector is indeterminate by a constant that must be chosen.

Example

Equations of motion:

$$M \begin{pmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{pmatrix} + K \begin{pmatrix} \theta_1 \\ \theta_2 \end{pmatrix} = 0$$

Setting $\vec{\theta} \equiv (\theta_1, \theta_2)$

$$M\ddot{\vec{\theta}} + K\vec{\theta} = 0$$

Premultiplying by M^{-1} and defining $A = M^{-1}K$,

$$\ddot{\vec{\theta}} + A\vec{\theta} = 0$$

Try solutions of the form $\vec{\theta} = \vec{C}e^{i\omega t}$. The constant vector $\vec{C}$ determines the "mode shape:" $\theta_1/\theta_2 = C_1/C_2$. The exponential gives the time-dependent amplitude.

Plugging the assumed solution into the equations of motion:

$$-\omega^2 \vec{C} + A\vec{C} = 0 \qquad \Longrightarrow \qquad A\vec{C} = \omega^2 \vec{C}$$

So the mode shape $\vec{C}$ is an eigenvector of A and the corresponding eigenvalue gives the square of the frequency.

There will be two different eigenvectors $\vec{C}$, hence two mode shapes and two corresponding frequencies.

Note: we may lose symmetry in the above procedure. There are better ways to do this.

Procedure

To find the eigenvalues and eigenvectors of a matrix A,

1. Find the zeros of the determinant $|A - \lambda I|$ (i.e. of matrix A with $-\lambda$ added to each main diagonal element.) (The book uses $\lambda I - A$. This is very error-prone, and I do not recommend it.) For an $n \times n$ matrix A, $|A - \lambda I|$ is an n-th degree polynomial in λ. From it, we can find n eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$, (which do not all need to be distinct, however.)

2. When the eigenvalues are found, for each eigenvalue λ_i the corresponding eigenvector(s) can be found as the basis of the null space of $A - \lambda_i I$. *Note: Do not leave undetermined coefficients in eigenvectors. This is counted as an error.*

Eigenvector Basis

Examples:

- decomposing motion along the fundamental modes;

- writing solid body motion along the principal axes;

- separation of variables;

- improving numerical schemes;

- ...

Diagonalization:

If we use the eigenvectors $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$ of a matrix A as a new basis, so that the transformation matrix P contains the eigenvectors:

$$P = (\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n),$$

then the transformed matrix A' is much simpler than the original A. In particular, it is diagonal:

$$A' = \begin{pmatrix} \lambda_1 & 0 & 0 & \cdots & 0 \\ 0 & \lambda_2 & 0 & \cdots & 0 \\ 0 & 0 & \lambda_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda_n \end{pmatrix}$$

Reason: for any arbitrary vector

$$\vec{w} = w_1\big|_S \vec{v}_1 + w_2\big|_S \vec{v}_2 + \ldots + w_n\big|_S \vec{v}_n$$

then

$$A\vec{w} = w_1\big|_S \lambda_1 \vec{v}_1 + w_2\big|_S \lambda_2 \vec{v}_2 + \ldots + w_n\big|_S \lambda_n \vec{v}_n$$

So A increases the first coordinate in the eigenvector basis by λ_1, the second by λ_2, etcetera. That is exactly what the diagonal matrix A' does with the vector of coefficients $(w_1\big|_S, w_2\big|_S, \ldots, w_n\big|_S)$.

Remember that the relationship between A and A' is

$$A = PA'P^{-1} \text{ or } A' = P^{-1}AP$$

Note: If an $n \times n$ matrix A has less than n independent eigenvectors, it is *not* diagonalizable. It is called *defective.* Most matrices are however diagonalizable:

- As long as all n-eigenvalues are distinct, the matrix is diagonalizable.

- *Normal* matrices, which commute with their transpose, $AA^H = A^H A$, always have a complete set of *orthonormal* eigenvectors anyway.

- Even if the matrix has less than n different eigenvalues and is not normal, it *might* still be diagonalizable.

9.47

1 9.47(a), §1 Asked

Given:

$$A = \begin{pmatrix} 5 & 6 \\ -2 & -2 \end{pmatrix}$$

Asked: All eigenvalues and linearly independent eigenvectors.

2 9.47(a), §2 Solution

Eigenvalues:

$$|A - \lambda I| = \begin{vmatrix} 5 - \lambda & 6 \\ -2 & -2 - \lambda \end{vmatrix} = \lambda^2 - 3\lambda + 2 = 0$$

There are two roots: $\lambda_1 = 1$ and $\lambda_2 = 2$

The eigenvector corresponding to λ_1 satisfies

$$(A - \lambda_1 I)\vec{v}_1 = 0 = \begin{pmatrix} 5 - 1 & 6 \\ -2 & -2 - 1 \end{pmatrix} \vec{v}_1$$

Solving using Gaussian elimination:

$$\begin{pmatrix} 4 & 6 & | & 0 \\ -2 & -3 & | & 0 \end{pmatrix} \quad \begin{matrix} (1) \\ (2) \end{matrix} \implies \begin{pmatrix} \boxed{4} & 6 & | & 0 \\ 0 & 0 & | & 0 \end{pmatrix} \quad \begin{matrix} (1) \\ (2') = 2(2) + (1) \end{matrix}$$

Equation (1) gives $v_{1x} = -\frac{3}{2}v_{1y}$. In order to get a vector, instead of a set of possible vectors, one component must be arbitrarily chosen. *Remember: undetermined constants in eigenvectors are not allowed.* To get simple numbers, take $v_{1y} = -2$, then $v_{1x} = 3$:

$$\vec{v}_1 = \begin{pmatrix} 3 \\ -2 \end{pmatrix}$$

Check:

$$A\vec{v}_1 \overset{?}{=} \lambda_1 \vec{v}_1 \qquad \begin{pmatrix} 5 & 6 \\ -2 & -2 \end{pmatrix} \begin{pmatrix} 3 \\ -2 \end{pmatrix} = \begin{pmatrix} 3 \\ -2 \end{pmatrix}$$

Note: the null space of the matrix above is

$$\begin{pmatrix} v_{1x} \\ v_{1y} \end{pmatrix} = \begin{pmatrix} -\frac{3}{2} \\ 1 \end{pmatrix} v_{1y}$$

so $(-\frac{3}{2}, 1)$ would also have been an acceptable eigenvector, just messier.

The eigenvector corresponding to λ_2 satisfies

$$(A - \lambda_2 I)\vec{v}_2 = 0 = \begin{pmatrix} 5-2 & 6 \\ -2 & -2-2 \end{pmatrix} \vec{v}_1$$

Solving using Gaussian elimination:

$$\left(\begin{array}{cc|} 3 & 6 \\ -2 & -4 \end{array} \right) \quad \begin{array}{c} (1) \\ (2) \end{array} \implies \left(\begin{array}{cc|} 3 & 6 \\ 0 & 0 \end{array} \right) \quad \begin{array}{c} (1) \\ (2') = 3(2) + 2(1) \end{array}$$

Choosing $v_{2y} = 1$, then $v_{2x} = -2$:

$$\vec{v}_2 = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$$

3   9.47(b), §1 Asked

Given:

$$A = \begin{pmatrix} 5 & 6 \\ -2 & -2 \end{pmatrix}$$

with eigenvalues and corresponding eigenvectors

$$\lambda_1 = 1, \ \vec{v}_1 = \begin{pmatrix} 3 \\ -2 \end{pmatrix} \qquad \lambda_2 = 2, \ \vec{v}_2 = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$$

Asked: A matrix P such that $A' = P^{-1}AP$ is diagonal.

4   9.47(b), §2 Solution

- The matrix $P^{-1}AP$ is the new matrix A' after a basis transformation described by a transformation matrix P.

- To get A' diagonal, we want to take the new basis to be the eigenvectors $\vec{v}_1$ and $\vec{v}_2$ of A.

- A transformation matrix consists of the new basis vectors expressed in terms of the old basis, so:

$$P = (\vec{v}_1 \ \vec{v}_2) = \begin{pmatrix} 3 & -2 \\ -2 & 1 \end{pmatrix}$$

Check:

$$A' = P^{-1}AP = \begin{pmatrix} 3 & -2 \\ -2 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 5 & 6 \\ -2 & -2 \end{pmatrix} \begin{pmatrix} 3 & -2 \\ -2 & 1 \end{pmatrix}$$

$$= \frac{1}{-1} \begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix}^{T} \begin{pmatrix} 3 & -4 \\ -2 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$$

5   9.47(c), §1 Asked

Given:

$$A = \begin{pmatrix} 5 & 6 \\ -2 & -2 \end{pmatrix}$$

with eigenvalues and corresponding eigenvectors

$$\lambda_1 = 1, \ \vec{v}_1 = \begin{pmatrix} 3 \\ -2 \end{pmatrix} \qquad \lambda_2 = 2, \ \vec{v}_2 = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$$

Asked: A^6 and $A^4 - 5A^3 + 7A^2 - 2A + 5$

6   9.47(c), §2 Solution

Do it first in the eigenvector basis!

$$A' = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \implies A'^6 = \begin{pmatrix} 1^6 & 0 \\ 0 & 2^6 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 64 \end{pmatrix}$$

This works for diagonal matrices only.

Now transform back:

$$A^6 = PA'^6 P^{-1} = \begin{pmatrix} 3 & -2 \\ -2 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 64 \end{pmatrix} \begin{pmatrix} 3 & -2 \\ -2 & 1 \end{pmatrix}^{-1}$$

$$A^6 = \begin{pmatrix} 3 & -128 \\ -2 & 64 \end{pmatrix} \frac{1}{-1} \begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix}^{T} = \begin{pmatrix} 253 & 378 \\ -126 & -188 \end{pmatrix}$$

Note that this is very different from

$$\begin{pmatrix} 5^6 & 6^6 \\ (-2)^6 & (-2)^6 \end{pmatrix}$$

(Answer in the book is for A^{10}

$$A' = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \implies A'^4 - 5A'^3 + 7A'^2 - 2A' + 5I =$$

$$\begin{pmatrix} 1\,1^4 - 5\,1^3 + 7\,1^2 - 2\,1 + 5 & 0 \\ 0 & 2^4 - 5\,2^3 + 7\,2^2 - 2\,2 + 5 \end{pmatrix} = \begin{pmatrix} 6 & 0 \\ 0 & 5 \end{pmatrix}$$

$$A^4 - 5A^3 + 7A^2 - 2A + 5I = \begin{pmatrix} 3 & -2 \\ -2 & 1 \end{pmatrix} \begin{pmatrix} 6 & 0 \\ 0 & 5 \end{pmatrix} \begin{pmatrix} 3 & -2 \\ -2 & 1 \end{pmatrix}^{-1}$$

$$= \begin{pmatrix} 18 & -10 \\ -12 & 5 \end{pmatrix} \frac{1}{-1} \begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix}^T = \begin{pmatrix} 2 & -6 \\ 2 & 9 \end{pmatrix}$$

7   9.47(d), §1 Asked

Given:

$$A = \begin{pmatrix} 5 & 6 \\ -2 & -2 \end{pmatrix}$$

with eigenvalues and corresponding eigenvectors

$$\lambda_1 = 1, \ \vec{v}_1 = \begin{pmatrix} 3 \\ -2 \end{pmatrix} \qquad \lambda_2 = 2, \ \vec{v}_2 = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$$

Asked: A matrix B so that $B^2 = A$

8   9.47(d), §2 Solution

Do it first in the eigenvector basis!

$$A' = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \implies B' = \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{2} \end{pmatrix}$$

This works for diagonal matrices only.

Now transform back:

$$B = PB'P^{-1} = \begin{pmatrix} 3 & -2 \\ -2 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{2} \end{pmatrix} \begin{pmatrix} 3 & -2 \\ -2 & 1 \end{pmatrix}^{-1}$$

$$B = \begin{pmatrix} 3 & -2\sqrt{2} \\ -2 & \sqrt{2} \end{pmatrix} \frac{1}{-1} \begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix}^T = \begin{pmatrix} -3+4\sqrt{2} & -6+6\sqrt{2} \\ 2-2\sqrt{2} & 4-3\sqrt{2} \end{pmatrix}$$

9.48(c)

1   9.48(c), §1 Asked

Given:
$$A = \begin{pmatrix} 1 & 2 & 2 \\ 1 & 2 & -1 \\ -1 & 1 & 4 \end{pmatrix}$$

Asked: All eigenvalues and linearly independent eigenvectors.

2   9.48(c), §2 Solution

Eigenvalues:
$$0 = |A - \lambda I| = \begin{vmatrix} 1 - \lambda & 2 & 2 \\ 1 & 2 - \lambda & -1 \\ -1 & 1 & 4 - \lambda \end{vmatrix}$$

$$= -\lambda^3 + 7\lambda^2 - 15\lambda + 9 = -(\lambda - 1)(\lambda - 3)^2 = 0$$

There is a single root: $\lambda_1 = 1$ and a double root $\lambda_2 = \lambda_3 = 3$

Eigenvectors corresponding to $\lambda_1 = 1$ satisfy

$$(A - \lambda_1 I)\vec{v}_1 = 0 = \begin{pmatrix} 1 - 1 & 2 & 2 \\ 1 & 2 - 1 & -1 \\ -1 & 1 & 4 - 1 \end{pmatrix} \begin{pmatrix} v_{1x} \\ v_{1y} \\ v_{1z} \end{pmatrix}$$

Solving using Gaussian elimination:

$$\begin{pmatrix} 0 & 2 & 2 & \big| & 0 \\ 1 & 1 & -1 & \big| & 0 \\ -1 & 1 & 3 & \big| & 0 \end{pmatrix} \qquad \begin{matrix} (1) \\ (2) \\ (3) \end{matrix}$$

$$\implies \begin{pmatrix} 1 & 1 & -1 & \big| & 0 \\ 0 & 2 & 2 & \big| & 0 \\ -1 & 1 & 3 & \big| & 0 \end{pmatrix} \qquad \begin{matrix} (1') = (2) \\ (2') = (1) \\ (3) \end{matrix}$$

$$\implies \begin{pmatrix} 1 & 1 & -1 & \big| & 0 \\ 0 & 2 & 2 & \big| & 0 \\ 0 & 2 & 2 & \big| & 0 \end{pmatrix} \qquad \begin{matrix} (1') \\ (2') \\ (3') = (3) + (1') \end{matrix}$$

$$\implies \begin{pmatrix} \boxed{1} & 1 & -1 & \Big| & 0 \\ 0 & \boxed{2} & 2 & \Big| & 0 \\ 0 & 0 & 0 & \Big| & 0 \end{pmatrix} \qquad \begin{matrix} (1') \\ (2') \\ (3'') = (3') - (2') \end{matrix}$$

Equation (2') gives $v_{1y} = -v_{1z}$ and then (1') gives $v_{1x} = 2v_{1z}$.

The general solution space is:

$$\begin{pmatrix} v_{1x} \\ v_{1y} \\ v_{1z} \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} v_{1z}$$

We *choose* $v_{1z} = 1$ to get

$$\vec{v}_1 = \begin{pmatrix} v_{1x} \\ v_{1y} \\ v_{1z} \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}$$

Eigenvectors corresponding to $\lambda_2 = \lambda_3 = 3$ satisfy

$$(A - \lambda_2 I)\vec{v}_2 = 0 = \begin{pmatrix} 1-3 & 2 & 2 \\ 1 & 2-3 & -1 \\ -1 & 1 & 4-3 \end{pmatrix} \begin{pmatrix} v_{2x} \\ v_{2y} \\ v_{2z} \end{pmatrix}$$

Solving using Gaussian elimination:

$$\begin{pmatrix} -2 & 2 & 2 & \Big| & 0 \\ 1 & -1 & -1 & \Big| & 0 \\ -1 & 1 & 1 & \Big| & 0 \end{pmatrix} \qquad \begin{matrix} (1) \\ (2) \\ (3) \end{matrix}$$

$$\implies \begin{pmatrix} \boxed{-2} & 2 & 2 & \Big| & 0 \\ 0 & 0 & 0 & \Big| & 0 \\ 0 & 0 & 0 & \Big| & 0 \end{pmatrix} \qquad \begin{matrix} (1) \\ (2') = 2(2) + (1) \\ (3') = 2(3) - (1) \end{matrix}$$

Equation (1') gives $v_{2x} = v_{2y} + v_{2z}$. There are two unknown parameters.

The general solution space is:

$$\begin{pmatrix} v_{2x} \\ v_{2y} \\ v_{2z} \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} v_{2y} + \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} v_{2z}$$

We need *two* independent eigenvectors to span the space corresponding to this multiple root.

We can use the two vectors above, which means choosing $v_{2y} = 1$ and $v_{2z} = 0$ for one, and $v_{2y} = 0$ and $v_{2z} = 1$ for the other. That gives

$$\vec{v}_{2a} = \begin{pmatrix} v_{2ax} \\ v_{2ay} \\ v_{2az} \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \qquad \vec{v}_{2b} = \begin{pmatrix} v_{2bx} \\ v_{2by} \\ v_{2bz} \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

If the three vectors $\vec{v}_1$, $\vec{v}_{2a}$, and $\vec{v}_{2b}$ are used as basis, A becomes diagonal. So despite the multiple root, this A is still diagonalizable. But if the solution space for the second eigenvalue would have been one-dimensional, the matrix would not have been diagonalizable.

y

Symmetric Matrices

Definition

A matrix A is symmetric if $A^T = A$.

Examples:

- mass and stiffness matrices found from the Lagrangian equations;

- finite element methods for structures, fluids, ...;

- inertia matrices of solid bodies;

- ...

Diagonalization:

- Symmetric matrices have *real* eigenvalues.

- Symmetric matrices *always* have a complete set of independent eigenvectors.

- These eigenvectors are (or at least can be taken to be) orthonormal.

For symmetric matrices, in this class you are *required* to orthonormalize the eigenvectors. As long as the null space of each eigenvalue has only one basis vector, this simply means normalizing the eigenvactor to length one (i.e. divide by its length.) If the null space has multiple basis vectors however, you will need to apply Gram-Schmidt on them or equivalent.

In either case, the result is that the eigenvectors to are an orthonormal set that we can indicate as $\hat{\imath}', \hat{\jmath}', \hat{k}', \ldots$, and is no more than a rotated coordinate system. In other words, *symmetric matrices are diagonalized by merely rotating the coordinate system.*

As was mentioned in chapter 2, since the transformation matrix $P = (\hat{\imath}', \hat{\jmath}', \hat{k}', \ldots)$ has orthonormal columns, it is called an orthonormal matrix. For any orthonormal matrix

$$P^{-1} = P^T$$
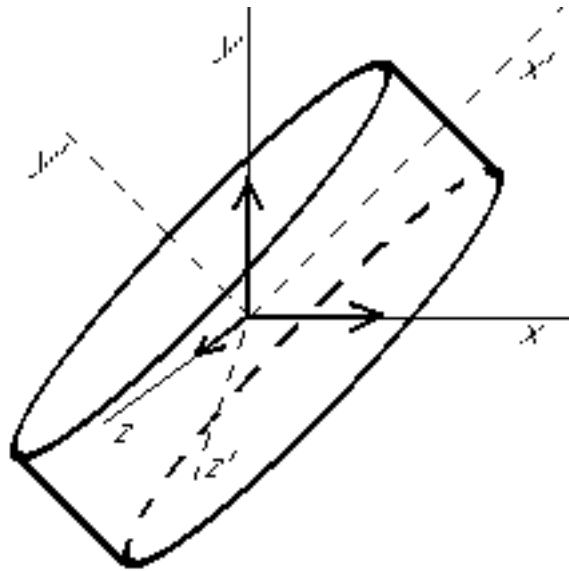
Example:

Kinetic energy of a solid body:

$$T = \tfrac{1}{2}\vec{v}_{\text{cg}}^T m \vec{v}_{\text{cg}} + \tfrac{1}{2}\vec{\omega}^T I \vec{\omega}$$

$$I = \begin{pmatrix} \int (y^2 + z^2)\, \mathrm{d}m & \int xy\, \mathrm{d}m & \int xz\, \mathrm{d}m \\ \int xy\, \mathrm{d}m & \int (x^2 + z^2)\, \mathrm{d}m & \int yz\, \mathrm{d}m \\ \int xz\, \mathrm{d}m & \int yz\, \mathrm{d}m & \int (x^2 + y^2)\, \mathrm{d}m \end{pmatrix}$$

where the x, y, z axis system has its origin at the center of gravity.

By rotating the x, y, z axis system to the principal axes of the body, the inertia matrix I becomes diagonal.

For a disk:

If you write the inertia matrix for a disk and find the eigenvalues, you will find one single eigenvalue and one double eigenvalue, giving the moments of inertia along the principal axes.

The eigenvector corresponding to the single eigenvalue will be in the y' direction; just normalize it to length one to give $\hat{j}'$.

The eigenvector solution space corresponding to the double eigenvalue will have two independent basis vectors. Use Gram-Schmidt on them to orthonormalize them, that will poduce your $\hat{i}'$ and $\hat{k}'$.

The axis system $\hat{i}', \hat{j}', \hat{k}'$ are the principal axes of the disk.

9.56(a)

1 9.56(a), §1 Asked

Given:
$$A = \begin{pmatrix} 5 & 4 \\ 4 & -1 \end{pmatrix}$$

Asked: The orthonormal transformation matrix P so that $A' = P^{-1}AP$ is diagonal.

2 9.56(a), §2 Solution

Given:
$$A = \begin{pmatrix} 5 & 4 \\ 4 & -1 \end{pmatrix}$$

Eigenvalues:
$$|A - \lambda I| = \begin{vmatrix} 5 - \lambda & 4 \\ 4 & -1 - \lambda \end{vmatrix} = \lambda^2 - 4\lambda - 21 = 0$$

There are two roots: $\lambda_1 = -3$ and $\lambda_2 = 7$

The eigenvector corresponding to $\lambda_1 = -3$ satisfies

$$\left(\begin{array}{cc|} 8 & 4 \\ 4 & 2 \end{array} \right) \quad \begin{array}{c} (1) \\ (2) \end{array} \quad \Longrightarrow \quad \left(\begin{array}{cc|} 8 & 4 \\ 0 & 0 \end{array} \right) \quad \begin{array}{c} (1) \\ (2') = 2(2) - (1) \end{array}$$

Taking $v_{1y} = -2$, then $v_{1x} = 1$, giving an eigenvector (1,-2). Normalizing this vector to length one gives:
$$\vec{v}_1 = \begin{pmatrix} 1 \\ -2 \end{pmatrix} \Big/ \sqrt{1^2 + (-2)^2} = \begin{pmatrix} 1/\sqrt{5} \\ -2/\sqrt{5} \end{pmatrix}$$

The eigenvector corresponding to $\lambda_2 = 7$ satisfies

$$\left(\begin{array}{cc|} -2 & 4 \\ 4 & -8 \end{array} \right) \quad \begin{array}{c} (1) \\ (2) \end{array} \quad \Longrightarrow \quad \left(\begin{array}{cc|} -2 & 4 \\ 0 & 0 \end{array} \right) \quad \begin{array}{c} (1) \\ (2') = (2) + 2(1) \end{array}$$

Taking $v_{2y} = 1$, then $v_{2x} = 2$, giving after normalization:

$$\vec{v}_2 = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \Big/ \sqrt{2^2 + 1^2} = \begin{pmatrix} 2/\sqrt{5} \\ 1/\sqrt{5} \end{pmatrix}$$
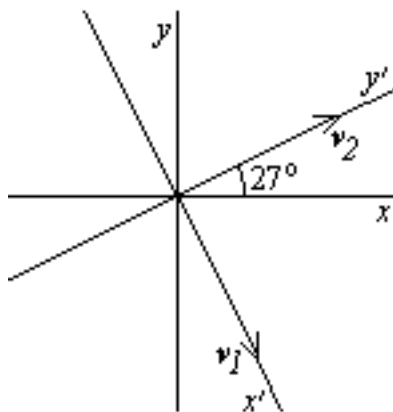
Finally:
$$P = (\vec{v}_1\,\vec{v}_2) = \begin{pmatrix} 1/\sqrt{5} & 2/\sqrt{5} \\ -2/\sqrt{5} & 1/\sqrt{5} \end{pmatrix}$$

Check:
$$P^{-1}AP = \begin{pmatrix} 1/\sqrt{5} & -2/\sqrt{5} \\ 2/\sqrt{5} & 1/\sqrt{5} \end{pmatrix} \begin{pmatrix} 5 & 4 \\ 4 & -1 \end{pmatrix} \begin{pmatrix} 1/\sqrt{5} & 2/\sqrt{5} \\ -2/\sqrt{5} & 1/\sqrt{5} \end{pmatrix} = \begin{pmatrix} -3 & 0 \\ 0 & 7 \end{pmatrix}$$

The diagonal form is what matrix A looks like in the coordinate system x', y' shown below:

Quadratic Forms

Examples:

- quadratic curves (circles, ellipses, hyperbolae, parabolae) and surfaces (spheres, spheroids, ellipsoids, cones, cylinders, ...);

- kinetic energy of solid bodies;

- potential energy near equilibria;

- ...

Matrix form:

$$
\begin{aligned}
\vec{x}^T A \vec{x} \;=\; & a_{11}x_1^2 + a_{12}x_1x_2 + a_{13}x_1x_3 + \ldots \\
+\; & a_{21}x_2x_1 + a_{22}x_2^2 + a_{23}x_2x_3 + \ldots \\
+\; & \ldots
\end{aligned}
$$

An orthonormal transformation leaves the quadratic form unchanged

$$
\vec{x}'^T A' \vec{x}' = \vec{x}^T P^{T^T} P^T A P\, P^T \vec{x} = \vec{x}^T A \vec{x}
$$

but can simplify the coefficients. On principal axes

$$
\vec{x}'^T A' \vec{x}' = a'_{11}{x'_1}^2 + a'_{22}{x'_2}^2 + \ldots
$$

9.58(b)

1 9.58(b), §1 Asked

Asked: Diagonalize

$$q(x, y) = 2x^2 - 6xy + 10y^2$$

2 9.58(b), §2 Solution

$$q = 2x^2 - 6xy + 10y^2$$

Find the matrix of coefficients:

$$A = \begin{pmatrix} 2 & -3 \\ -3 & 10 \end{pmatrix}$$

Eigenvalues:

$$|A - \lambda I| = \begin{vmatrix} 2 - \lambda & -3 \\ -3 & 10 - \lambda \end{vmatrix} = \lambda^2 - 12\lambda + 11 = 0$$

There are two roots: $\lambda_1 = 1$ and $\lambda_2 = 11$

The eigenvector corresponding to λ_1 satisfies

$$\begin{pmatrix} 1 & -3 \\ -3 & 9 \end{pmatrix} \quad \begin{matrix} (1) \\ (2) \end{matrix} \implies \begin{pmatrix} 1 & -3 \\ 0 & 0 \end{pmatrix} \quad \begin{matrix} (1) \\ (2') = (2) + 3(1) \end{matrix}$$

Taking $v_{1y} = 1$, then $v_{1x} = 3$, giving an eigenvector (3,1). Normalizing this vector to length one gives:

$$\vec{v}_1 = \begin{pmatrix} 3 \\ 1 \end{pmatrix} \Big/ \sqrt{3^2 + 1^2} = \begin{pmatrix} 3/\sqrt{10} \\ 1/\sqrt{10} \end{pmatrix} = \hat{i}'$$
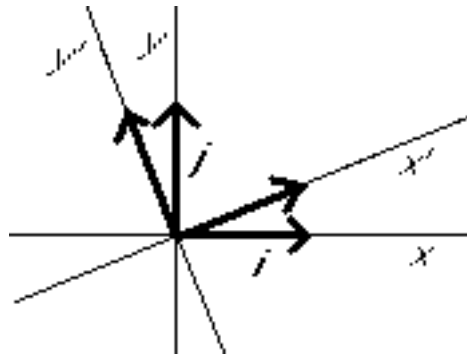
The eigenvector corresponding to λ_2 satisfies

$$\begin{pmatrix} -9 & -3 \\ -3 & -1 \end{pmatrix} \quad \begin{matrix} (1) \\ (2) \end{matrix} \implies \begin{pmatrix} -9 & -3 \\ 0 & 0 \end{pmatrix} \quad \begin{matrix} (1) \\ (2') = 3(2) - (1) \end{matrix}$$

Taking $v_{2y} = 3$, then $v_{2x} = -1$, giving after normalization:

$$\vec{v}_2 = \begin{pmatrix} -1 \\ 3 \end{pmatrix} \Big/ \sqrt{(-1)^2 + 3^2} = \begin{pmatrix} -1/\sqrt{10} \\ 3/\sqrt{10} \end{pmatrix} = \hat{j}'$$

Since $1/\sqrt{10} = \sin(18.4°)$, the new axes are rotated $18.5°$ counter clockwise from the old:

In the new coordinates,

$$q = x'^2 + 11y'^2$$

Note that lines of constant q are now seen to be elliptic.

Important note: It is seen that the quadratic form $\vec{x}^T A \vec{x}$ is always positive for nonzero $\vec{x}$. Symmetric matrices for which this is true are called *positive definite.* They have all positive eigenvalues. Similarly, if all eigenvalues are negative, a symmetric matrix is called *negative definite.* If all eigenvalues are positive or zero, it is called *positive semi-definite.*

Finite element codes for structures typically produce positive definite matrices, as do many other physical applications, such as the kinetic energy of a solid body. Definite matrices are typically easier to deal with in numerical applications than general matrices. For example, no pivoting is needed in the Gaussian elimination involving a definite matrix.